Extra Homework, Math 2406, Spring 2009

March 9, 2009

For this problem, I want you to give a proof of the following fact, using a certain set of ideas (listed below): Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a surjective linear transformation. Prove that $m \leq n$.

Now, there is a rather easy way to do this, by simply picking any basis $v_1, ..., v_n$ for $\mathbb{R}^n$, and then mapping them over to $\mathbb{R}^m$ via $T(v_1), ..., T(v_n)$. Clearly, $T(V)$ is contained in the span of these new vectors; and so, upon extracting a linearly independent subset of them, we have a basis for $\mathbb{R}^m$, having at most n elements.

But now I want you to give a "basis free" proof of this – i.e. a proof that does not require you to pick a basis $v_1, ..., v_n$ for $\mathbb{R}^n$. The tools I want you to use in doing so are as follows:

- First, use the fact that for a linear transformation $T : V \to W$, we have that

$$\begin{aligned} T \text{ injective} \quad &\Longleftrightarrow \quad T \text{ has a left} - \text{inverse} \\ T \text{ surjective} \quad &\Longleftrightarrow \quad T \text{ has a right} - \text{inverse.} \end{aligned}$$

Furthermore, you get to assume in each case that the inverse function is a linear transformation.

- Second, use the fact that

$$\text{rank}(T) \ + \ \text{nullity}(T) \ = \ n.$$

- Finally, use the fact that injective mappings have trivial kernels.

Now you might object to calling this a "basis free" proof, since some of the proofs of the results used above made use of bases, at least how we proved them. Maybe so... still, it is a good exercise in the use of inverses and kenels and such.