Final Exam, Math 2406

April 29, 2009

1. Consider the plane of all points $z \in \mathbb{R}^3$ parameterized by

$$z = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad t, u \in \mathbb{R}.$$

Find numbers a, b, c, d so that this plane is the set of all points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$ax + by + cz = d.$$

2. Let V be the vector space of all continuous function f defined on $[0, 1]$. Let S be the subset of these functions f such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx.$$

Prove that S is a subspace.

3. Consider the subspace spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 0 \\ -2 \\ -3 \end{bmatrix}.$$

Find an orthogonal basis that spans the same space.

4. Compute the rank and nullity of the following matrix

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 5 & 2 \end{bmatrix}.$$

5. Find a one-to-one parameterization of the set of solutions to the following system of equations:

$$\begin{aligned} 2x + y + z - w &= 3 \\ x - y - z - w &= -4 \\ x + y + 5z + 2w &= 3. \end{aligned}$$

That is, find vectors $v_0, v_1, ..., v_k$ so that each solution can be written uniquely as $v_0 + t_1 v_1 + \cdots + t_k v_k$, where the t_i are real numbers.

6. Consider the subspace $S \subseteq \mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}.$$

Find the projection of v onto S; that is, find coefficients a and b so that this projection is $av_1 + bv_2$.

7. Suppose that A is an $m \times n$ matrix, and that B is an $n \times p$ matrix. Prove that the rank of the product AB is at most equal to the rank of B. (Hint: First, explain why the kernel of B is contained in the kernel of AB. Then, explain why the nullity of B is at most the nullity of AB. Finally, find a way to use the rank-nullity formula to then solve the problem...)

8. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}.$$

9. Let V be the vector space of all 2×2 matrices. Note that V is four-dimensional. Consider the mapping $T : V \to V$ which sends a matrix A to its transpose A^t.

a. Prove that T is a linear transformation.

b. Show that $\lambda = 1$ and $\lambda = -1$ are its only eigenvalues (Hint: Don't compute $\det(C - \lambda I)$! Instead, write down what an eigenvalue is in terms of a general linear transformation $T : V \to V$, and then solve for λ that way. This is an easy problem if you know how to set it up properly!)

c. Find an eigenvector associated to $\lambda = 1$, and then find an eigenvector associated to $\lambda = -1$. (Again, this is easy if you set it up properly! Note that in this problem eigenvectors are 2×2 matrices.)

10. Consider the second-order differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y \;=\; 0,$$

subject to the initial conditions $y(0) = 1$, $y'(0) = 1$. Find $y(x)$ using the method of matrix exponentials.

11. Consider the following matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -4 & 1 & 1 \\ 0 & -4 & -1 & 3 \end{bmatrix}.$$

It turns out that this matrix has only the eigenvalue $\lambda = 2$. Given this, find the Jordan block matrix occuring in the Jordan Canonical Form; that is, we know that this matrix can be written as $V^{-1}JV$, where J is the Jordan block matrix – the problem is to find this matrix J.