

Math 1553 Worksheet: Lines and planes in  $\mathbb{R}^n$  and §1.1

Solutions

1. Which of the following equations are linear? Justify your answers.

a)  $3x_1 + \sqrt{x_2} = 4$

b)  $x_1 = x_2 - x_3 + 10x_4$ .

c)  $\pi x + \ln(13)y + z = \sqrt[3]{2}$

**Solution.**

a) No. The  $\sqrt{x_2}$  term makes it non-linear.

b) Yes.

c) Yes. The  $\sqrt[3]{2}$  term is just a constant. Don't be misled by the appearance of the natural logarithm:  $\ln(13)$  is just the coefficient for  $y$ .

If the second term had been  $\ln(13y)$  instead of  $\ln(13)y$ , then  $y$  would have been inside the logarithm and the equation would have been non-linear.

2. Find all values of  $h$  so that the lines  $x + hy = -5$  and  $2x - 8y = 6$  do *not* intersect.

**Solution.**

We can use standard algebra, row operations, or geometric intuition.

Using standard algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point  $(x, y)$  where

$$x + hy = -5$$

$$2x - 8y = 6.$$

Subtracting twice the first equation from the second equation gives us

$$x + hy = -5$$

$$0 + (-8 - 2h)y = 16.$$

If  $-8 - 2h = 0$  (so  $h = -4$ ), then the second line is  $0 \cdot y = 16$ , which is impossible. In other words, if  $h = -4$  then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if  $h \neq -4$ , then we can solve for  $y$  above:

$$(-8 - 2h)y = 16 \quad y = \frac{16}{-8 - 2h} \quad y = \frac{8}{-4 - h}.$$

We can now substitute this value of  $y$  into the first equation to find  $x$ :

$$x + hy = -5 \quad x + h \cdot \frac{8}{-4 - h} = -5 \quad x = -5 - \frac{8h}{-4 - h}.$$

Therefore, the lines fail to intersect if and only if  $\boxed{h = -4}$ .

Using row operations: Like the previous technique, let's see what happens if the lines intersect. We put the equations into augmented matrix form and use row operations.

$$\left[ \begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array} \right] \xrightarrow{R_2=R_2-2R_1} \left[ \begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array} \right].$$

If  $-8 - 2h = 0$  (so  $h = -4$ ), then the second equation is  $0 = 16$ , so our system has no solutions. In other words, the lines do not intersect.

If  $h \neq -4$ , then the second equation is  $(-8 - 2h)y = 16$ , so  $y = \frac{16}{-8 - 2h} = \frac{8}{-4 - h}$ , and  $x = -5 - hy = -5 - \frac{8h}{-4 - h}$ , so the lines intersect at  $\left(-5 - \frac{8h}{-4 - h}, \frac{8}{-4 - h}\right)$ .

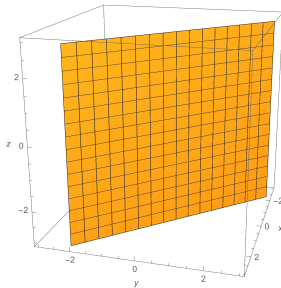
Therefore, our answer is  $h = -4$ .

Using intuition from geometry: Two non-identical lines in the  $xy$ -plane intersect if and only if they are not parallel. The second line is  $y = \frac{1}{4}x - \frac{3}{4}$ , so its slope is  $\frac{1}{4}$ . If  $h \neq 0$ , then the first line is  $y = -\frac{1}{h}x - \frac{5}{h}$ , so the lines are parallel when  $-\frac{1}{h} = \frac{1}{4}$ , which means  $h = -4$ . You can check that when  $h = -4$  the lines aren't identical. (And if  $h = 0$  then the first line is vertical so it isn't parallel to the second).

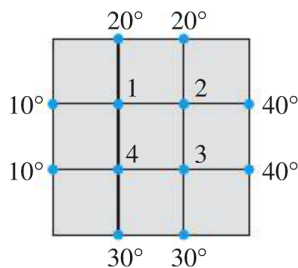
3. For each of the following, answer true or false. Justify your answer.
- Every system of linear equations has at least one solution.
  - There is a system of linear equations that has exactly 5 solutions.
  - If  $a$ ,  $b$ , and  $c$  are real numbers, then the equation  $ax + by = c$  for  $(x, y, z)$  in  $\mathbf{R}^3$  describes a line.

### Solution.

- False. Some examples from class and this worksheet have no solutions.
- False. There are only three possibilities: no solutions, exactly one solution, or infinitely many solutions.
- False. For example, in  $\mathbf{R}^3$ , the equation  $x + y = 1$  corresponds geometrically to a vertical plane. We could write the plane in parametric form as  $(t, 1 - t, z)$  where  $t$  and  $z$  vary among all real numbers.



4. The picture below represents the temperatures at four interior nodes of a mesh.



Let  $T_1, \dots, T_4$  be the temperatures at nodes 1 through 4. Suppose that the temperature at each node is the average of the four nearest nodes. For example,

$$T_1 = \frac{10 + 20 + T_2 + T_4}{4}.$$

- Write a system of four linear equations whose solution would give the temperatures  $T_1, \dots, T_4$ .
- Write an augmented matrix that represents that system of equations.

### Solution.

(a)

The first equation was given. The others are:

$$T_2 = (T_1 + 20 + 40 + T_3)/4, \quad \text{or} \quad 4T_2 - T_1 - T_3 = 60$$

$$T_3 = (T_4 + T_2 + 40 + 30)/4, \quad \text{or} \quad 4T_3 - T_4 - T_2 = 70$$

$$T_4 = (10 + T_1 + T_3 + 30)/4, \quad \text{or} \quad 4T_4 - T_1 - T_3 = 40$$

(b) To put this in matrix form, we need to put everything in order.

$$\begin{array}{cccccc} 4T_1 & - & T_2 & & & - & T_4 & = & 30 \\ -T_1 & + & 4T_2 & - & T_3 & & & = & 60 \\ & & -T_2 & + & 4T_3 & - & T_4 & = & 70 \\ -T_1 & & & - & T_3 & + & 4T_4 & = & 40 \end{array}$$

This gives the augmented matrix

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix}.$$

5. Consider the following three planes, where we use  $(x, y, z)$  to denote points in  $\mathbf{R}^3$ :

$$2x + 4y + 4z = 1$$

$$2x + 5y + 2z = -1$$

$$y + 3z = 8.$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

**Solution.**

We can isolate  $z$  in the third equation using algebra, but it is probably best to do using an augmented matrix and elementary row operations.

$$\left[ \begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array} \right] \xrightarrow{R_2=R_2-R_1} \left[ \begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array} \right] \xrightarrow{R_3=R_3-R_2} \left[ \begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array} \right].$$

The last line is  $5z = 10$ , so  $z = 2$ . Since we don't have much practice with row-reduction, we will use substitution to finish.

The second equation is  $y - 2z = -2$ , so  $y - 2(2) = -2$ , thus  $y = 2$ .

The first equation is  $2x + 4(2) + 4(2) = 1$ , so  $2x = -15$ , thus  $x = -\frac{15}{2}$ .

We have found that the planes intersect at the point  $\left(-\frac{15}{2}, 2, 2\right)$ .