Midterm I, Abstract Vector Spaces, Fall 2006

February 6, 2009

1. We have seen that the set of all functions that are differentiable on $\mathbb{R}$ form a vector space V. Now consider all such functions $f(x)$, differentiable on $\mathbb{R}$, which satisfy

$$f'(0) + f(0) = 0.$$

Show that the set of all such functions $f(x)$ is a subspace of V.

2. Suppose that x and y are two independent vectors in some vector space V equipped with an inner product. Further, suppose that e_1 and e_2 are two vectors satisfying

$$\begin{aligned} x &= (x, e_1)e_1 + (x, e_2)e_2; \text{ and,} \\ y &= (y, e_1)e_1 + (y, e_2)e_2. \end{aligned}$$

a. Show that this implies that every vector $v \in V$ satisfies

$$v = (v, e_1)e_1 + (v, e_2)e_2.$$

(Hint: Express v as a linear combination of x and y. Why does such a linear combination always exist?)

b. Show that e_1 and e_2 are orthonormal. (Hint: First, show they have length 1 by picking just the right vectors v in part a.)

3. Apply Gram-Schmidt to the vectors

$$v_1 = (1, 0, 1),\ v_2 = (1, 1, 1),\ v_3 = (1, -1, 3),$$

and write down the orthogonal vectors that result.

4. Verify that the following three vectors are orthogonal:

$$(1, 2, 3), \ (1, -2, 1), \ (4, 1, -2).$$

Then, express the vector

$$x \ = \ (3, -4, 2)$$

as a linear combination

$$x \ = \ c_1(1, 2, 3) + c_2(1, -2, 1) + c_3(4, 1, -2);$$

that is, find c_1, c_2, c_3.

5. Find a Cartesian equation for the plane through $(1, 1, 1)$ if a normal vector N makes angles $\pi/3$, $\pi/4$, and $\pi/3$ with the vectors $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$, respectively.