# Lattice points in a rectangle

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## 1 Introduction

Suppose that $L$ is a 2D lattice in $\mathbb{R}^2$, given by $L = \mathbb{Z}v_1 + \mathbb{Z}v_2$, where $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ are linearly independent. Furthermore suppose that we have a rectangle $R$ defined by

$$R \ := \ \{(x, y) \ : \ -X \leq x \leq X, \ -Y \leq y \leq Y\}.$$

In this exercise we will use Fourier analysis to uncover what relationships should exist between $v_1, v_2, X$, and $Y$ guaranteeing that any shift $t + R$ of $R$ always contains a lattice point.

Basically, what we can try to do is to take the Fourier transform of $1_R$ (the indicator function for the set $R$), and then use Fourier inversion to show that for each shift $t+L$ of the lattice $L$, this shift intersects the set $R$. Actually, it will not quite be enough to take the Fourier transform of just $1_R$, but instead we will need to work with a smoothed version of $1_R$ in order to make the Fourier transform have the requisite "decay properties". Basically, we will need that the Fourier transform of our function is in $L^1$.

It turns out that the dual group $\hat{\mathbb{R}}^n$ is isomorphic to $\mathbb{R}^n$, and that the additive characters $\chi : \mathbb{R}^n \to \mathbb{C}^*$ take the form $\chi(\vec{x}) = e^{2\pi i(\xi \cdot \vec{x})}$. Fourier transforms are then defined via

$$\hat{f}(\chi) \ := \ \int_{\mathbb{R}^n} f(\vec{x})\overline{\chi(\vec{x})}d\vec{x},$$

provided the integral exists and is finite, which will happen if $f \in L^1(\mathbb{R}^n)$. It is customary to represent this instead by

$$\hat{f}(\xi) \ := \ \int_{\mathbb{R}^n} f(\vec{x})e^{-2\pi i(\xi\cdot\vec{x})}d\vec{x} \ = \ \int_{\mathbb{R}^n} f(x_1, ..., x_n)e^{-2\pi i(\xi_1 x_1 + \cdots + \xi_n x_n)}d\vec{x},$$

where $\xi = (\xi_1, ..., \xi_n)$. We now arrive at the main fact we will need:

**Theorem 1 (Fourier Inversion Formula)** *Suppose $f : \mathbb{R}^n \to \mathbb{C}$ has the property that $f, \hat{f} \in L^1(\mathbb{R}^n)$. Then,*

$$f(\vec{x}) \ = \ \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i (\xi \cdot \vec{x})} d\vec{\xi}.$$

The way we will use this to help in our lattice point problem is as follows: first, note that the indicator function $1_R$ for our rectangle, although is itself an element of $L^1(\mathbb{R}^2)$, unfortunately has the property that $\hat{1}_R$ is *not* in $L^1$: we have that

$$\begin{aligned} \hat{1}_R(\xi_1, \xi_2) \ &= \ \int_{-X}^{X} \int_{-Y}^{Y} e^{-2\pi i (\xi_1 x + \xi_2 y)} dy dx \ = \ \left( \int_{-X}^{X} e^{-2\pi i \xi_1 x} dx \right) \left( \int_{-Y}^{Y} e^{-2\pi i \xi_2 y} dy \right) \\ &= \ \frac{\sin(2\pi \xi_1 X) \sin(2\pi \xi_2 Y)}{\pi^2 \xi_1 \xi_2}. \end{aligned}$$

And then we will have that this typically has size proportional to $1/|\xi_1 \xi_2|$, at least when $\xi$ is not too near 0, which implies that

$$\int_{\mathbb{R}^2 \setminus D} \frac{d\xi_1 d\xi_2}{|\xi_1 \xi_2|} \ = \ \infty,$$

where $D$ is any disk enclosing $(0, 0)$. This then means that

$$\int_{\mathbb{R}^2} |\hat{1}_R(\xi)| d\xi_1 d\xi_2 \ = \ \infty,$$

and therefore $\hat{1}_R \notin L^1(\mathbb{R}^2)$. Fortunately, there is a standard trick for handling this: smooth out the function $1_R$ a little bit by convolving it with another function. To carry out this plan we begin by defining the slightly smaller rectangle

$$R' \ := \ \{(x, y) \ : \ -(1-\varepsilon)X \leq x \leq (1-\varepsilon)X, \ -(1-\varepsilon)Y \leq y \leq (1-\varepsilon)Y\},$$

where eventually we will let $\varepsilon \to 0$, and then we define the measure

$$\mu(x, y) \ := \ \begin{cases} 1/4\varepsilon^2, & \text{if } -\varepsilon \leq x, y \leq \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Finally we let

$$f(x,y) \;:=\; 1_{R'} * \mu(x,y) \;=\; \int_{\mathbb{R}^2} 1_{R'}(u,v)\mu(x-u,y-v)dudv.$$

This function has the properties that

$$\mathrm{supp}(f)^* \;=\; \mathrm{supp}(1_R),\ \text{and for all } (x,y)\in\mathbb{R}^2, 0\le f(x,y)\le 1,$$

where the $*$ here means we include the boundary of the set. [1]

**Problem 1.** Verify that $\hat{f}\in L^1(\mathbb{R}^2)$ and that

$$\hat{f}(\xi_1,\xi_2) \;=\; \frac{\sin(2\pi\xi_1(1-\varepsilon)X)\sin(2\pi\xi_2(1-\varepsilon)Y)\sin(2\pi\xi_1\varepsilon)\sin(2\pi\xi_2\varepsilon)}{4\pi^4\varepsilon^2\xi_1^2\xi_2^2}$$

The next step is to choose large – but not too large – values for paramters $N_1, N_2$ and then to compute

$$F = F(N_1,N_2,t) \;:=\; \sum_{\substack{-N_1\le j\le N_1\\ -N_2\le k\le N_2}} f(t+jv_1+kv_2),$$

which will be a lower bound for the number of lattice points in $L$ contained in the set $R-t$ (the translate of our rectangle $R$ by $-t$).

**Problem 2.** Using Theorem 1 show that

$$F \;=\; \int_{\mathbb{R}^2} e^{-2\pi i(\xi\cdot t)}\hat{f}(\xi)\frac{\sin(2\pi(N_1+1/2)(\xi_1x_1+\xi_2y_1))}{\sin(\pi(\xi_1x_1+\xi_2y_1))} \frac{\sin(2\pi(N_2+1/2)(\xi_1x_2+\xi_2y_2))}{\sin(\pi(\xi_1x_2+\xi_2y_2))}d\xi_1d\xi_2.$$

Clearly we can assume that $t\in\mathbb{R}^2$ is contained in a fundamental parallelogram $\Gamma_0$ for the lattice $\mathbb{Z}v_1+\mathbb{Z}v_2$. [2] And, it turns out that the above

[1] Typically one convolves by an even smoother function than $\mu$; alternatively, one can convolve with $\mu$ multiple times – that is, for a certain rectangle $R''$ one considers $1_{R''}*\mu*\mu*\cdots*\mu$.

[2] Given basis vectors $v_1, v_2$ for the lattice, a fundamental parallelogram is the set of all linear combinations $t_1v_1+t_2v_2$, where $t_1,t_2\in[0,1]$.

approach is more than adequate to show that $F$ is "large" when $t$ is near enough to 0; however, having looked ahead I can see that we will need to smooth out the patch of the lattice we are considering to handle all values of $t$ in such a fundamental parallelogram. [3] When this is done, it turns out that we recover the so-called "Poisson Summation Formula" as we will see below. First, though, it is worth getting a feel for why the above formula (where the lattice is *not* smoothed) turns out to be harder to work with. To this end, we define $G(\xi)$ to be the ratio of products of sines in the integral in problem 2 above. Note that $G(\xi) \geq 0$ for all $\xi \in \mathbb{R}^2$; and furthermore $G$ is periodic (the proper term is 'doubly periodic') by shifts contained in the lattice dual to $\mathbb{Z}v_1 + \mathbb{Z}v_2$.[4] In other words, if $\sigma \in \mathbb{R}^2$ satisfies $\sigma \cdot v_1, \sigma \cdot v_2 \in \mathbb{Z}$ then

$$G(\xi + \sigma) \ = \ G(\xi).$$

What this means is that if we let $\Gamma$ be a fundamental parallelogram for this dual lattice, then if $\mathcal{R} \subseteq \mathbb{R}^2$ can be covered by at most $K$ shifts of $\Gamma$, then

$$\int_{\mathcal{R}} G(\xi) d\xi \ \leq \ K \int_{\Gamma} G(\xi) d\xi,$$

where equality holds if $\mathcal{R}$ is exactly a disjoint union of $K$ shits of $\Gamma$. This nicely motivates the following problem:

**Problem 3.** Show that

$$\int_{\Gamma} G(\xi) d\xi \ \ll \ \text{area}(\Gamma)(\log N_1)(\log N_2).$$

Let me help you out: fix vectors $u_1, u_2 \in \mathbb{R}^2$ such that

$$u_1 \cdot v_1 \ = \ 1 \ = \ u_2 \cdot v_2, \text{ and } u_1 \cdot v_2 = 0 = u_2 \cdot v_1.$$

Then, the lattice dual to the one spanned by $v_1$ and $v_2$ has basis $u_1, u_2$, and we can choose $\Gamma$ to be

$$\Gamma \ := \ \{t_1 u_1 + t_2 u_2 \ : \ t_1, t_2 \in [0, 1]\}.$$

[3] By 'patch' I mean $jv_1 + kv_2$ where $|j| \leq N_1$ and $|k| \leq N_2$.

[4] This dual lattice consists of all $\sigma \in \mathbb{R}^2$ satisfying $\sigma \cdot v_1, \sigma \cdot v_2 \in \mathbb{Z}$.

It follows that if $u_1 = (\alpha_1, \beta_1)$ and $u_2 = (\alpha_2, \beta_2)$ then by the theory of integrating factors (and multidimensional integrals) we have

$$\begin{aligned} \int_\Gamma G(\xi) d\xi &= \left| \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \right| \int_{[0,1]^2} G(t_1 \alpha_1 + t_2 \alpha_2, t_1 \beta_1 + t_2 \beta_2) dt_1 dt_2 \\ &= \text{area}(\Gamma) \int_{[0,1]^2} G(t_1 u_1 + t_2 u_2) dt_1 dt_2. \end{aligned}$$

We then observe that

$$G(t_1 u_1 + t_2 u_2) = \frac{\sin(2\pi(N_1 + 1/2)t_1)}{\sin(\pi t_1)} \cdot \frac{\sin(2\pi(N_2 + 1/2)t_2)}{\sin(\pi t_2)};$$

Now finish the problem...

As we said before, we would like to be able to work with all translates $t \in \Gamma_0$, and it turns out that what goes wrong is that mass of $G(\xi)$ isn't too strongly concentrated near the lattice points $\mathbb{Z}u_1 + \mathbb{Z}u_2$ – this will mess up various estimates along the way that I won't bother to discuss further. To fix this problem we will consider the following "smoothed version" of our counting problem: first, we will no longer use $N_1, N_2$, but will use a single parameter $N = N_1 = N_2$. Then, we define

$$\begin{aligned} F_2(N, t) &:= \binom{2N}{N}^{-2} \int_{\mathbb{R}^2} e^{-2\pi i(\xi \cdot t)} \hat{f}(\xi) (e^{-\pi i(\xi \cdot v_1)} + e^{\pi i(\xi \cdot v_1)})^{2N} (e^{-\pi i(\xi \cdot v_2)} + e^{\pi i(\xi \cdot v_2)})^{2N} d\xi \\ &= 2^{4N} \binom{2N}{N}^{-2} \int_{\mathbb{R}^2} e^{-2\pi i(\xi \cdot t)} \hat{f}(\xi) \cos(\pi(\xi \cdot v_1))^{2N} \cos(\pi(\xi \cdot v_2))^{2N} d\xi. \end{aligned}$$

It is easy to see that

$$F_2 = \sum_{-N \le j,k \le N} \lambda_{j,k} f(t + j v_1 + k v_2),$$

where we won't even bother to say exactly what these $\lambda_{j,k}$ are, except to say that $0 \le \lambda_{j,k} \le 1$ and that for $(j, k)$ near enough to $(0, 0)$ we will have $\lambda_{j,k} \sim 1$ – so, by letting $N \to \infty$ we will essentially get

$$\sum_{j,k \in \mathbb{Z}} f(t + j v_1 + k v_2).$$

Notice that, unlike with $F = F(N_1, N_2, t)$, in defining $F_2$ we *started* on the Fourier side and then worked backwards to see what it meant on the "physical side". This is a common approach.

Now let us define

$$G_2(\xi) \;=\; 2^{4N} \binom{2N}{N}^{-2} \cos(\pi(\xi \cdot v_1))^{2N} \cos(\pi(\xi \cdot v_2))^{2N},$$

which is the analogue of the function $G(\xi)$ for this "smoothed lattice". [5] Consider then the integral $\int_\Gamma G_2(\xi) d\xi$. Unlike with $\int_\Gamma G(\xi) d\xi$, *here* we will have that most of the integral's mass comes from those $\xi$ near lattice points: for example, suppose $\xi$ is near $(0,0)$ and write $\xi_1 = c_1/\sqrt{N}$ and $\xi_2 = c_2/\sqrt{N}$. Then, upon applying a Maclaurin expansion to those cosines above, we get:

$$G_2(\xi) \;\ll\; N(1 - O(c_1^2/N))^{2N}(1 - O(c_2^2/N))^{2N} \;=\; N \exp(-O(\|c\|^2)).$$

We will get a similar estimate for when $\xi$ is near $u_1, u_2$ and $u_1 + u_2$, which are the other lattice points on the boundary of $\Gamma$.

What this means is that for large $N$

$$\begin{aligned} \sum_{j,k \in \mathbb{Z}} f(t + jv_1 + kv_2) \;&\approx\; \int_{\mathbb{R}^2} e^{-2\pi i(\xi \cdot t)} \hat{f}(\xi) G_2(\xi) d\xi \\ &\approx\; \delta \sum_{j,k \in \mathbb{Z}} e^{-2\pi i((ju_1 + ku_2)\cdot t)} \hat{f}(ju_1 + ku_2), \qquad (1) \end{aligned}$$

where $\delta$ is basically just the integral of $G_2$ near a lattice point of $\mathbb{Z}u_1 + \mathbb{Z}u_2$ (we are using the fact that $G_2$ is periodic by elements of this lattice) – that is, we can just let

$$\delta \;\sim\; \int_{\|\xi\| < 1/N^{1/3}} G_2(\xi) d\xi,$$

say. In the following problem you will work out what $\delta$ equals (upon letting $N \to \infty$).

**Problem 4.** To work out $\delta$, first set $t = 0$ and let $X = Y$, and note that the sum of $f(jv_1 + kv_2)$ is *essentially* the number of lattice points of $\mathbb{Z}v_1 + \mathbb{Z}v_2$ inside the box $[-X, X] \times [-X, X]$, at least for $\varepsilon > 0$ small. The

[5] Note that among its many properties is that if $\sigma \in \mathbb{Z}u_1 + \mathbb{Z}u_2$ then $G_2(\xi) = G_2(\xi + \sigma)$.

number of such lattice points is clearly approximately $|2X|^2/\mathrm{Vol}(\Gamma_0)$. On the other hand, as $X \to \infty$ the last sum in (1) tends to be dominated by the term with $(j,k) = (0,0)$. And now, putting all this together, show that $\delta = 1/\mathrm{Vol}(\Gamma_0) = \mathrm{Vol}(\Gamma)$.

What we have just shown is that

$$\sum_{j,k \in \mathbb{Z}} f(t + jv_1 + kv_2) \;=\; \mathrm{Vol}(\Gamma_0)^{-1} \sum_{j,k \in \mathbb{Z}} e^{-2\pi i((ju_1+ku_2)\cdot t)} \hat{f}(ju_1 + ku_2),$$

which is an exact equation, surprisingly arrived at using only analytic inequalities! [6] Another interesting feature is that although we started out using particular choices $v_1, v_2, u_1, u_2$ for our bases, this equation is really basis-independent, since it can be interpreted as sums over lattice points in two different lattices. This is a common feature of many analytic arguments: one starts non-canonically by choosing a basis, but then in the final analysis this choice disappears and all that remains are expressions involving global properties of the lattice (or vector space, or subgroup, or ...).

[6] And for all I know this is a new proof of the Poisson Summation formula for $\mathbb{R}^2$.