Math 2406, Midterm 1 Solutions, Spring 2009

February 24, 2009

1.

First, it is obvious that if T is the identity map, then the set V under $\oplus$ and the usual scalar multiplication, forms a vector space, since in this case $\oplus$ is the same as $+$, and we already know V is a vector space under $+$.

Conversely, suppose that T is not the identity map. Then, there exists $v \in V$ such that $T(v) \neq v$. So, by properties of vector spaces, if the set V with $\oplus$ and scalar multiplication forms a vector space,

$$2v \ = \ (1+1)v \ = \ 1v \oplus 1v \ = \ v \oplus v \ = \ T(v+v) \ = \ 2T(v) \ \neq \ 2v,$$

contradiction.

One can also try to prove this last claim a little differently by making use of O: if $T(v) \neq v$, then

$$v \ = \ v \oplus 0 \ = \ T(v+0) \ = \ T(v),$$

contradiction. But, there is one little problem with this that needs to be ironed out: one needs to prove that the 0 in V using $+$ is the same 0 in V using $\oplus$ (it need not be the case that the additive identity in one context is the same in another). This can be proved by letting O_2 denote the 0 in the set V with the operation $\oplus$. Then, assuming that this is a vector space, by properties of vector spaces we have

$$0_2 \ = \ v \oplus (-1v) \ = \ T(v + (-1v)) \ = \ T(0) \ = \ 0.$$

So, the 0's are the same, and we are done.

2. The claim is clearly true for the base case $n = 1$. Assume, for proof by induction, that

$$T(n) \ \le \ n \log_2(n)$$

holds for all $1 \le n \le k$. Then, since $1 \le (k+1)/2 < k+1$, we have

$$\begin{aligned} T(k+1) \ &= \ 2T(\lfloor (k+1)/2 \rfloor) + (k+1) \\ &\le \ 2\lfloor (k+1)/2 \rfloor \log_2(\lfloor (k+1)/2 \rfloor) \ + \ (k+1) \\ &\le \ (k+1)\log_2((k+1)/2) + (k+1) \\ &= \ (k+1)\log_2((k+1)/2) + (k+1)\log_2(2) \\ &= \ (k+1)\log_2(k+1), \end{aligned}$$

by properties of logs (specifically, $\log_2(ab) = \log_2(a) + \log_2(b)$). So, the induction step is proved, and we are done.

3. First, write

$$L_1 \ := \ \{P + tA \ : \ t \in \mathbb{R}\}, \ L_2 \ := \ \{Q + uB \ : \ u \in \mathbb{R}\}, \text{ where } A, B \neq \vec{0}.$$

Then,

$$L_1 + L_2 \ = \ \{(P+Q) + tA + uB \ : \ t, u \in \mathbb{R}\}.$$

By the definition, this is a plane if and only if A and B are independent. And again by definition, we know that L_1 and L_2 are parallel if and only if A and B are dependent; and, taking the contrapositive, we have A and B are independent if and only if L_1 and L_2 are not parallel. Q.E.D.

4. You can look these up. So, I will not bother to define the terms.

5. The Gram-Schmidt vectors are

$$(1, 1, 0, 1), \ (1/3, -2/3, 2, 1/3), \ (-5/14, 14, -2/7, -1/7, 9/14).$$