Study Guide for Math 2406 Final, Fall 2006

April 29, 2009

Here are a few items that you should know for the final exam, which will be open note.

• Know some basic facts about vectors, such as that the length of a vector $v = (v_1, ..., v_n)$, denoted by $|v|$, is $\sqrt{v_1^2 + \cdots + v_n^2}$. Know the definition of parallel and orthogonal vectors.

• Know some basic facts about planes and lines. For example, recall that a plane in n dimensions is the set of all points of the form

$$\{P_0 + tv_1 + uv_2 \ : \ u, t \in \mathbb{R}\}.$$

Here, P_0 is a point on the plane and v_1, v_2 are vectors that "sweep out" the plane. Recall that a line can be expressed as the set of all points of the form

$$\{P_0 + tv \ : \ t \in \mathbb{R}\},$$

where P_0 is any point on the line.

• Recall that in $\mathbb{R}^3$ we have an alternate way of expressing the set of points on a plane: A plane is the set of all points $(x, y, z) \in \mathbb{R}^3$ that satisfy

$$ax + by + cz \ = \ d,$$

where (a, b, c) is the normal vector of the plane. Recall also how to find a, b, c, d: We can let $(a, b, c) = v_1 \times v_2$, where $\times$ here denotes the cross product.

• Recall some basic facts about cross product. For example, the cross product of vectors (a, b, c) and (d, e, f) can be found by computing the determinant

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix},$$

where i, j, k denote the standard unit vector. Since a, b, c, d, e, f are given constants, this determinant can be written as $t_1 i + t_2 j + t_3 k$, where t_1, t_2, t_3 are scalars.

• Know how to find the distance from a given point to a plane: The nearest point on the plane to the given point will be the projection of the given point onto the plane. There are several ways to find this projection point, but perhaps the simplest is just to find the point P on the plane such that the vector $P - b$ (here, b is the given point) is parallel to the normal vector to the plane. We know that $P - b$ is parallel to the normal vector if and only if the dot product

$$(P - b) \cdot u_i \;=\; 0,\; i = 1, ..., k,$$

where $u_1, ..., u_k$ span the plane.

• Know how to compute the volume of a parallelopiped using determinants. One of the key facts that goes in to the derivation of this formula is the fact that

$$|u \times v| = \; |u||v|| \sin \theta|,$$

where θ is the angle between the vectors u and v.

• Know the definition of linearly independent vectors. Know the definition of linear dependent. Know how to determine whether a set of vectors is linearly independent (Write down the matrix whose columns are those vectors, and then row reduce. if the rank is n, then they are independent).

• Know how to prove that $v_1, ..., v_n$ are orthogonal (with respect to any particular inner product) implies that they are linearly independent.

• Know, and know how to prove, some basic facts about dimension. Recall that the dimension of a subspace (or vector space) is the size of any basis for the subspace. It is a central theorem in linear algebra that all bases have the same size. Know how to produce a basis by building up a list of linearly independent vectors until they span all the whole subspace. One of the key ingredients in some of these proofs is as follows: If $v_1, ..., v_n$ are linearly independent, but do not span all of S, then every vector v outside of the span of $v_1, ..., v_n$ is linearly independent from $v_1, ..., v_n$; so, any such remaining vector can be adjoined to our list to produce an even large list $v_1, ..., v_n, v$ of linearly independent vectors.

• Know the key fact that if S is a subspace of T, and T has dimension n, then S has dimension at most n. Furthermore, S has dimension exactly n if and only if S and T are equal.

• Know that a set of three points in $\mathbb{R}^3$ are linearly dependent if and only if they lie on the same plane through the origin.

• Regarding the previous bullet, one way to tell that the three points lie on the same plane is to use the scalar triple product: Recall that the scalar triple product of three vectors A, B, C is $A \cdot (B \times C)$. We have that A, B, C lie on the same plane if and only if $A \cdot (B \times C) = 0$. Note that the scalar triple product is a number, and not a vector.

To verify the connection beween scalar triple products and linear dependence, note that A, B, C lie if and only if A lies in the plane swept out by B, C; and, this holds if and only if A is orthogonal to the normal to the plane swept out by B, C (and containing 0). The normal to this plane is $B \times C$. And, A is orthogonal to this normal if and only if $A \cdot (B \times C) = 0$, whence the connection between scalar triple products and linear dependence.

• Know the definition of a vector space (you will not have to know all nine of the axioms, but it is a good idea to know roughly what they are, and how to verify them).

• Know how to prove that V is a subspace of W. Recall that all one needs to show that is: 1) V is non-empty; and, 2) That if $u, v \in V$, then $au + bv \in V$.

• Know some of the standard examples and counterexamples of subspaces. For example: The kernel of a matrix A is a subspace, as is the image of the matrix A (which is $T(V)$, where T is the linear transformation associated to the matrix A; here, $T : V \to W$). The set of all polynomials of degree at most n is a subspace, while the set of polynomials of degree exactly n is not a subspace.

• Know the basic properties of inner products. Know how to prove that a function $T : (V, V) \to \mathbb{R}$ is an inner product on a real vector space. Know the definition of "orthogonality" with respect to this inner product.

• Know a few examples of inner products: For example, the usual dot product is an inner product, as is the integral $\int_0^1 f(x)g(x)dx$, where the underlying vector space is the set of all functions that are inegral on $[0, 1]$.

• Know the definition of a norm $||\cdot||$: We have that $||v|| \geq 0$, and equals 0 if and only if $v = 0$; we have the triangle inequality $||v+w|| \leq ||v|| + ||w||$.

• Know how to produce a norm given an inner product. In the case where the inner product is defined over the real numbers (as we have done above), then we have that $||u|| =< u, u >^{1/2}$, where $< a, b >$ is our inner product between the vectors a and v.

When we have an inner product defined for our vector space, we say that the space is an "inner product space" or "Hilbert Space"; and, when we have a norm for the vector space, then we say that it is a "normed vector space".

• Know some basic facts, as well as how to prove them, about inner products: We have that a, b are orthogonal with respect to the inner product if and only if $||a + b||^2 = ||a||^2 + ||b||^2$. Also know the Cauchy-Schwarz inner product, which says that $| < a, b > | \leq ||a||||b||$ (here, the norms are those induced by the inner product).

• Know how the Gram-Schmidt process works to construct an orthogonal basis for a subspace.

• Know what it means for T to be a "linear transformation" from $\mathbb{R}^n \to \mathbb{R}^m$ (it means that for vectors $x_1, ..., x_n$ and scalars $a_1, ..., a_n$, $T(a_1x_1 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$). Know how to prove that a given mapping is a linear transformation. Also, know how to solve the folloiwng type of problem: Suppose that T is given to be a linear transformation, and the values of $T(x_1)$,...,$T(x_n)$ are known; then, find $T(v)$ where v is some linear combination of $x_1, ..., x_n$.

• Know that multiplying a matrix times a column vector is a linear transformation.

• Know how to interpret a linear transformation in terms of matrices: Given T, and given a basis $v_1, ..., v_k$ for V, form the matrix A whose columns are $T(v_1), ..., T(v_k)$. Then, given a k-dimensional vector $x = (x_1, ..., x_k)$, written as a column vector, interpret Ax to be $x_1 v_1 + \cdots + x_k v_k$. This is how matrix multiplication is usually *defined*.

• Know what it means for a column vector to represent a vector with respect to a given basis. For example, if v_1, v_2, v_3 is one basis for a subspace, and we say that "$x = (x_1, x_2, x_3)$ (written as a column vector) is a vector with respect to the basis v_1, v_2, v_3", we mean that the vector is $x_1 v_1 + \cdots + x_3 v_3$.

• Know how to write the matrix for T with respect to different bases: Perhaps it is initially given with respect to the standard bases, and then you want to switch to different sets of bases for both the starting and ending spaces.

• Know how to find a matrix V such that if x is a vector with respect to one set of basis vectors, then Vx produces a representation of a vector with respect to a different set of vectors.

• Know the definition of the orthogonal complement of a subspace S. This orthogonal complement is denoted by $S^\perp$, and equals the set of all vectors $s' \in \mathbb{R}^n$ ($\mathbb{R}^n$ is the ambient space) that are orthogonal to every vector of S; that is, $s' \in S^\perp$ if and only if $s' \cdot s = 0$ for every $s \in S$.

• Know the basic fact that $\mathbb{R}^n = S + S^\perp$; that is, for each $v \in \mathbb{R}^n$ there exists $s \in S$ and $s' \in S^\perp$ such that $v = s + s'$.

• Not only does $\mathbb{R}^n = S + S^\perp$, but in fact if the dimension of S is k, then the dimension of $S^\perp$ is $n - k$; so, we have

$$\dim(S) \ + \ \dim(S^\perp) \ = \ n.$$

Thus, not only can every vector $v \in \mathbb{R}^n$ be written as $s + s'$ as above, but, in fact, the $s \in S$ and $s' \in S^\perp$ must be unique.

• Know what is meant by the projection of a vector v onto a subspace S: Write v uniquely as $s + s'$, $s \in S$ and $s' \in S^{\perp}$. Then, this s is the "projection of v onto s".

Another way to find this projection is as follows: Find $s \in S$ such that $v - s$ is orthogonal to every basis vector of S.

• Know basic facts about matrices, such as: If we say that "A is an $m \times n$ matrix", then we mean that it has m rows and n columns. When we write $A_{i,j}$ we mean the entry in the ith row and jth column. The matrix A^t, called the "transpose of A" is $n \times m$, and its i, j entry is the j, i entry of A.

• Know the different ways of describing matrix multiplication: It can be described in terms of dot products (a row vector dot a column vector), or linear combinations of column vectors.

• Know some basic facts about matrix multiplication and addition. Here are a few: If A and B are both $m \times n$ matrix, then the matrix $C = A + B$ is the $m \times n$ matrix whose entries are the sums of respective entries of A and B; matrix multiplication is associative $A(BC) = (AB)C$; we have a distributive rule $A(B + C) = AB + AC$; finally, if c is a scalar, then cA is the matrix gotten by multiplying every element of A by c.

• Know how to write the dot product of two vectors x, y as a matrix product, where if we write them as column vector $[x]$ and $[y]$, then $x \cdot y = [x]^t[y]$. Here $[x]^t$ means the transpose of the column vector $[x]$.

• Know the connection between the composition of linear transformations and matrix multiplication.

• Know how to solve a system $Ax = b$ using row reduction. Know how to find a "one-to-one parameterization" of the solution set. Know the terms "pivots", "pivot columns", and how to find them.

Know the difference between "row-reduced form" and "Row-reduced echelon form".

• Know the definition of the kernel of a matrix. Know how to find a one-to-one parameterization of the kernel using row reduction.

• Know the definition of the rank of a matrix A, and how to find it. (It equals the number of pivot columns).

• Know how to find the inverse of a matrix by putting the left-side of the augmented matrix $[A|I]$ into row-reduced *echelon* form (the right-hand-side will then be the inverse) – note the emphasis on the word echelon.

• Know some basic facts about matrix inverses, such as: If A has both a left-inverse and a right-inverse, then A must be a square matrix and the left-inverse and right-inverse must be equal. Also, $A = n \times n$ is invertible (i.e. has both a left and right inverse) if and only if it has trivial kernel (i.e. the kernel consists only of the 0 vector) if and only if it has rank n.

• Given a matrix A, know how to find a matrix R such that RA is row-reduced. Know how to use this to efficiently parameterize solutions to $Ax = b$ for the same A but different vectors b. This matrix R corresponds to the row operations performed on A to get it in row-reduced form, and can be found as follows: Starting with the augmented matrix $[A|I]$, we row reduce so that A is put into row-reduced form U and then the augmented matrix will be $[U|R]$.

• Know the definition of the image of a matrix A, denoted $\text{img}(A)$. It is the set of all output vectors $y = Ax$; that is, it is $\{Ax \ : \ x \in \mathbb{R}^n\}$. We can interpret this in terms of linear transformations thusly: If we let $T : V \to W$ be the linear transformation corresponding to A, then $T(V)$ is the image of A.

• Know that if you take a maximal collection of linearly independent column vectors from the matrix A, then you have a basis for the image of A. This maximal number of columns equals the number of pivots when A is row reduced, and it also equals the dimension of the image of A. For obvious reasons, sometimes we refer to the image of A as the "column space of A".

• Know that the dimension of the column space of A is refered to as the rank of the matrix A, and denoted $\text{rank}(A)$.

• Know that the dimension of the kernel of A is called the nullity of A, and denoted $\text{nullity}(A)$.

• Know the rank-nullity formula: Given an $m \times n$ matrix A we have that $\text{rank}(A) + \text{nullity}(A) = n$.

• Know how to apply the rank-nullity formula to some basic problems.

• Know how to use the rank-nullity formula to prove that

$$\dim(S) + \dim(S^{\perp}) \;=\; n,$$

where $\mathbb{R}^n$ is the ambient space that S and $S^{\perp}$ lie in. The proof amounts to letting A be the matrix whose rows are a basis for S. Note that if A has size $m \times n$, then we are saying that the ambient space is $\mathbb{R}^n$ and that $\dim(S) = m$. The kernel of this matrix is $S^{\perp}$; and so, $\text{nullity}(A) = \dim(S^{\perp})$.

• Know basic properties of determinants: A determinant function is a mapping for the set of $n \times n$ matrices A (with possibly complex number entries) to the complex numbers, and satisfies three axioms. Know the standard axioms (not the ones in the book): d is multilinear, alternating, and assigns the identity matrix value 1.

• Know that d is unique, and therefore we denote it as $\det(A)$, as in "THE determinant of A".

• Know basic properties. A few are: $\det(A) = \det(A^t)$; the determinant of a block matrix is the product of the determinants of the blocks; the determinant of an upper triangular matrix is the product of the diagonal entries.

• Know that the i, j minor of the matrix A is gotten by striking out the ith row, jth column. This minor is denoted $A_{i,j}$. Then, the adjoint matrix for A is the matrix whose i, j entry $(-1)^{i+j}\det(A_{i,j})$. This matrix is denoted $\text{adj}(A)$. It is the inverse of A up to a multiple $\det(A)$, in the sense that $A\text{adj}(A) = \det(A)I$.

• Know how to find the determinant of a matrix by expanding in terms of matrix minors.

• Know how to calculate the determinant by row-reduction. The determinant will equal $(-1)^k$ times the product of diagonal entries in the row reduced version of A, where k is the number of row interchanges needed to row reduce. Here, the only row operations allowed are: 1) Interchange two rows; 2) Add a multiple of one row to another.

• Know the definition of eigenvalues and eigenvectors. Know a few basic proofs, such as that λ is an eigenvalue if and only if $\det(A - \lambda I) = 0$.

• Know how to diagonalize a matrix (assuming that it *can* be diagonalized); i.e. put it in the form $V \Lambda V^{-1}$, where Λ is a diagonal matrix.

• Know a few basic applications of eigenvalues and eigenvectors: 1) Two diagonalizable matrices commute if and only if they share the same eigenvectors (but not necessarily eigenvalues); 2) Know how to rapidly compute a power of a diagonalizable matrix; 3) Know how to express Fibonacci numbers in terms of eigenvalues of a certain matrix.

• Know the Cayley-Hamilton Theorem, and know how to prove it in the special case where all the eigenvalues are distinct.

• Know the definition of a Hermitian matrix: A is Hermitian if and only if $A^* = A$, where A^* is the "Hermitian conjugate of A", and is computed by taking the transpose of A and then the complex conjugate entries of that transpose.

• Know the definition of "unitary matrix": We have that A is unitary if and only if $A^*A = I$. A is unitary if and only if all the columns are an orthonormal set of vectors.

• Know the Spectral Theorem: If A is a Hermitian matrix, then it can be diagonalized, where all the eigenvalues are real numbers; furthermore, A can be written as $U^* \Lambda U$, where U is a unitary matrix (i.e. $U^*U = I$) and Λ is a diagonal matrix.

• Note that this last part of the Spectral Theorem – that the matrix U is unitary – is a consequence of the fact that the eigenvectors of a Hermitian matrix are mutually orthogonal (with respect to the usual complex inner product).

• A consequence of the Spectral Theorem for complex matrices is the following: If $A = A^t$ (i.e. A is symmetric), and A has all real entries, then $A = U^t A U$, where $U^t U = I$. Furthermore, the eigenvectors of A are orthogonal.

• In the spectral theorem (Hermitian case) the columns of U^* are the eigenvectors, normalized so that each column has length 1.

• Know the meaning of the matrix exponential e^A. If A can be diagonalized, know how to write e^A in terms of e^{λ_i}, instead of as an infinite series.

• Know how to apply matrix exponentials to solve second-order linear differential equations.

• Know the definion of the eigenspace E_λ, which is kernel of $A - \lambda I$.

• Know what the Jordan Canonical form of a matrix looks like, and know how to compute the sizes of the Jordan blocks J_i.