Solution to problem 2, set 2, Math 3215

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Let's recall the problem:

> An army interrogator, knowing that lie detector tests are only 80% reliable (meaning that if a liar is given the test, the test responds "liar" 80% of the time; and if a truth teller is given the test, the test responds "truth teller" 80% of the time), believes that testing several suspects at once is the key to upping the accuracy of the test.
>
> Suppose you know that 3% of the *suspected* terrorists really are terrorists, and that whenever a suspected terrorist is given the test, they always say "No, I am not a terrorist."
>
> Calculate the probability that if 10 individuals are drawn at random from the whole population of suspected terrorists, and are tested, then at least one of them is a terrorist, given that the tests say 8 of them are liars and 2 of them are truth tellers.

Solution. Let's let E be the event that at least one of the suspected terrorists is actually a terrorist; and let G be the event that "the tests say 8 of them are liars and 2 of them are truth tellers". We wish to compute $\mathbb{P}(E|G)$, which is the same as $1 - \mathbb{P}(\overline{E}|G)$.

Now,

$$\mathbb{P}(\overline{E}|G) \ = \ \frac{\mathbb{P}(G|\overline{E})\mathbb{P}(\overline{E})}{\mathbb{P}(G)},$$

(this is the baby Bayes's Theorem).

The probability of the event $\overline{E}$ – the event that *none* of the suspects are actually terrorists – is the same as the probability that "the first suspect isn't a terrorist, and the second isn't a terrorist, and... and the 10th one isn't a terrorist". Since the probability that each suspect is a *not* a terrorist is $1 - 3\% = 0.97$, and since whether or not a suspect is a terrorist is independent of whether other suspects are terrorists (we assume people are drawn independently from the pool of suspects), we have that $\mathbb{P}(\overline{E}) = (0.97)^{10} = 0.737424....$

It is more delicate (and difficult) to compute the probability of G: think of it like a coin flip experiment, where you flip a pair of coins 10 times. On each of the 10 rounds the first coin represents whether the suspect is a terrorist or not – H for 'terrorist' and T for 'not terrorist' – while the second coin represents whether the machine indicates they are a truth-teller or a liar – say, H for 'liar' and T for 'truth teller'.

During a given round of two flips, we either have HH, TT, HT and TH as possible outcomes. Based on the 3% and 80% figures given to us, we have that

$$\mathbb{P}(HH) = (0.03)(0.8),\ \mathbb{P}(HT) = (0.03)(0.2),\ \mathbb{P}(TH) = (0.97)(0.2),\ \mathbb{P}(TT) = (0.97)(0.8).$$

The event that "the test says an individual is a terrorist" corresponds to $\{HH, TH\}$ in our coin flip experiment; and so the probability the machine will say this is $p = (0.03)(0.8) + (0.97)(0.2) = 0.218$. If we let X be the number among 10 randomly selected individuals that the machine says are terrorists, we will have that X is binomially distributed with $n = 10$ and $p = 0.218$; so,

$$\mathbb{P}(G) \;=\; \mathbb{P}(X = 8) \;=\; \binom{10}{8} p^8 (1-p)^2 \;=\; 0.00014037...$$

Lastly, we compute $\mathbb{P}(G|\overline{E})$: in other words, given that none are terrorists, what is the probability that the machine says 8 of them are, and 2 are not? If we let Y be the number that the test says are terrorists given that none are, then Y clearly has a binomial distribution with parameters $n = 10$ and $p = 0.2$, so that

$$\mathbb{P}(G|\overline{E}) \;=\; \mathbb{P}(Y = 8) \;=\; \binom{10}{8} p^8 (1-p)^2 \;=\; 0.000073728.$$

Putting all these numbers together, we get

$$\mathbb{P}(E|G) \;=\; 1 - \frac{\mathbb{P}(G|\overline{E})\mathbb{P}(\overline{E})}{\mathbb{P}(G)} \;=\; 1 - \frac{0.000073728 \cdot 0.737424}{0.00014037} \;=\; 0.612675...$$

So, there is only slightly better than 50-50 chance that at least one of the suspects is a terrorist!