Midterm 1 for Math 3225

1. Suppose that A is a finite set. Fix an element $a \in A$, and consider the sequence

$$a,\ \phi(a),\ (\phi \circ \phi)(a),\ (\phi \circ \phi \circ \phi)(a), ...$$

where *phi* is an injection from A to A. Prove that there exists an integer $n \geq 1$ such that

$$\phi^n(a)\ =\ (\phi \circ \phi \circ \cdots \circ \phi)(a)\ =\ a.$$

(Hint: It is farily easy to show that the sequence above must settle down to a cycle; however, it is not immediately obvious that that cycle returns to a.)

2. Suppose (Ω, Σ, P) is a probability space.

a. State the Kolmogorov axioms that P must satisfy.

b. If $A \subseteq B$, where $A, B \in \Sigma$, show that $P(A) \leq P(B)$. Give an example Ω, Σ, and P where A is a proper subset of B (i.e. $A \neq B$), and yet $P(A) = P(B)$.

3.

a. State the chain rule (or product rule, as it is sometimes called) for events.

b. Suppose that an urn contains 10 blue balls, 10 black balls, and 10 red balls. You select three balls from the urn, without replacement. What is the probability you selected one of each of the three colors (i.e. all three balls are of different colors).

4.

a. State Bayes's theorem for events $C_1, ..., C_k$.

b. A cancer test has reliability r, where $0 < r \leq 1$, which means that: If a person has cancer, then the test answers correctly, "Cancer", $r \times 100\%$ of the time; and if a person does not have cancer, then the test answers "No Cancer" $r \times 100\%$ of the time. Given that the test responds "Cancer", you want there to be a 95% chance that the person taking the test indeed has cancer. Determine r as a function of x (the percent of the population with cancer).

5.

a. Define the probability density function for a continuous random variable. Define the cumulative distribution function for a continuous random variable, and express it in terms of the probability density function.

b. Suppose that x is a random variable on $[1, \infty)$, and suppose that the probability density function $f(x)$ associated to x satisfies

$$x^\theta\ <\ f(x)\ <\ 2x^\theta.$$

Prove that $\theta < -1$.