MATH 1553
PRACTICE MIDTERM 3 (VERSION A)

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to chapter 3 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to chapters 3 and 5.

Problem 1. [2 points each]

In this problem, if the statement is always true, circle **T**; if it is always false, circle **F**; if it is sometimes true and sometimes false, circle **M**.

a) **T** **F** **M** If A is a 3×3 matrix with characteristic polynomial $-\lambda^3 + \lambda^2 + \lambda$, then A is invertible.

b) **T** **F** **M** A 3×3 matrix with (only) two distinct eigenvalues is diagonalizable.

c) **T** **F** **M** If A is diagonalizable and B is similar to A, then B is diagonalizable.

d) **T** **F** **M** A diagonalizable $n\times n$ matrix admits n linearly independent eigenvectors.

e) **T** **F** **M** If $\det(A) = 0$, then 0 is an eigenvalue of A.

Solution.

a) False: $\lambda = 0$ is a root of the characteristic polynomial, so 0 is an eigenvalue, and A is not invertible.

b) Maybe: it is diagonalizable if and only if the eigenspace for the eigenvalue with multiplicity 2 has dimension 2.

c) True: if $A = PDP^{-1}$ with D diagonal, and $B = CAC^{-1}$, then

$$B = C(PDP^{-1})C^{-1} = (CP)D(CP)^{-1},$$

so B is also similar to a diagonal matrix.

d) True: by the Diagonalization Theorem, an $n\times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

e) True: if $\det(A) = 0$ then A is not invertible, so $Av = 0v$ has a nontrivial solution.

Problem 2.

[2 points each]

Give an example of a 2×2 real matrix A with each of the following properties. You need not explain your answer.

a) A has no real eigenvalues.

b) A has eigenvalues 1 and 2.

c) A is invertible but not diagonalizable.

d) A is diagonalizable but not invertible.

e) A is a rotation matrix with real eigenvalues.

Solution.

a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

e) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Problem 3.

Consider the matrix
$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}.$$

a) [4 points] Find the eigenvalues of A, and compute their algebraic multiplicities.

b) [4 points] For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) [2 points] Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, why not?

Solution.

a) We compute the characteristic polynomial by expanding along the second row:

$$f(\lambda) = \det\begin{pmatrix} 4-\lambda & 2 & -4 \\ 0 & 2-\lambda & 0 \\ 2 & 2 & -2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 4-\lambda & -4 \\ 2 & -2-\lambda \end{pmatrix}$$
$$= (2-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda-2)^2$$

The roots are 0 (with multiplicity 1) and 2 (with multiplicity 2).

b) First we compute the 0-eigenspace by solving $(A - 0I)x = 0$:

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric vector form of the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the 0-eigenspace is $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Next we compute the 2-eigenspace by solving $(A - 2I)x = 0$:

$$A - 2I = \begin{pmatrix} 2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & 2 & -4 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric vector form for the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the 2-eigenspace is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

c) We have produced three linearly independent eigenvectors, so the matrix is diagonalizable:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}.$$

Problem 4.

Consider the matrix

$$A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}.$$

a) [3 points] Find the (complex) eigenvalues of A.

b) [2 points] For each eigenvalue of A, find a corresponding eigenvector.

c) [3 points] Find a rotation-scaling matrix C that is similar to A.

d) [1 point] By what factor does C scale?

e) [1 point] By what angle does C rotate?

Solution.

a) The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} 3-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix} = \lambda^2 + 1.$$

Its roots are the eigenvalues $\lambda = \pm i$.

b) First we find an eigenvector corresponding to the eigenvalue i by solving the equation $(A - iI)x = 0$.

$$A - iI = \begin{pmatrix} 3-i & -5 \\ 2 & -3-i \end{pmatrix}.$$

We know that this matrix is not invertible, since i is an eigenvalue; hence the second row must be a multiple of the first, so a row echelon form for A is $\begin{pmatrix} 3-i & -5 \\ 0 & 0 \end{pmatrix}$. The parametric form of the solution is $(3-i)x = 5y$, so an eigenvector is $\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

The second eigenvalue $-i$ is the complex conjugate of the first, so it admits the complex conjugate $\begin{pmatrix} 5 \\ 3+i \end{pmatrix}$ as an eigenvector.

c) If $\lambda = a + bi$ is an eigenvector, then A is similar to the rotation-scaling matrix $C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Choosing $\lambda = -i$ means $a = 0$ and $b = -1$, so $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

d) The scaling factor is $|\lambda| = |-i| = 1$.

e) The argument of $\lambda = -i$ is $-\pi/2$, so the matrix C rotates by $+\pi/2$.

Problem 5. [10 points]

Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^3.$$

Solution.

The determinant of the cube is the cube of the determinant, so we start by computing

$$\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}.$$

This is a big, complicated matrix, so it's easiest to use row reduction.

$$\begin{aligned}
\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix} &= \det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{pmatrix} && \text{(row replacements)} \\
&= \det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{pmatrix} && \text{(row replacements)} \\
&= -\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{pmatrix} && \text{(row swap)} \\
&= -1 \cdot 1 \cdot 1 \cdot 10 && \text{(triangular matrix)} \\
&= -10.
\end{aligned}$$

Thus

$$\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^3 = -1000.$$

[Scratch work]