Math 4107 Midterm 1 Solutions, Fall 2009

October 18, 2009

1. You can look up these definitions yourself.

2. There are many ways to solve this problem. We present two here:

Way 1. Let H be the subgroup of order 35. By Lagrange, we know that every element of H has order either $1, 5, 7$ or 35. If some element has order 5 or 35, we are done (if g has order 35, then g^7 has order 5).

So, let us assume every element has order 1 or 7. So, since the identity is the only element of order 1, every non-identity element has order 7. If g_1 and g_2 are two elements of order 7, then either the cyclic group $(g_1) = (g_2)$ or $(g_1) \cap (g_2) = \{e\}$. What this means is that the subgroups of order 7 are essentially disjoint, apart from intersecting in the identity; so, letting $A_1, ..., A_k$ be all these different subgroups of order 7, together they contribute $6k$ elements of order 7, and 1 element of order 1. But this implies

$$35 \ = \ |H| = 1 + 6k,$$

which cannot hold because $35 - 1 = 34$ is not divisible by 3.

Way 2. We know that if A and B are subgroups of a finite group G, then

$$|AB| \ = \ |A| \cdot |B| / |A \cap B|;$$

So, if A and B are our subgroups of order 15 and order 35, respectively, we have

$$|A \cap B| \ = \ 15 \cdot 35 / |AB| \ \geq \ 15 \cdot 35 / |G| \ = \ 5.$$

So, A and B have an element in common other than just the identity. By Lagrange, the order of this element must divide both $|A|$ and $|B|$, implying that it must be 5.

3. There are several ways to solve this problem as well. I will present two below.

Way 1. First, the dihedral group D_n can be represented by the set

$$\{x^0, x^1, x^2, ..., x^{n-1}, y, yx, yx^2, ..., yx^{n-1}\},$$

where y represents any of the flips, and where x denotes a rotation by $2\pi/n$ counterclockwise. The rules governing how we multiply in this group are as follows:

$$yx^j = x^{-j}y, \; x^i x^j = x^{i+j}, \text{ and } y^2 = e.$$

So, thinking of

$$D_3 := \{x_1^0, x_1^1, x_1^2, y_1, y_1x_1, y_1x_1^2\},$$

and

$$D_6 := \{x_2^0, x_2^1, ..., x_2^5, y_2, y_2x_2, ..., y_2x_2^5\},$$

we can pick out the following subgroup of D_6 that is isomorphic to D_3:

$$H := \{x_2^0, x_2^2, x_2^4, y_2, y_2x_2^2, y_2x_2^4\}.$$

Clearly,

$$\begin{aligned} \varphi : H &\rightarrow D_3 \\ x_2^2 &\rightarrow x_1 \\ y_2 &\rightarrow y_1 \end{aligned}$$

forms an isomorphism.

Way 2. The second way is much more geometric: Take the hexagon on which D_6 acts, and starting with any vertex, connect every other vertex by a line segment. Note that you have now formed an isosceles triangle. The subgroup of rotations of D_6 by $0, 120$ and 240 degrees map this triangle to itself; and, the flips through the vertices of this triangle about the axis bisecting the side opposite these vertices, also are flips that fix the hexagon. Clearly, then, we have identified a subgroup of D_6 isomorphic to D_3 – it is that subgroup that fixes this inner isosceles triangle.

4. You can compute this yourself.

5. If H is a group of order 4, then by Lagrange each of its elements have order $1, 2$, or 4. If the group has an element of order 4, then it is cyclic; otherwise, every element has order 1 (the identity) or 2. So, we just need to locate subgroups H of S_4 that are either cyclic, or have every non-identity element of order 2.

First, let us consider the cyclic subgroup. It is easy to see that these are represented by

$$\begin{aligned} H_1 &:= \{e, (1\ 2\ 3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4\ 3\ 2)\} \\ H_2 &:= \{e,\ (1\ 2\ 4\ 3),\ (1\ 4)(2\ 3),\ (1\ 3\ 4\ 2)\} \\ H_3 &:= \{e,\ (1\ 3\ 2\ 4),\ (1\ 2)(3\ 4),\ (1\ 4\ 2\ 3)\}. \end{aligned}$$

To pin down the other subgroups of order 4, note that there are basically two kinds of elements in S_4 of order 2:

$$\text{those of the form } (a\ b), \text{ and those of the form } (a\ b)(c\ d).$$

Clearly, each of these groups must have at least one of the latter type, and so it is not too hard to see that these subgroups are

$$\begin{aligned} H_4 &:= \{e, (1\ 2)(3\ 4),\ (1\ 2),\ (3\ 4)\} \\ H_5 &:= \{e, (1\ 3)(2\ 4),\ (1\ 3),\ (2\ 4)\} \\ H_6 &:= \{e, (1\ 4)(2\ 3),\ (1\ 4),\ (2\ 3)\} \\ H_7 &:= \{e, (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}. \end{aligned}$$

Because this problem was so complex (harder than I intended), if you even wrote down just one of each kind (cyclic versus those isomorphic to $C_2 \times C_2$), I gave you credit.