Some notes on the Poisson distribution

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1 Introduction

The Poisson distribution is one of the most important that we will encounter in this course – it is right up there with the normal distribution. Yet, because of time limitations, and due to the fact that its true applications are quite technical for a course such as ours, we will not have the time to discuss them. The purpose of this note is not to address these applications, but rather to provide source material for some things I presented in my lectures, which are not covered in your book.

At the end of each section in what follows, I will let you know what material you should know for your next exams. You are, of course, also required to know all the material appearing in your text related to Poisson random variables.

2 Basic properties

Recall that X is a Poisson random variable with parameter λ if it takes on the values $0, 1, 2, ...$ according to the probability distribution

$$p(x) \ = \ P(X = x) \ = \ \frac{e^{-\lambda}\lambda^x}{x!}.$$

By convention, $0! = 1$.

Let us verify that this is indeed a legal probability density function (or "mass function" as your book likes to say) by showing that the sum of $p(n)$ over all $n \geq 0$, is 1. We have

$$\sum_{n=0}^{\infty} p(n) \ = \ e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}.$$

This last sum clearly is the power series formula for e^λ, so

$$\sum_{n=0}^{\infty} p(n) \;=\; e^{-\lambda}e^{\lambda} \;=\; 1,$$

as claimed.

A basic fact about the Poisson random variable X (actually, two facts in one) is as follows:

Fact. $E(X) = V(X) = \lambda$.

Let us just prove here that $E(X) = \lambda$: As with showing that $p(x)$ is a legal pdf, this is a simple exercise in series manipulation. We begin by noting that

$$E(X) \;=\; \sum_{n=0}^{\infty} np(n) \;=\; \sum_{n=1}^{\infty} n \cdot \frac{e^{-\lambda}\lambda^n}{n!}.$$

Noting that

$$n! \;=\; n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1,$$

we see that we may cancel the n with the $n!$ to leave an $(n-1)!$ in the denominator. So, pulling out the $e^{-\lambda}$ and cancelling the n, we deduce that

$$E(X) \;=\; e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \;=\; \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}.$$

(Note here that the term $n = 1$ has a $0!$ in the denominator.)

Upon setting $m = n - 1$, we find that

$$E(X) \;=\; \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \;=\; \lambda e^{-\lambda} e^{\lambda} \;=\; \lambda,$$

and we are done.

Take home message. You should know everything from this section.

3 A sum property of Poisson random variables

Here we will show that if Y and Z are independent Poisson random variables with parameters λ_1 and λ_2, respectively, then $Y+Z$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Before we even begin showing this, let us recall what it means for two discrete random variables to be "independent": We say that Y and Z are independent if

$$P(Y = y, Z = z) \;=\; P(Y = y \text{ and } Z = z) \;=\; P(Y = y)P(Z = z).$$

This is not really the "official" definition of independent random variables, but it is good enough for us.

Letting now

$$X \;=\; Y + Z,$$

we have that

$$\begin{aligned} P(X = x) \;=\; P(Y + Z = x) \;&=\; \sum_{\substack{y,z \geq 0 \\ y+z=x}} P(Y = y, Z = z) \\ &=\; \sum_{\substack{y,z \geq 0 \\ y+z=x}} P(Y = y)P(Z = z) \\ &=\; \sum_{\substack{y,z \geq 0 \\ y+z=x}} \frac{e^{-\lambda_1}\lambda_1^y}{y!} \cdot \frac{e^{-\lambda_2}\lambda_2^z}{z!} \\ &=\; e^{-(\lambda_1+\lambda_2)} \sum_{\substack{y,z \geq 0 \\ y+z=x}} \frac{\lambda_1^y \lambda_2^z}{y!z!}. \end{aligned}$$

Now we require a little trick to finish our proof. We begin by recognizing that the $y!z!$ is the denominator of

$$\frac{x!}{y!z!} \;=\; \binom{x}{y} \;=\; \binom{x}{z}.$$

So, if we insert a factor $x!$ in the sum, then we will have some binomial coefficient. Indeed, we have

$$P(X = x) \;=\; \frac{e^{-(\lambda_1+\lambda_2)}}{x!} \sum_{\substack{y,z \geq 0 \\ y+z=x}} \binom{x}{y} \lambda_1^y \lambda_2^z.$$

Notice that from the binomial theorem we deduce that this last sum is just

$$(\lambda_1 + \lambda_2)^x.$$

So,

$$P(X = x) \;=\; \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^x}{x!},$$

and therefore X has a Poisson distribution with parameter $\lambda_1+\lambda_2$, as claimed.

Take home message. I expect you to know this fact about sums of Poisson random variables. **BUT**, I do not expect you to know the proof.

4 An application

Now we consider the following basic problem, which appeared on an exam I gave in years past. It assumes you know what a "Poisson process" is, which although mentioned in your book, is not discussed in great detail. I will expect you to know enough to at least be able to solve this problem if it were put on an exam.

Problem. The number of people arriving at a fast food drive thru (or through) in any given 2 minute interval obeys a Poisson process with mean 1. Suppose that the waiters can only process 3 orders in any given 4 minute interval (also, assume the waiters can process an order instantaneously, but are limited in how many they can process in given time intervals, as indicated). What is the expected number of people that leave the drive thru with their orders filled in any given 4 minute interval?

Solution. Let X be the number of people that arrive in the 4 minute interval. We may write $X = Y + Z$, where Y is the number arriving in the first two minutes of the interval, and where Z is the number arriving in the last two minutes. Clearly, Y and Z are assumed to be independent, and thus X is Poisson with parameter $1 + 1 = 2$. So,

$$P(X = x) \;=\; e^{-2}2^x/x!.$$

Now let W denote the number of orders that are processed. Clearly,

$$\begin{array}{rcl} X = 0 & \Longrightarrow & W = 0 \\ X = 1 & \Longrightarrow & W = 1 \\ X = 2 & \Longrightarrow & W = 2 \\ X \geq 3 & \Longrightarrow & W = 3. \end{array}$$

So, the probability distribution for W is given by

$$P(W = w) = \begin{cases} e^{-2}, & \text{if } w = 0; \\ 2e^{-2}, & \text{if } w = 1; \\ 2e^{-2}, & \text{if } w = 2; \\ 1 - 5e^{-2}, & \text{if } w = 3. \end{cases}$$

We have now that

$$\begin{aligned} E(W) &= 0 \cdot P(W = 0) + 1 \cdot P(W = 1) + 2 \cdot P(W = 2) + 3 \cdot P(W = 3) \\ &= 2e^{-2} + 4e^{-2} + 3 - 15e^{-2} \\ &= 3 - 9e^{-2}. \end{aligned}$$

Take home message. I expect you to be able to solve a problem at this level on an exam.