Midterm 1, Math 110

1. Suppose that A, B and C are $n \times n$ matrices with

$$A = BC.$$

Prove that if the rank of B equals n, then

$$\operatorname{rank}(A) = \operatorname{rank}(C).$$

2. Prove that if A and B are two $n \times n$ nilpotent matrices which commute with one another, then $A + B$ is likewise nilpotent.

3. Let

$$M = \begin{bmatrix} \alpha & \beta & c \\ -\beta & \alpha & d \\ 0 & 0 & 1 \end{bmatrix},$$

where $\alpha^2 + \beta^2 = 1$. Find a matrix L such that

$$LM \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

for all real numbers x, y. (In fact, L will be the inverse of M.)

Hint: You have seen a matrix of type M. Ask youself how it acts on the column vector $(x, y, 1)$.

4. Suppose that M is a $3 \times n$ matrix, and suppose we perform the following set of row operations on M:

1. Interchange rows 2 and 3.
2. Then, add row 1 to row 3.
3. Then, interchange rows 1 and 3.
4. Finally, multiply row 2 by -3.

Find a 3×3 matrix E such that left-multiplication of the matrix M by the matrix E (so, we compute EM) has the same affect on the matrix M as performing the above row operations.

5. Write down the normal form of the 4×4 matrix $M = [m_{i,j}]$, whose entries are given by $m_{i,j} = i + j - 2$.