Math 3215 Alternate Midterm 2
Selected Solutions

July 27, 2009

2.

a.

$$M_X(t) = \int_{-1}^{1} e^{xt}|x|dx = \int_{-1}^{0} e^{xt}(-x)dx + \int_{0}^{1} e^{xt}xdx$$
$$= \int_0^1 (e^{-yt} + e^{yt})ydy.$$

Now we do integration by parts, letting $u = y$ and $dv = e^{-yt} + e^{yt}$, so that the integral becomes

$$y(-e^{-yt}/t + e^{yt}/t)\big|_0^1 - \int_0^1 (e^{yt}/t - e^{-yt}/t)dy$$
$$= e^t/t - e^{-t}/t - (e^{yt}/t^2 + e^{-yt}/t^2)\big|_0^1$$
$$= e^t/t - e^{-t}/t - e^t/t^2 - e^{-t}/t^2 + 2/t^2.$$

b. The third moment is easily seen to be 0, by direct computation. And this can be backed up by finding the coefficient of t^3 in the above expansion – the coefficient is

$$\frac{1}{4!} - \frac{1}{4!} - \frac{1}{5!} - \frac{1}{5!} = 0 = \mathbb{E}(X^3)/3!.$$

3. We just need to solve for c so that

$$\int_0^2 \int_0^1 (x^2 + cxy)dxdy = 1.$$

Computing the inner integral, we find that it gives

$$x^3/3 + cx^2y/2\Big|_0^1 \;=\; 1/3 + cy/2.$$

Then computing the outer integral, we get

$$y/3 + cy^2/4\Big|_0^2 \;=\; 2/3 + c.$$

So,

$$c \;=\; 1/3.$$

4.

a. Since X and Y are independent, we know that the mass function $f(x,y)$ satisfies

$$f(x,y) \;=\; \mathbb{P}(X=x)\mathbb{P}(Y=y) \;=\; (e^{-1}/x!)\begin{cases} 1/4, & \text{if } Y=0; \\ 1/2, & \text{if } Y=1; \\ 1/4, & \text{if } Y=2. \end{cases}$$

b. We have that

$$\mathbb{E}(X|Z=0) \;=\; \sum_{x=0}^{\infty} x\mathbb{P}(X=x|Z=0) \;=\; \sum_{x=1}^{\infty} x\mathbb{P}(X=x, Z=0)/\mathbb{P}(Z=0).$$

For $x \geq 1$ we have

$$\mathbb{P}(X=x, Z=0) \;=\; \mathbb{P}(X=x, Y=0) \;=\; 1/4ex!.$$

Also, note that

$$\begin{aligned}\mathbb{P}(Z=0) \;=\; \mathbb{P}(X=0 \text{ or } Y=0) \;&=\; \mathbb{P}(X=0)+\mathbb{P}(Y=0)-\mathbb{P}(X=0,Y=0) \\ &=\; 1/e + 1/4 - 1/4e.\end{aligned}$$

So, the answer is

$$\frac{(1/4e)\sum_{x=1}^{\infty} x/x!}{\mathbb{P}(Z=0)} \;=\; \frac{1/4}{1/e+1/4-1/4e} \;=\; \frac{e}{3+e}.$$

5. We have that

$$\begin{aligned}
\mathrm{V}(X+Y) &= \mathbb{E}((X+Y)^2) - (\mathbb{E}(X+Y))^2 \\
&= (\mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2)) - (\mu_X + \mu_Y)^2 \\
&= (\mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2)) - \mu_X^2 - 2\mu_X\mu_Y - \mu_Y^2 \\
&= (\mathbb{E}(X^2) - \mu_x^2) + (\mathbb{E}(Y^2) - \mu_y^2) + 2(\mathbb{E}(XY) - \mu_X\mu_Y).
\end{aligned}$$

Now applying the basic identities

$$\mathrm{V}(Z) \;=\; \mathbb{E}(Z^2) - \mu_Z, \quad \text{and} \quad \mathrm{Cov}(U,V) \;=\; \mathbb{E}(UV) - \mu_U\mu_V,$$

we are done.