Math 2406, Midterm 1, Spring 2009

February 24, 2009

Instructions: You may use a calculator for the exam, but it must be non-programmable.
Honor Code: I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.
Signed (your name):

1. Suppose that V is a vector space, and that $T : V \to V$ is a linear transformation. Now we define a funny kind of addition operation on pairs of vectors in V as follows: given $v, w \in V$, the addition will be

$$v \oplus w \ = \ T(v + w) = Tv + Tw.$$

Of course, scalar multiplication will be defined in the usual way over V.

Show that the set of vectors in V, under the operation $\oplus$ and the usual scalar multiplication, forms a vector space if and only if T is the identity map. [1]

2. Use mathematical induction to prove that if $T(n)$ is the sequence satisfying

$$T(1) = 0, \text{ and } T(n) = 2T(\lfloor n/2 \rfloor) + n \text{ for } n \geq 2,$$

then

$$T(n) \ \leq \ n \log_2(n), \text{ for } n \geq 1,$$

where $\log_2$ denotes the logarithm to the base 2. [2] Note that the first few terms are

$$T(1) = 0, \ T(2) = 2, \ T(3) = 3, \ T(4) = 8, \ T(5) = 9, \ ...$$

[1] By "identity map" we just mean that $T(v) = v$ for all $v \in V$.

[2] Also, the "floor function" $\lfloor x \rfloor$ for $x \geq 0$ just means "round down to the nearest integer"; so, $\lfloor 1.5 \rfloor = 1$ and $\lfloor 2 \rfloor = 2$.

3. Suppose that L_1 and L_2 are lines in $\mathbb{R}^n$, $n \geq 2$. Show that

$$L_1 + L_2 \ := \ \{a + b \ : \ a \in L_1, \ b \in L_2\}$$

is a plane if and only if L_1 and L_2 are not parallel.

4. Define the following terms

a. norm induced by an inner product (give a formula).

b. $X^\perp$, where X is a subspace of some ambient vector space V.

c. scalar triple product (give a formula).

d. Explain the difference between type I and type II induction.

e. Explain what it means for two planes in $\mathbb{R}^n$ to be parallel.

5. Use Gram-Schmidt to find an orthogonal basis spanned by

$$(1, 1, 0, 1), \ (1, 0, 2, 1), \ (1, 2, -2, 2) \ \in \ \mathbb{R}^4.$$