Homework 3, Analytic Number Theory

February 6, 2007

1. We showed (or will show) in class that if the Riemann Hypothesis is true, then

$$\sum_{n \leq x} \Lambda(n) \ = \ x + O(x^{1/2} \log^2 x).$$

In this problem, you will prove the converse; that is, suppose that we have such an estimate on this sum over the von-Mangoldt function. Then, show that $\zeta(s)$ has no zeros ρ satisfying $\mathrm{Re}(\rho) > 1/2$. The way to do this is to produce a meromorphic continuation of

$$-\frac{\zeta'(s)}{\zeta(s)} \ = \ \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}$$

from $\mathrm{Re}(s) > 1$ to the region $\mathrm{Re}(s) > 1/2$, with only a simple pole at $s = 1$. If $s = 1$ is indeed the only place where ζ'/ζ has a singularity in $\mathrm{Re}(s) > 1/2$, then it means that $\zeta(s)$ has no zeros in this same region, which we know implies the Riemann hypothesis, because by the functional equation, all the zeros of $\zeta(s)$ are symmetric about the line $\mathrm{Re}(s) = 1/2$.

The key to solving this problem is just some clever use of integration-by-parts (i.e. partial summation), in much the same way that we proved the analytic continuation

$$\zeta(s) \ = \ \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}} ds,$$

where $\{x\}$ denotes the fractional part of x.

2. In this problem you will use contour methods to esimate the sum

$$\sum_{n \leq x} \varphi(n)$$

to within a good error. Although for this particular sum we already know good estimates, which were provided, for example, by the convolution method, it is a good idea to become familar with the contour methods, as they usually provide the sharpest estimates for these sorts of quantities. [1]

Step 1. Suppose that $f(s)$ and $g(s)$ are zeta functions that take the form

$$f(s) \;=\; \sum_{n=1}^{\infty} \frac{a_n}{n^s},\;\; g(s) \;=\; \sum_{n=1}^{\infty} \frac{b_n}{n^s}.$$

We think of the coefficients a_n and b_n of these series as functions from the natural numbers to the complex numbers; that is, $a_n = a(n)$ and $b_n = b(n)$. Now consider the zeta function $f(s)g(s)$, which can be written as

$$f(s)g(s) \;=\; \sum_{n=1}^{\infty} \frac{c_n}{n^s}.$$

Show that the coefficients c_n satisfy

$$c(n) \;=\; (a * b)(n) \;=\; \sum_{d|n} a(d)b(n/d).$$

Step 2. Recalling that

$$\varphi(n) \;=\; (\mu * \mathrm{id})(n) \;=\; \sum_{d|n} \mu(d)(n/d),$$

expalin why for $\mathrm{Re}(s) > 2$,

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} \;=\; \frac{\zeta(s-1)}{\zeta(s)}.$$

Step 3. Using the formula

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} ds \;=\; \delta(y) + E(y, T),$$

[1] Indeed, a key part of Goldston, Pintz, and Yildrim's famous recent work on small gaps between primes made use of the vast power of contour methods to evaluate certain difficult, technical sums in intermediate steps of their proof. Contour methods are also a small part of Green and Tao's famous work on primes in arithmetic progression.

where

$$\delta(y) \;=\; \begin{cases} 0, & \text{if } 0 < y < 1; \\ 1/2, & \text{if } y = 1; \\ 1, & \text{if } y > 1. \end{cases}$$

and

$$|E(y,T)| \;\leq\; \begin{cases} y^c \min(1, T^{-1}|\log y|^{-1}), & \text{if } y \neq 1; \\ cT^{-1}, & \text{if } y = 1, \end{cases}$$

explain why for any real number $c > 2$,

$$\sum_{n \leq x} \varphi(n) \;=\; \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\frac{\zeta(s-1)}{\zeta(s)}ds \;+\; F(x,T),$$

where $F(x,T)$ is a certain "error term". How does $E(x,T)$ grow as a function of x and T? In order to answer this question you will need to pay attention to the following facts:

- First, you will need the basic estimate $\varphi(n) \leq n$.

- And second, you will need to pay attention to the $|\log y|^{-1}$ in the $E(y,T)$, as it can become significant for certain ranges of y. In particular, suppose that $y = x/n$, where say $n = x - j < x$, $j \geq 1$. Then, we have by Taylor expansion that

$$\frac{1}{|\log y|} \;=\; \frac{1}{|\log(1 - j(x-j)^{-1})|} \;<\; \frac{x}{j}.$$

Step 4. Suppose we take $c = 3$ and $T = x^4$ in step 3, and then extend the contour to a rectangle with edges

$$\begin{aligned} E_1 &\;:\; 3 - iT \to 3 + iT \\ E_2 &\;:\; 3 + iT \to 3/2 + iT \\ E_3 &\;:\; 3/2 + iT \to 3/2 - iT \\ E_4 &\;:\; 3/2 - iT \to 3 - iT. \end{aligned}$$

Give good upper bounds for the quantities

$$\left|\frac{1}{2\pi i}\int_{E_i} \frac{x^s}{s}\frac{\zeta(s-1)}{\zeta(s)}ds\right|$$

over $i = 2, 3$ and 4. Two key things you will need use and pay attention to are:

• For $\mathrm{Re}(s) \geq 1/2$, and $|s-1| > \epsilon$, we have that $|\zeta(s)| < C_1$, for some absolute constant $C_1 = C_1(\epsilon)$ that depends only on ϵ (I proved this in class at one point – it follows from one of the analytic continuation formulas for $\zeta(s)$). Basically, this is saying that if you are in a certain "right-half-plane", and if you are "not too near $s = 1$, where $\zeta(s)$ has a simple pole", then $\zeta(s)$ is bounded in magnitude.

• For $\mathrm{Re}(s) \geq 3/2$, we have that $|1/\zeta(s)| \leq C_2$, for some absolute constant C_2; in fact,

$$C_2 \ = \ \prod_{p \text{ prime}} \left(1 + \frac{1}{p^{3/2}}\right).$$

Basically, this is telling you that "not only does $\zeta(s)$ have no zeros for $\mathrm{Re}(s) \geq 3/2$, but in fact $\zeta(s)$ cannot even get too near to 0 in this region."

Step 5. Use Cauchy's integral formula to show that the integral around the whole rectangular contour equals

$$\mathrm{Res}_{s=2} \frac{x^s}{s} \frac{\zeta(s-1)}{\zeta(s)} \ = \ \frac{x^2}{\zeta(2)} \ = \ \frac{6x^2}{\pi^2}.$$

Step 6. Combine all the above steps together, to deduce that

$$\sum_{n \leq x} \varphi(n) \ = \ \frac{6x^2}{\pi^2} + O(x^{3/2} \log x).$$

***Step 7.** This should be considered a "bonus step", as it requires a lot more work: Explain how you could have gotten a sharper estimate, of say $O(x \log x)$ or so, by replacing the 3/2 in the above contour with, say, $1 + 1/\log x$, or some such thing.

****Step 8.** This is *really* a bonus step, and I don't adivse you spend much time on it: What sort of estimates can you get for the sum of $\varphi(n)$ over $n \leq x$ if you assume the Riemann hypothesis? Note that, if you assume RH, you can extend the left-most real coordinate of the contour from 3/2 (or $1 + 1/logx$) to, say, $1/2 + \epsilon$, for arbitrary $\epsilon > 0$. In principle, you should be able to exploit this to give sharper esimates for error terms, if you are clever.