The two slips of paper puzzle

September 30, 2008

Recall this puzzle from class today:

Puzzle. You pick two distinct numbers at random, and write the first on a slip of paper, and then write the other on another slip of paper. You turn the papers face down so that I cannot see them, and scramble them any way you wish. I randomly select one of the two slips, turn it over, and then try to guess whether it is the larger or the smaller of the two numbers. It turns out that I can guess correctly more than 50% of the time, which seems impossible.

Solution. After I have turned over the random slip of paper, I select a random number according to the standard normal distribution (actually, any continuous distribution with positive density on all the real numbers, works for the purposes of the solution). Say that X is the random number so generated, and that Z is the number on the slip of paper that I turned over. We will think of this X as standing in for the number on the slip of paper that I don't see.

If $Z < X$, then I report that the number on the paper (which is Z) I see is the smaller of the two numbers; and if $X \leq Z$, I report that the number on the paper is larger than the two numbers.

What we now show is that, for any given pair of initial numbers on the slips of paper – say they were A and B, where $A < B$ – this procedure gives the correct answer with probability exceeding 50%: Given values for A and B (which I am not allowed to know), there is a 50% chance that the number I select is $Z = A$ and a 50% chance that $Z = B$. So,

$$\begin{aligned} \mathrm{P}(\text{I win}) &= \mathrm{P}(\text{I win} \mid Z = A)\mathrm{P}(Z = A) \ + \ \mathrm{P}(\text{I win} \mid Z = B)\mathrm{P}(Z = B) \\ &= \mathrm{P}(Z < X \mid Z = A) \cdot (1/2) + \mathrm{P}(Z \geq X \mid Z = B) \cdot (1/2). \end{aligned}$$

So, to show that we win with probability exceeding 1/2, we just need to show that

$$P(Z < X \mid Z = A) \ + \ P(Z \geq X \mid Z = B) \ > \ 1.$$

Using ϕ to denote the cumulative distribution of a normal random variable with mean 0 and variance 1, we find that

$$P(Z < X \mid Z = A) \ = \ 1 - \phi(A), \text{ and } P(Z \geq X \mid Z = B) \ = \ \phi(B).$$

So, since $A < B$ the sum of these is

$$1 - \phi(A) + \phi(B) \ > \ 1.$$

Comment 1. One objection here is that we cannot really pick random numbers according to the normal distribution. There are ways to fix this through the use of a certain coin flip experiments, which I will not bother to discuss.

Comment 2. One can object to the problem on practical grounds: Can you ever really run the experiment so that very very large numbers can come up? Perhaps it would take until the end of time to select a number $X > 10$, if we use normal distributions as in our above example. Maybe so, but from a theoretical standpoint, the solution I gave is correct!