Study sheet for Math 4107 midterm 2, Fall 2009

November 10, 2009

- Know all the material covered up to and including the first midterm.

- Know what an automorphism and inner-automorphism are, and know that they form a group under composition. Know that the group of inner-automorphisms on a group G is isomorphic to G/Z, where Z is the center. Know how to show that every finite group G of order greater than 2 (including now groups of even order) has a non-trivial automorphism. Know that $\text{Aut}((\mathbb{Z}/n\mathbb{Z})^+)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$.

- Know the definition of S_n and A_n (as I covered in class – S_n is the symmetric group on $\{1, 2, ..., n\}$, while A_n is the corresponding alternating group). Know how to write every permutation as a product of disjoint cycles, and know how to use cycle notation. Know that every element of A_n is a product of 3-cycles. Know the general form of the conjugate of a permutation when written in disjoint cycle form – it basically has the same cycle structure. Know how many equivalence classes of permutations there are under conjugation. Know that A_n is simple for $n \geq 5$, and roughly know how to prove this for $n = 5$. Know the sign of a permutation and how to prove that it is well-defined (recall that it is the determinant of the corresponding permutation matrix). Know how to enumerate the subgroups of S_4.

- Know the definition of a group action, in terms of homomorphisms. Given that a group G acts on a set X, know what the stabilizer subgroup of an element $x \in X$, denoted by $\text{Stab}(x)$, means; also, know

what is meant by the orbit of x, denoted by $\mathrm{Orbit}(x)$. Know the Orbit-Stabilizer Theorem, which asserts that

$$|G| \;=\; |\mathrm{Stab}(x)| \cdot |\mathrm{Orbit}(x)|.$$

- Know several different types of group actions, such as "left-multiplication" action in the proof of Cayley's theorem, conjugation, left-multiplication of cosets, etc. Know how to use group actions to prove such things as that a subgroup H of G is normal (remember the $\varphi : G \to S_{[G:H]}$ trick). Know the definition of the class equation, and how to use it to prove the following: Every p-group (a group of order p^k, for some k) has a non-trivial center; every group of order p^2 is abelian; the Sylow I theorem; and that if p (p prime) divides the order of a group (abelian or otherwise), then that group has an element of order p

- Know the statements of the Sylow theorems (I will only hold you responsible for knowing how to prove the first one), and know how to use them to do the following: Show that certain subgroups are normal by running through the possibilities; show that groups of certain orders (like 15) are abelian; classify the different types of groups of a given order. Sometimes you may need to use the Sylow theorems in combination with the Orbit-Stabilizer theorem and some enumeration of elements of a given order, to solve certain problems (i.e. it can be quite tricky just what sort of deductions one can make using the Sylow theorems).

- Know what a double coset is, and know how to interpret it as an equivalence relation (as we did in class).

- Know what the normalizer of a subgroup H is, and what the centralizer of an element x is. Know that the normalizer of H is the whole group if and only if the subgroup is normal; and, the centralizer of x is the whole group if and only if the element is in the center. Know how to prove something like the following: If at least half of all pairs $(x, y) \in G \times G$ commute, then G is abelian. Can you do the same for "at least half" replaced by "more than a fourth"?

- Know the definition of a direct product, and how to use it to decompose certain groups into direct products of their Sylow groups (when

possible). Know that a group is a direct product of $H_1 \cdots H_k$ implies it is isomorphic to $H_1 \times \cdots \times H_k$.

- Know the statement of the Fundamental Theorem of Finite Abelian Groups. Know how to use it to enumerate the number of abelian groups of a given order. It might be helpful (for test problems) to have some vague idea of how to prove it. Know the number of abelian groups of order p^n, p prime.

- Know the definition of a ring, and know some basic examples.