Math 3770 Final Exam, Fall 2008

December 31, 2008

1. Define the following terms.

a. State the "chain rule" for the probability of k events.

b. Define the expectation of a continuous random variable (write down the definition as a formula involving an integral).

c. Moment generating function of a random varible X.

d. Liklihood function.

e. Type I error.

2. Suppose that X and Y are binomial random variables with parameters n and $p = 1/2$; further, assume that they are independent. Let $Z = X + Y$. Prove that Z is binomial with parameters $2n$ and $p = 1/2$. (Hints: There are two standard ways to approach this. One way is to write X and Y in terms of Bernoullis. Another way is to write $P(Z = z)$ as $\sum_{x=0}^{z} P(X = x, Y = z - x)$. The first way is the easiest.)

3. Prove that if X and Y are independent events, then so are $\overline{X}$ and $\overline{Y}$ (these are the complements of the events X and Y).

4. Two ants start out at position 0, and after each round, each either stays where it is or moves right one unit. Assume that the decision that each ant makes at each round is independent of the other, and that each chooses to either stay where it is or move right with probability 1/2. What is the probability that after 5 rounds (the number of times each has stayed put plus the number of times moved is 5), both ants are on the same point on the number line?

5. A scientist on Mars (in the year 2050) wishes to transmit one bit of information to another one on Earth (i.e. a single '0' or '1'). Because of various noises introduced along the way from Mars to Earth, the scientist on Mars sends his bit of information three times – in other words, if the bit is '1', he sends '111'; and, if the bit is '0', he sends '000'. The scientist on Earth then computes the "majority function" of the received bits: If the sent bits are '111', maybe the scientist on Earth *receives* '101'. He then notes that since there are more 1's than 0's (two 1's and one 0), the message sent was likely '1'.

Now suppose that the scientist on Earth really does receive the signal '101'. What is the probability that the sent message was '1', given that the probability of any given bit being flipped is 1/3, given that the noise in each bit is independent of the noise in the other bits, and given that the scientist on Mars wasn't picky about which bit to send (he choose to send 0 or 1 with equal probability)? (Hint: This is a Bayes's theorem problem, though it doesn't look like one. Let X be the event that 1 was the intended message, and let A be the event that the received message was '101'. Compute $P(X|A)$.)

6. Suppose that X is some random variable, and that the following numbers form a random sample of values of X:

$$1,\ 2,\ 3,\ 2, 4.$$

a. Compute $\overline{X}$.
b. Compute the sample variance.
c. Compute the median.
d. Compute the upper fourth, lower fourth, and fourth spread.

7. You decide to determine the percent of Georgians who smoke by doing a survey of 100 randomly selected Georgians (the same person is allowed to be polled twice). Let X be the number among the 100 who smoke. Note that X is binomial with parameters $n = 100$ and p, where p is the percent of Georgians who smoke.

Suppose that the observed value of X was 20. Using the fact that X can be approximated by a normal distribution ($(X - 100p)/10\sqrt{p(1-p)}$ is roughly $N(0,1)$), determine a 95% confidence interval for p.

8. Assume that the energy in a particular pulse of electricity in a computer circuit has a normal distribution with mean μ and variance $\sigma^2 = 1$. The specs for the circuit claim that $\mu = 2$, and you wish to test this. So, let

$$H_0 \ : \ \mu = \mu_0, \text{ where } \mu_0 = 2,$$

and then let

$$H_a \ : \ \mu \neq \mu_0.$$

Assume that $\alpha = 0.05$ (the probability of making a Type I error); and, assume that after making 25 random observations of the energy, the average of those 25 values is $\overline{X} = 2.1$.

a. Determine the rejection region.

b. Decide whether the hypothesis should be rejected or not.

9. Suppose that (X, Y) is a pair of random variables having joint pdf given by

$$f(x,y) \ = \ \begin{cases} ce^{-x-y}, & \text{if } (x,y) \in R; \\ 0 & \text{otherwise,} \end{cases}$$

where R is the intersection of the regions $x > 0$, $y > 0$, $y < x$.

Determine c.

10. Suppose that one has the following set of data points:

$$(1,1), \ (2, 0.9), \ (3, 1.1), \ (4, 4).$$

These are assumed to fit a linear model

$$Y \ = \ \beta_1 x + \beta_0 + \epsilon,$$

where for each $x = 1, 2, 3, 4$ we have that $\epsilon = N(0, \sigma^2)$.

Compute the standard least squares estimate for β_1 and β_0 by writing down the SSE function, and setting partials with respect to β_0 and β_1 to 0.