Midterm 1 for Math 3225

Instructions: Work problems 1 through 5 in class; and then work problems 6 through 8 after class, and turn them in by next Wednesday.

1. Suppose A and B are sets, and $\phi : A \to B$ is a map.

a. State what it means for ϕ to be injective, surjective, and bijective.

b. Show that ϕ is bijective if and only if ϕ is invertible.

2.

a. Suppose Ω is a sample space. Define what it means for Σ to be a σ-algebra on Ω. Also, state what it means for P to be a probability measure on Σ.

b. Suppose μ is the Lebesgue measure on $\mathbf{R}$, the reals. Explain why $\mu(\mathbf{Q}) = 0$, where $\mathbf{Q}$ is the set of rational numbers. (Hint: What is $\mu(\{p\})$ for an arbitrary $p \in \mathbf{R}$?)

3.

a. An urn contains 10 black balls and 10 white balls. You randomly select a ball from the urn. If it is black, you stop. If it is white, you keep that white ball, and then select another from among the 19 remaining balls in the urn. If *that* ball you just drew is black, you stop. If it is white, you continue drawing balls until a black one is found. What is the probability that you stop after the third draw? (i.e. the 3rd draw is a black ball.)

b. Suppose you do the same thing as in part a, except that you replace the balls you draw after each step. So, at the beginning of each draw, there will be 20 balls (10 black and 10 white) in the urn to choose from. For this case, what is the probability you stop on the third draw?

4.

a. State the following: The inclusion-exclusion identity for two events; the "product rule" for conditional probabilities; and the form of Bayes's Theorem for events $C_1, ..., C_k$ that partition Ω.

b. Suppose you have a collection of days. 10 percent are rainy days, and 90 percent are dry days. If you select one of the rainy days at random, there is a 20 percent chance it will be a spring day; and if you select a dry day at random, there is an 80 percent chance it *won't* be a spring day. If you select a spring day at random, what is the probability it was a rainy day?

5.

a. Define the probability density function of a continuous random variable x; also, define the cumulative distribution function associated to x, and express it in terms of the p.d.f. for x.

b. Suppose x is a random variable on $[0, 1]$ such that for $t \in [0, 1]$, $P(x \leq t) = t^2$. Find a p.d.f. for x. Also, if $y = x^2$, find the cumulative distribution function for y. In a single word, describe how y is distributed.

Midterm 1 for Math 3225, Take-home Portion

6. Suppose A, B, C, and D are independent events (as we defined in class). Show that

$$P(A \cap (B \cup C \cup D)) \; = \; P(A)P(B \cup C \cup D).$$

7. Show that the map

$$\phi \; : \; \mathbf{N} \times \mathbf{N} \; \longrightarrow \; \mathbf{N},$$

which sends

$$(a, b) \; \longrightarrow \; a + \frac{(a+b-2)(a+b-1)}{2},$$

is a bijection. Here $\mathbf{N}$ is the set of positive integers. (Note: On a posted HW solution I had an erroneous version of this map.) Hint: Drawthe 5×5 square of numbers (a, b), $1 \leq a, b \leq 5$, and see where each (a, b) gets sent.

8. In a certain "pick 3" lottery a person selects a number from among 1000 possibilities $\{000, ..., 999\}$. Let us suppose each number is equally likely. Now suppose 500 people play the game, and each picks a number. From the lottery commission's perspective, the worst thing that could happen is if the 500 people conspire and each picks a different number. If they do this, then the chance that at least one of them wins (and splits his winnings with the other 499 people he conspires with) is obviously $1/2$. Now suppose the 500 people *don't* conspire, and each picks his or her number independently of the other players. Show that the probability that at least one of the players picks the winning number is approximately $1 - 1/\sqrt{e}$. In working this problem, state what your sample space is, what the probability measure is, and any assumptions you make.