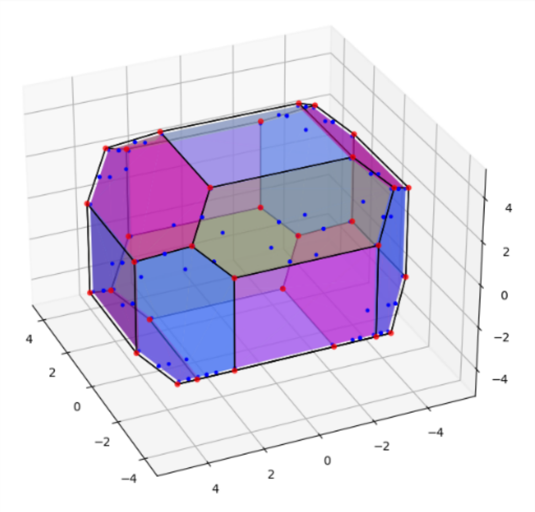
Why the Thurston Metric is (Not) like L^∞

Assaf Bar-Natan

Dec. 11, 2022



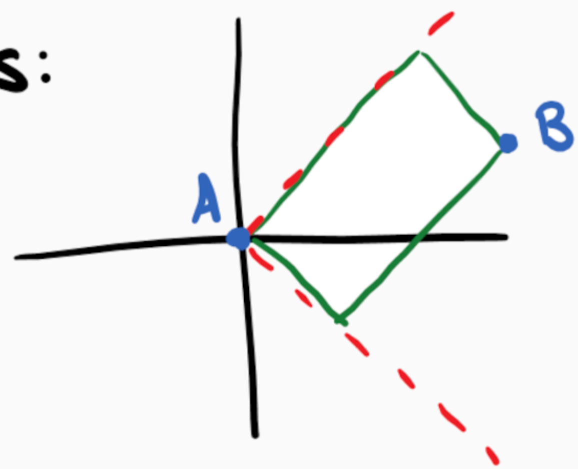
Tech Topology 2022

Properties of L^∞

L^∞ metric on $\mathbb{R}^2$

$$d\left((x_1, x_2),(y_1, y_2)\right) = \max\left(|x_1 - x_2|, |y_1 - y_2|\right)$$

Geodesics:



- Sometimes Unique
- Can get far apart
- In a cone defined by unique geodesics

Teichmüller Space

$S =$  $; \chi(S) < 0$

A marking on S is a homeo. $f : S \to X$

← Hyperbolic surface

← Topological surface

$[f, X] \sim [g, Y]$ if $\exists h : X \to Y$ isometry with hf homotopic to g

↑ markings ↑

$$\begin{tikzcd} & X \arrow[dd, dashed, "\wr h"] \\ S \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & Y \end{tikzcd}$$

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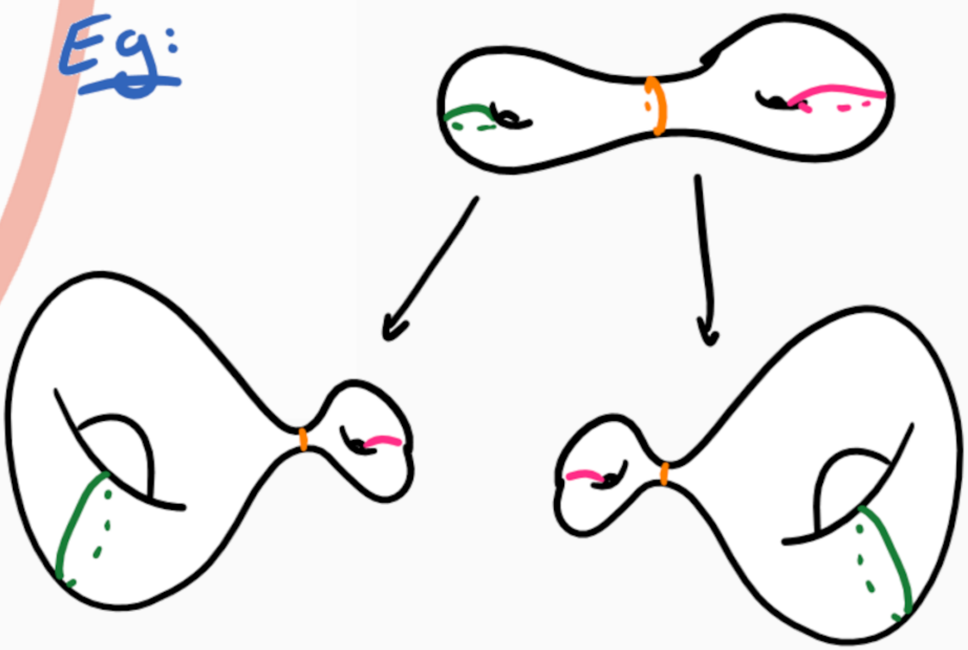

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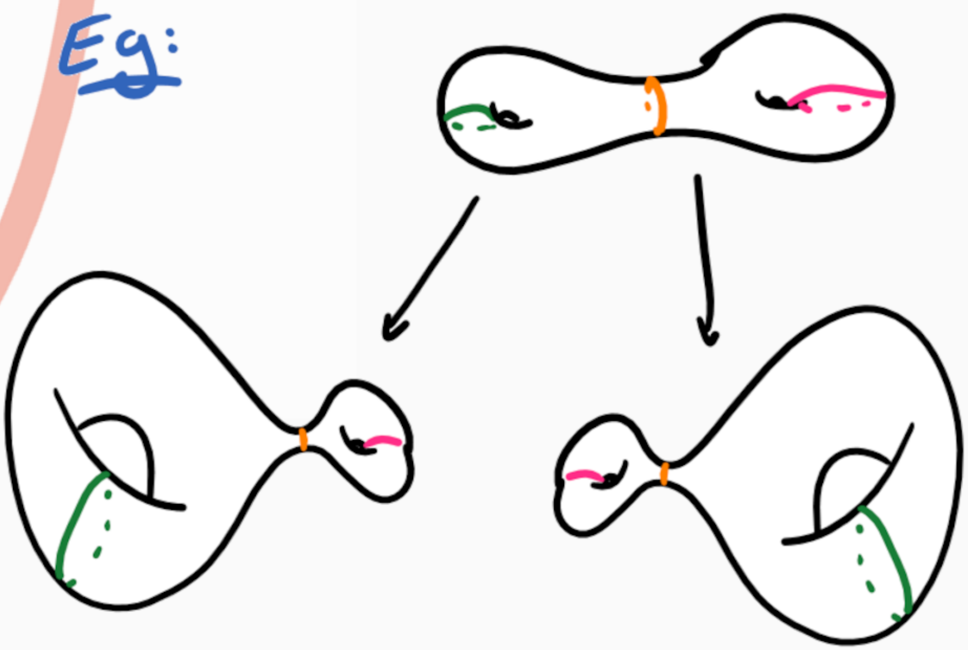

Definition:

$T(S) = \text{Markings}/\sim$

The Thurston Metric

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$$= \log \inf_{\substack{f \sim id \\ f: X \to Y}} \|Df\|$$

$f \in C^1$
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Lemma: $\forall X, Y, Z \in T(S)$

$$d_{Th}(X,Y) = 0 \Leftrightarrow X = Y \qquad d_{Th}(X,Z) \leq d_{Th}(X,Y) + d_{Th}(Y,Z)$$

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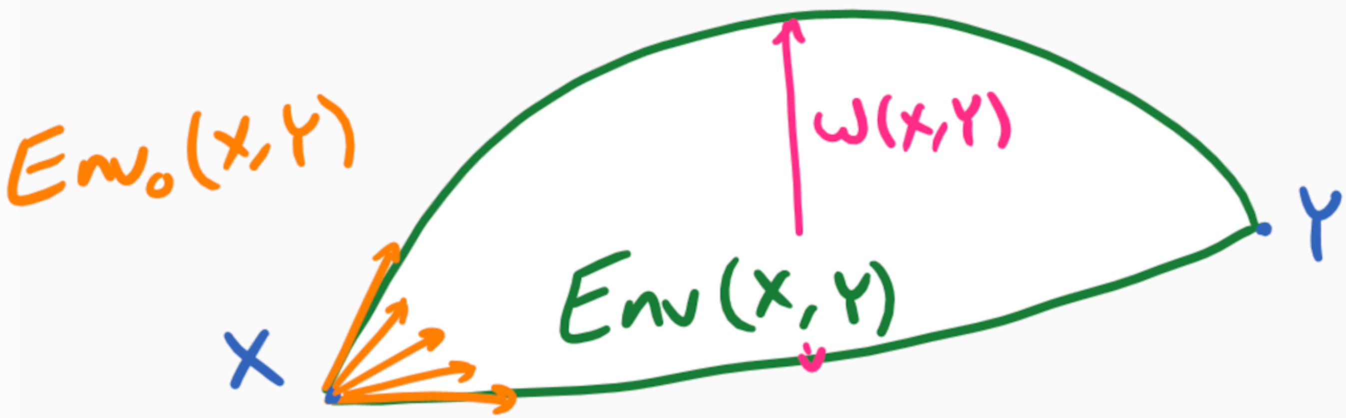
⚠ d_{Th} is asymmetric

Geodesic Envelopes

Thurston '86: d_{Th} is geodesic

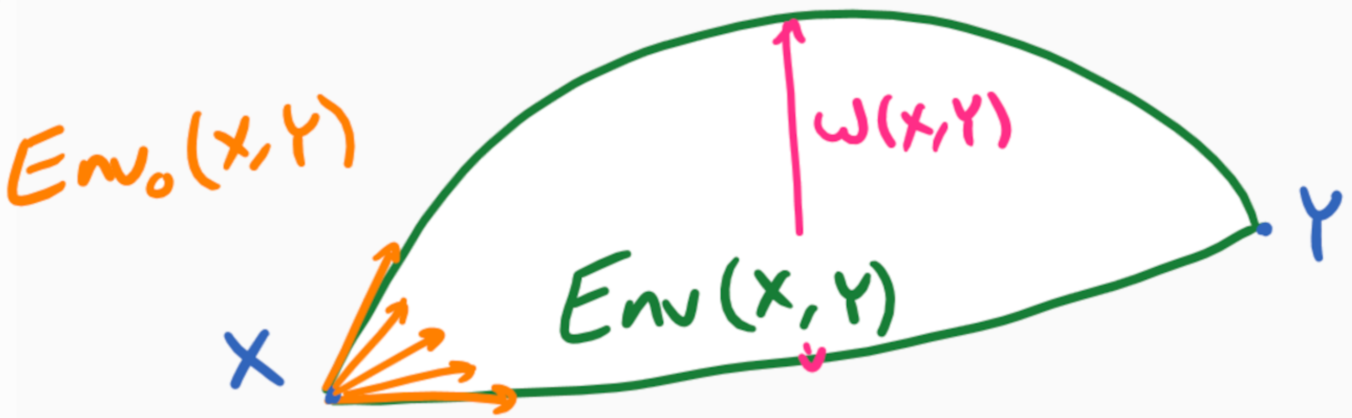


$$\omega(X,Y) = \sup_{t,\gamma,\gamma'} d_{Th}(\gamma(t), \gamma'(t))$$

paths from X to Y

Geodesic Envelopes

Thurston '86: d_{Th} is geodesic



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paths from X to Y

Thm (Dumas, Lenzhen, Rafi, Tao '19)

Egs where $\omega(X,Y) \to \infty$

Stretch Paths

A lamination is a Hausdorff limit of closed curves

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"$\sup_{\alpha} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}$ is realized on a lamination $\Lambda(X,Y)$"

"$\Lambda(X,Y)$ big $\Rightarrow$ unique geodesic"

Stretch Paths

A lamination is a Hausdorff limit of closed curves

"$\sup_{\alpha} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}$ is realized on a lamination $\Lambda(X,Y)$"

"$\Lambda(X,Y)$ big $\Rightarrow$ unique geodesic"

If $\omega(X,Y)=0$, the path from X to Y is a stretch path

d_{Th} is like L^{∞}

v is a **stretch vector** if it's the derivative of a stretch path v is a stretch vector if it's the derivative of a stretch path

<u>Thm (B-N, '22)</u>

initial vectors of geodesics from X to Y

Convex hull

Stretch vectors @ X

$$\mathrm{Env}_0(X,Y) = CH(SV_X)$$

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Thm (B-N, '22)

If $S =$  or , $\exists D > 0$

For any $X, Y \in T(S)$

$$\omega(X, Y) \leq D$$

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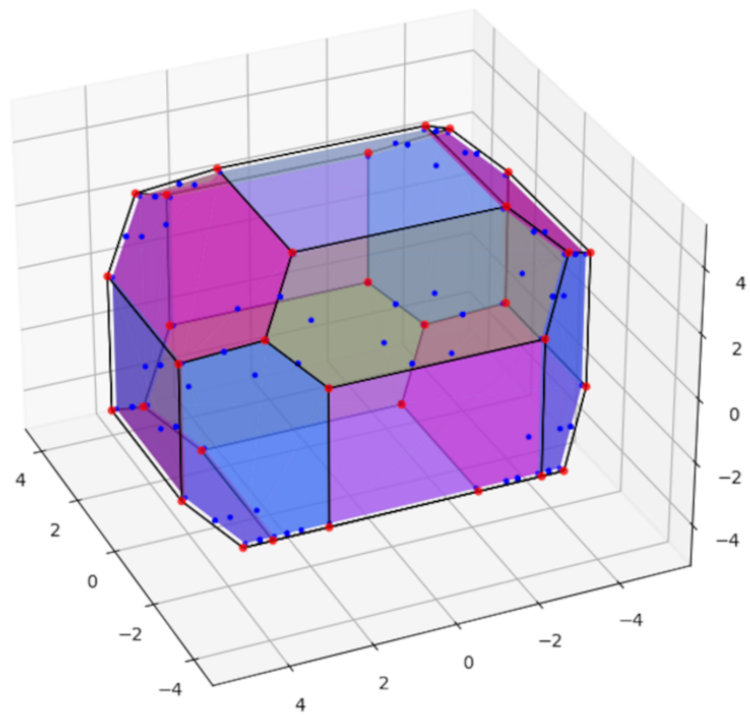
Hopes and Dreams



Current goal:
When is $w(X,Y)$ bdd uniformly?

Boundaries at Infinity

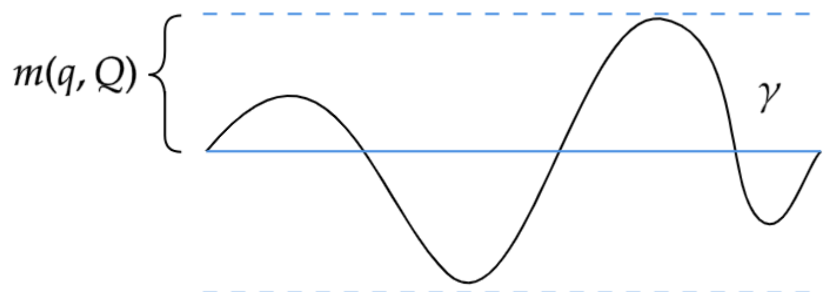
Vivian He

University of Toronto

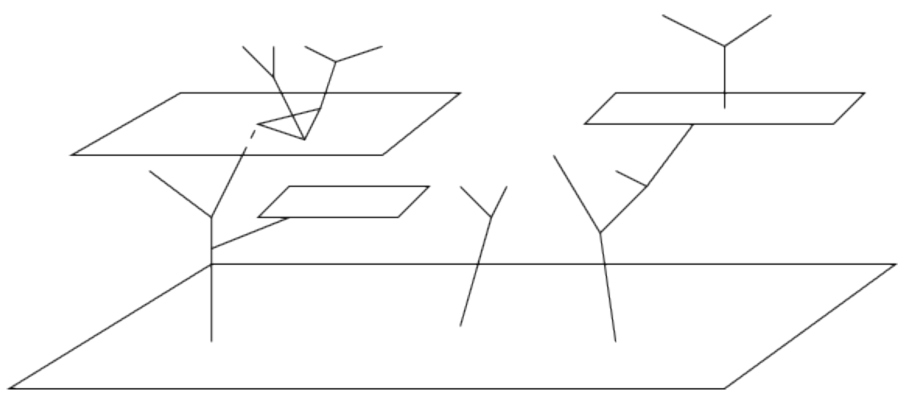
December 2022

The Sublinearly Morse Boundary

Morse: invariant under quasi-isometry



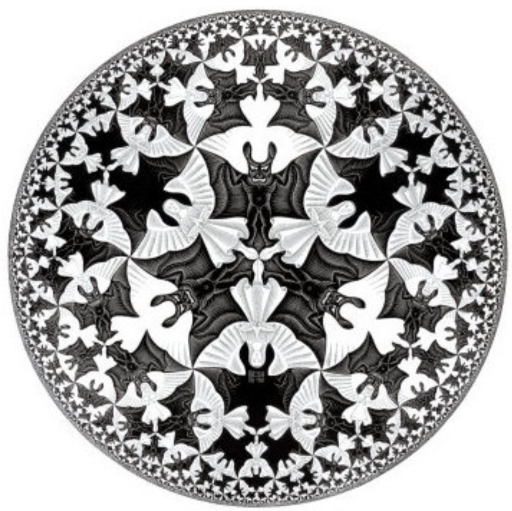
Sublinear: random walks converge to the sublinearly Morse boundary

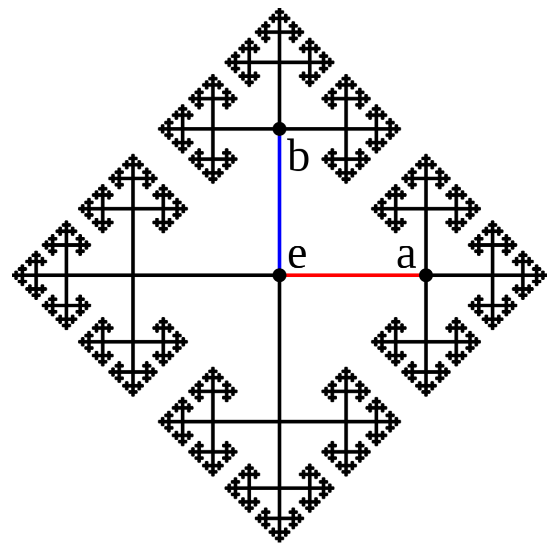


The Gromov Boundary

Definition

The Gromov boundary of a hyperbolic metric space X is the set $\partial X = \{[\gamma] \mid \gamma \text{ is a geodesic ray }\}$.





The topology is generated by the following open neighbourhoods around $[\gamma]$:

$$U([\gamma], r) = \left\{ [\gamma'] \mid \liminf_{s,t \to \infty} (\gamma(s), \gamma'(t))_o \geq r \right\}.$$

Quasi-isometry

Definition

Let (X_1, d_1) and (X_2, d_2) be metric spaces. A function $f : X_1 \to X_2$ is a quasi-isometry if there exists constants $q \geq 1$ and $Q \geq 0$ such that for any two points $x, y \in X_1$,

$$\frac{1}{q} d_1(x, y) - Q \leq d_2(f(x), f(y)) \leq q d_1(x, y) + Q.$$

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$$\frac{1}{q} d_1(x, y) - Q \leq d_2(f(x), f(y)) \leq q d_1(x, y) + Q.$$

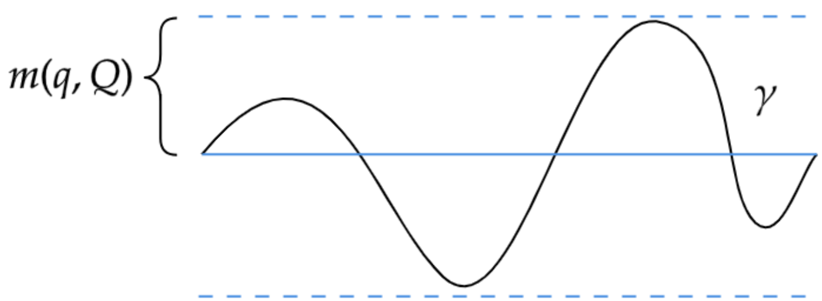
Theorem

Let X_1, X_2 be hyperbolic metric spaces, and let $f : X_1 \to X_2$ be a quasi-isometry. Then f induces a homeomorphism on the Gromov boundaries ∂X_1 and ∂X_2.

Quasi-isometry

Theorem (Morse lemma)

Let X be a hyperbolic space, and γ a (q, Q)–quasi-geodesic in X. Then there is a constant $m(q, Q)$ such that γ is in the $m(q, Q)$–neighbourhood of the geodesic segment connecting its endpoints.



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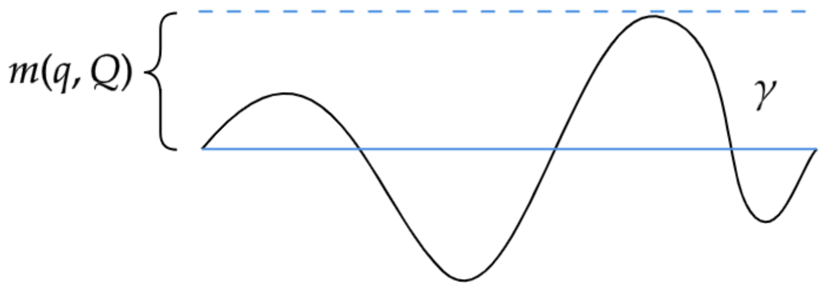
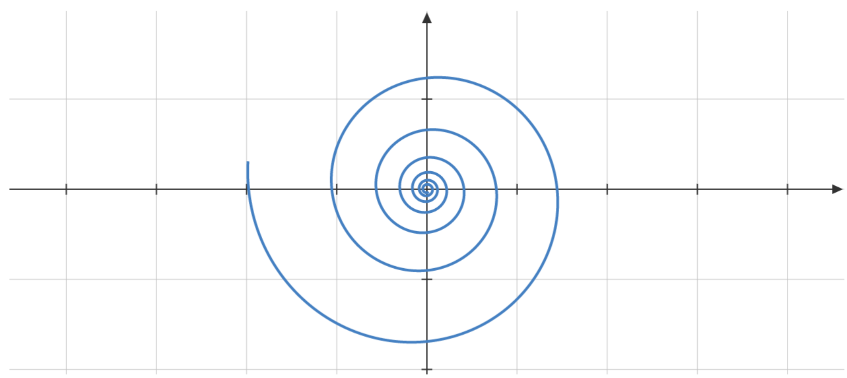
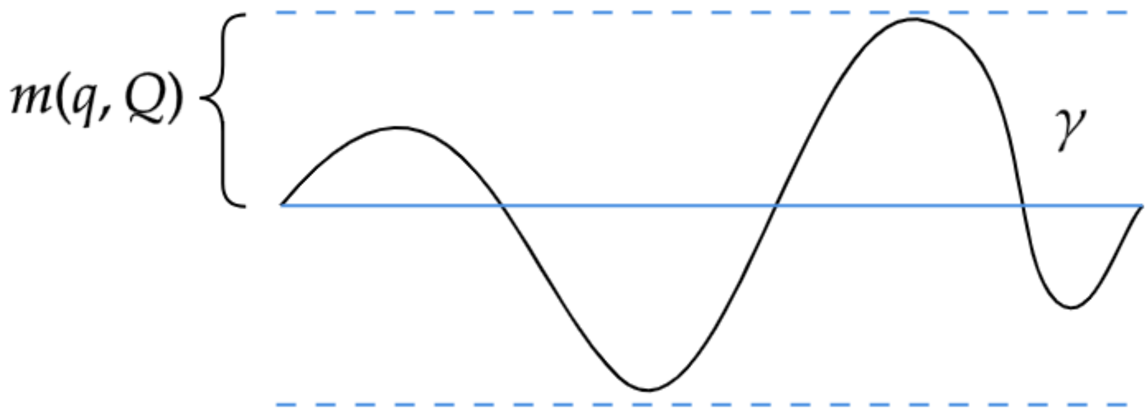




The Morse Boundary

Definition

A geodesic γ is M–Morse if any quasi–geodesics with endpoints on γ is contained in the M–nbhd γ.



Definition (Cashen-Mackay, Charney-Sultan, Cordes)

The Morse boundary of a geodesic metric space X is the set $\partial X = \{[\gamma] \mid \gamma \text{ is a } M\text{–Morse (quasi–)geodesic ray for some } M\}$.

Random Walks

Theorem (Kaimanovich)

In a hyperbolic group G, almost all sample paths $\{x_n\}$ of the random walk (G, μ) converge to a (random) point in the Gromov boundary.

The Tree of Flats

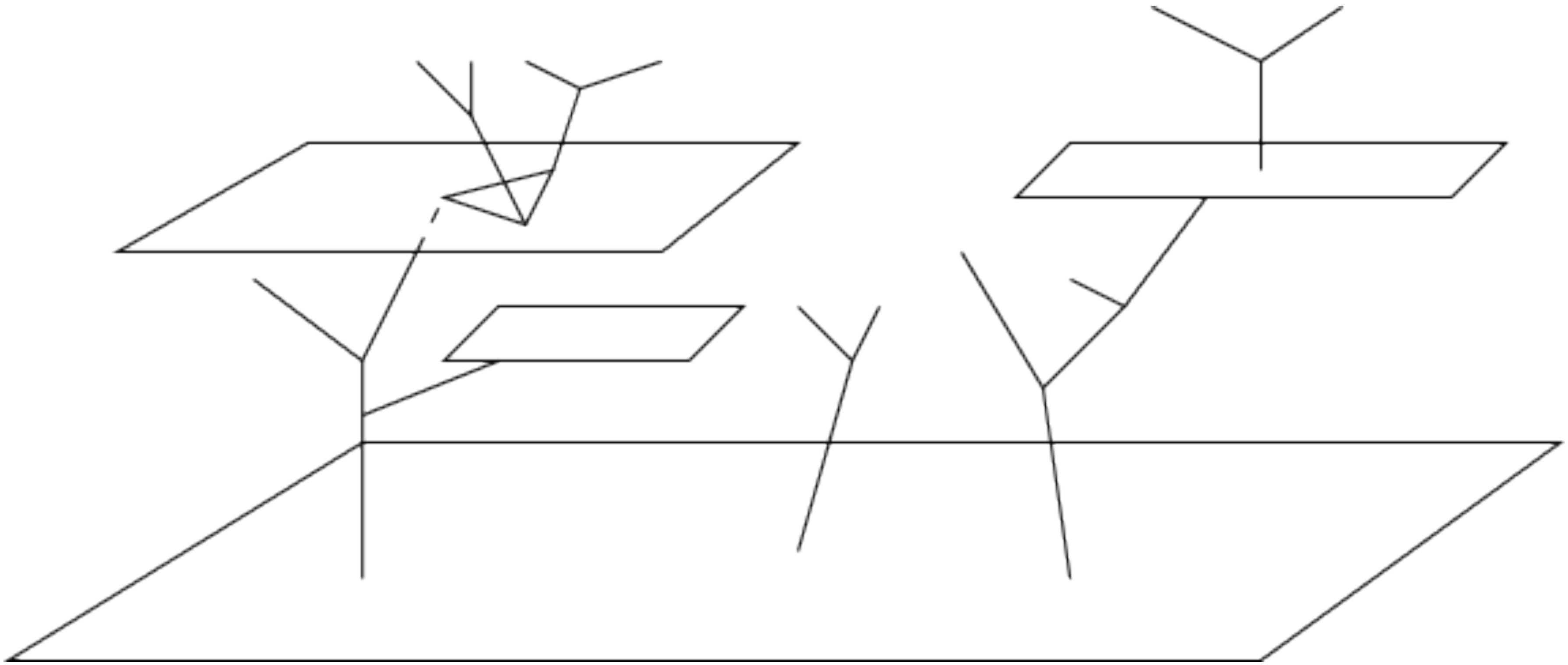
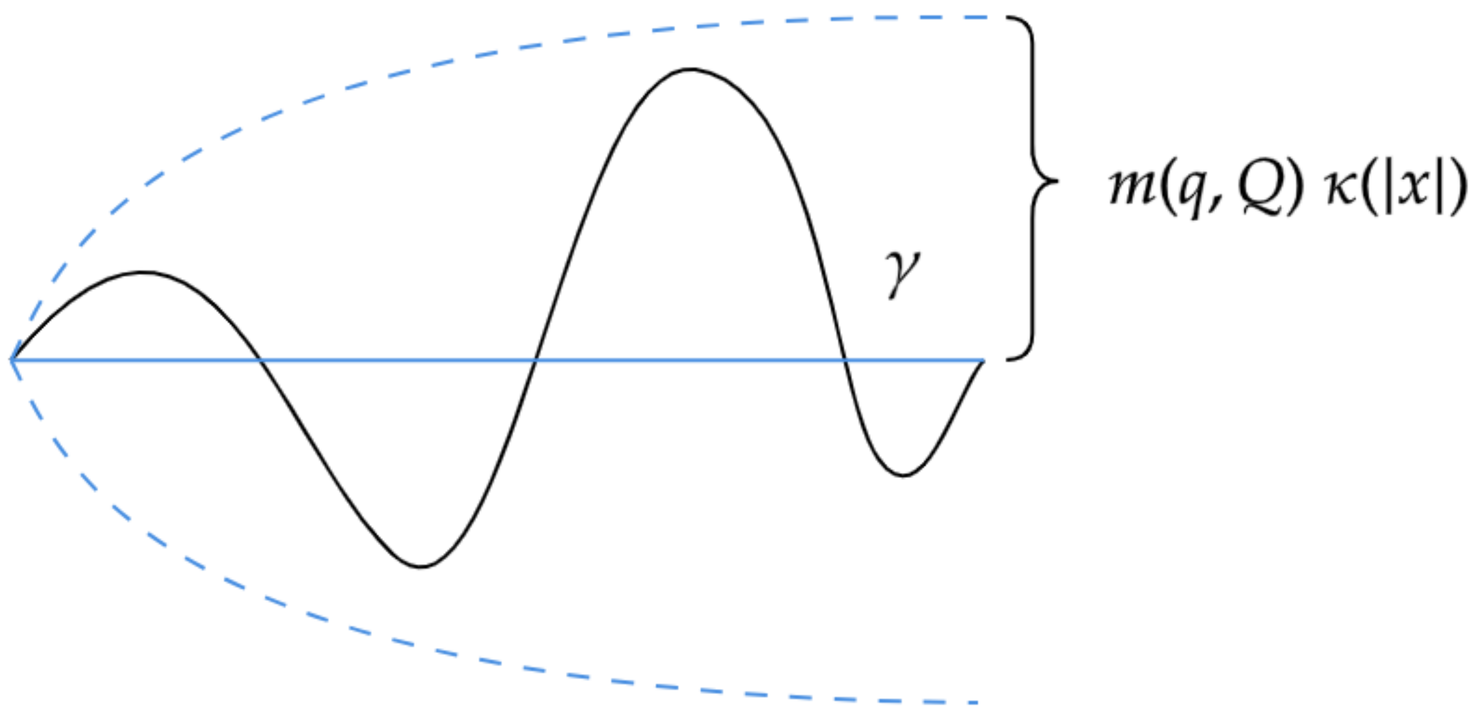


Image by Alex Sisto

The Sublinearly Morse Boundary

Definition (Qing-Rafi-Tiozzo)

A quasi-geodesic γ is sublinearly Morse if every quasi-geodesic β with endpoints on γ is contained in the $M\kappa(|x|)$–nbhd of γ, where M is a constant and κ is a sublinear function.



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Theorem (H.)

The sublinearly Morse boundary contains the Morse boundary as a topological subspace.

Summary: Boundaries

	Gromov Boundary	Morse Boundary	Sublinearly Morse Boundary
Compact	✓	✗	✗
Metrizable	✓	✓	✓
Invariant Under Quasi-Isometries	✓	✓	✓
Random Walk Converges	✓	✗	✓

Disk Configuration Spaces and Representation Stability

Nicholas Wawrykow
University of Michigan

Disk Configuration Spaces

Definition

For a manifold X with metric g, the ordered configuration space of n open unit-diameter disks in (X, g) is

$$\left\{(x_1, \ldots, x_n) \in (X, g)^n | d_g(x_i, x_j) \geq 1 \text{ and } \mathbb{D}_{\frac{1}{2}}(x_i) \subset (X, g)\right\}$$

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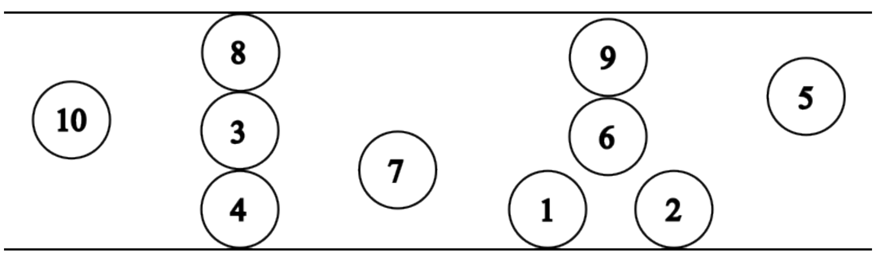
- Unlike the ordered configuration space of points $F_n(X)$, geometry matters!
- Easiest interesting example: $\mathrm{conf}(n,w)$ the ordered configuration space of n open unit-diameter disks in the infinite Euclidean strip of width w



A point in $\mathrm{conf}(10,3)$

Representation Stability

- The symmetric group S_n acts on $\mathrm{conf}(n, w) \Rightarrow H_k\big(\mathrm{conf}(n, w)\big)$ is an S_n-representation

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- First, what happens in the case of ordered configuration spaces of points?

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- First, what happens in the case of ordered configuration spaces of points?

Theorem (Church–Ellenberg–Farb 12, Miller–Wilson 19)

If X is a non-compact connected finite type manifold of dimension at least 2, then, for $n > 2k$, the decomposition of $H_k\big(F_n(X); \mathbb{Q}\big)$ into a direct sum of irreducible S_n-representations is determined by the decomposition of $H_k\big(F_m(X); \mathbb{Q}\big)$ into a direct sum of irreducible S_m-representations for every $m \leq 2k$.

- This is *representation stability*

Representation Stability for Conf(n,w)

Representation Stability for Conf(*n*,*w*)

Theorem (W. 22)

For $w \geq 2$ and $n > 3k$, upper bounds for the multiplicities of the irreducible S_n-representations in the direct sum decomposition of $H_k\big(\mathrm{conf}(n,w);\mathbb{Q}\big)$ are determined by the multiplicities of the irreducible S_m-representations in the direct sum decomposition of $H_k\big(\mathrm{conf}(m,w);\mathbb{Q}\big)$ for every $m \leq 3k$.

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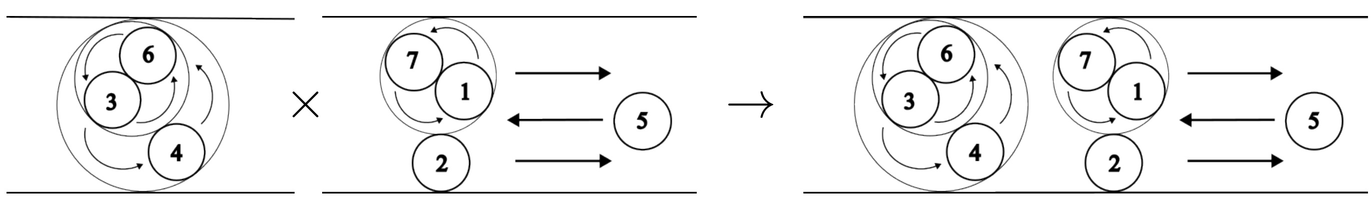
Proof sketch:

- $H_*\big(\mathrm{conf}(\bullet, w); \mathbb{Q}\big)$ is a *twisted algebra* (Alpert–Manin 21)



$H_2\big(\mathrm{conf}(3,3)\big)$ $\times$ $H_2\big(\mathrm{conf}(4,3)\big)$ $\to$ $H_4\big(\mathrm{conf}(7,3)\big)$

Representation Stability for Conf(n,w)

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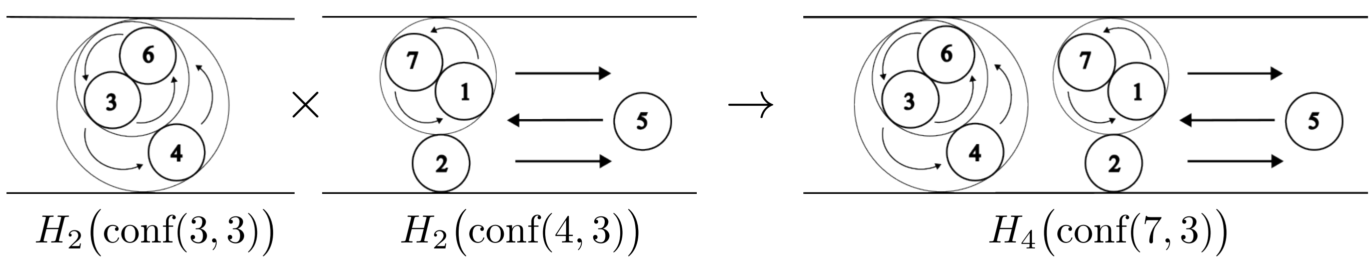


- Find a nice finite presentation

The Legendrian Unknot in a tight contact 3-manifold.

Eduardo Fernández (UGA)

Dec, 2022
Tech Topology

Joint work with J. Martínez-Aguinaga and Francisco Presas

Eliashberg-Fraser Theorem.

Theorem (Eliashberg-Fraser)

Two Legendrian unknots in a tight contact 3-manifold (M, ξ) are Legendrian isotopic iff they have the same tb *and* Rot*. Even more, every Legendrian unknot is obtained by the unique Legendrian unknot $L^{(0,-1)}$ with* $\mathrm{Rot} = 0$ *and* $\mathrm{tb} = -1$ *by a finite sequence of stabilizations.*

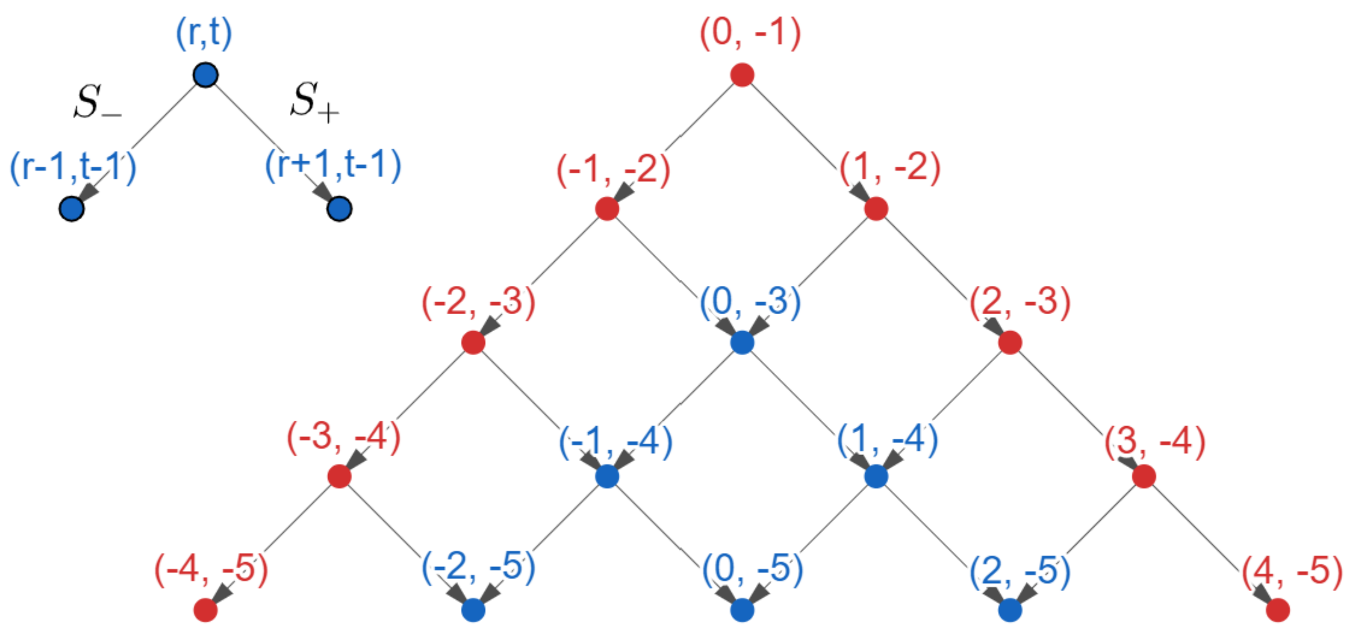


Figure: Eliashberg-Fraser Tartaglia Triangle.

The space of parametrized long Legendrian unknots.

Fix $(p, v) \in \mathbb{S}(\xi)$. Let $\mathfrak{Emb}_{(p,v)}(M)$ be the space of embedddings $\gamma : \mathbb{S}^1 \to M$ of long unknots into M; i.e. $(\gamma(0), \gamma'(0)) = (p, v)$. Let $\mathfrak{Leg}^{(r,t)}_{(p,v)}(M, \xi)$ be the subspace of $\mathfrak{Emb}_{(p,v)}(M)$ conformed by Legendrian unknots with $\mathrm{Rot} = r$ and $\mathrm{tb} = t$. Do note that both spaces are connected.

Theorem (F,Martínez-Aguinaga,Presas. 20/21)

If (M, ξ) is tight and $(|r|, t) = (-1 - t, t)$ then the natural inclusion

$$\mathfrak{Leg}^{(r,t)}_{(p,v)}(M, \xi) \hookrightarrow \mathfrak{Emb}_{(p,v)}(M)$$

is a homotopy equivalence.

Corollary

The space of parametrized Legendrian unknots with $\mathrm{tb} = -1$ in $(\mathbb{S}^3, \xi_{\mathrm{std}})$ is homotopy equivalent to the space of parametrized Legendrian great circles $\mathrm{U}(2)$.

The space of smooth parametrized unknots in $\mathbb{S}^3$ is homotopy equivalent to the space of parametrized great circles $V_{4,2}$ (Hatcher, Smale Conjecture).

About the proof.

Let $\mathfrak{Emb}(\mathbb{D}^2, M)$ be the space of smooth embeddings of disks that are fixed near the boundary bounding a Legendrian unknot with $(\mathrm{Rot}, \mathrm{tb}) = (r, t)$ and $\mathfrak{Emb}^{(r,t)}_{\mathrm{std}}(\mathbb{D}^2, (M, \xi))$ be the subspace of convex disks with fixed characteristic foliation (pick your favourite one).
There is a commutative diagram

$$\begin{tikzcd}
\mathfrak{Emb}^{(r,t)}_{\mathrm{std}}(\mathbb{D}^2, (M,\xi)) \arrow[r] \arrow[d, hook] & \Omega\mathfrak{Leg}^{(r,t)}_{(p,v)}(M,\xi) \arrow[d, hook] \\
\mathfrak{Emb}(\mathbb{D}^2, M) \arrow[r] & \Omega\mathfrak{Emb}_{(p,v)}(M)
\end{tikzcd}$$

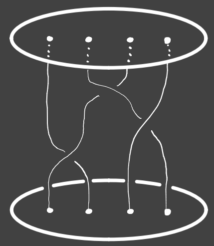
in which the horizontal arrows are h.e.

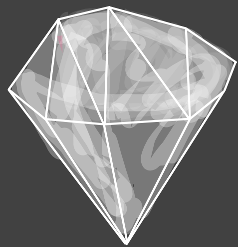
About the proof.

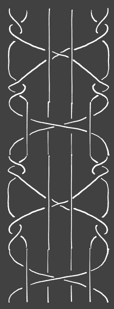
We are reduced to check that $\mathfrak{Emb}^{(r,t)}_{\mathrm{std}}(\mathbb{D}^2,(M,\xi)) \hookrightarrow \mathfrak{Emb}(\mathbb{D}^2, M)$ is a h.e.

- The condition on (r,t) that we have imposed imply that the inclusion is **dense**.
- **Locally** (in a tubular neighbourhood of a smooth disk) the problem is solved as a consequence of Eliashberg-Mishachev and Hatcher works.
- To globalize we build a **microfibration** with fiber the space of isotopies joining a smooth disk with a convex one in a tubular neighbourhood of the smooth disk. The fiber is $\neq \emptyset$ because of the density property and contractible. Therefore, we have a fibration with contractible fiber. $\square$

Thanks for listening!







The Burau Representation and "Shapes of Polyhedra"

Ethan Dlugie

UC Berkeley

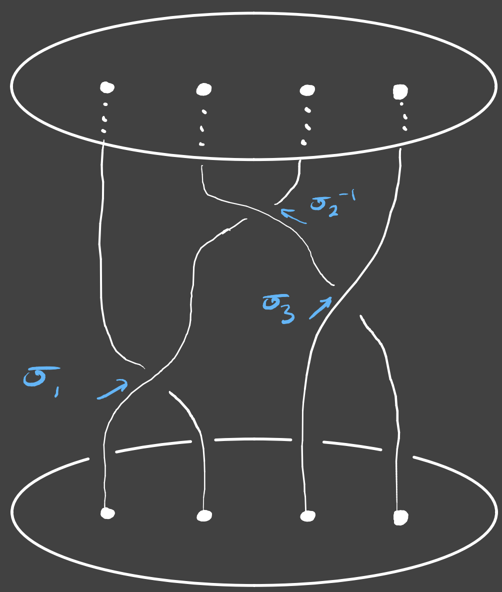
@

Tech Topology

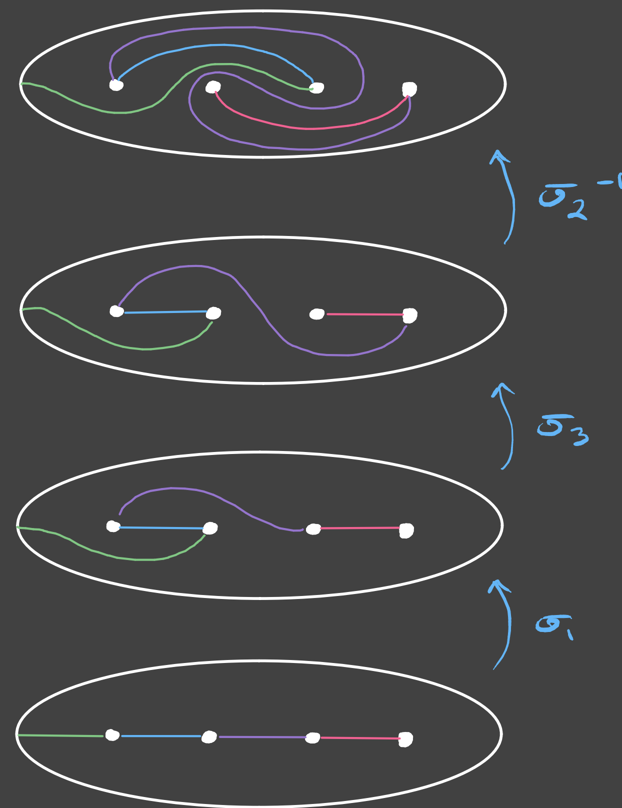
December 11, 2022

Braid groups : two perspectives

Braid diagrams



Mapping class groups



Theorem $B_n \approx MCG(D_n)$

The Burau representation

It's a representation of braid groups

$$\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm}])$$

via

$$\beta_n(\sigma_i) = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

The Burau representation $\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm}])$

Question: Is it faithful?

n	2	3	4	5	6	...
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Question: Is it faithful?

n	2	3	4	5	6	...
faithful?	yes	yes		no	no	...

Bigelow '99

The Burau representation $\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm}])$

Question: Is it faithful?

n	2	3	4	5	6	...
faithful?	yes	yes	??	no	no	...

Open question

The Burau representation $\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm}])$

Open Question: Is β_4 faithful?

Why care?

The Burau representation $\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm}])$

Open Question: Is β_4 faithful?

Why care?

Thm (Ito '14)
If $\ker \beta_4 \neq 0$, then Jones polynomial does not detect the unknot.

Main Theorem

Theorem* (Dlugie)

$\ker \beta_4 \leq \langle\langle \sigma_i{}^d \rangle\rangle$ for $d = 5, 6, 8$

and

$\ker \beta_4 \leq \langle\langle \Delta^2, \sigma_i{}^d \rangle\rangle$ for $d = 7, 10, 12, 18$

where Δ^2 = full twist

* Thanks to Nancy Scherich for helpful conversations!

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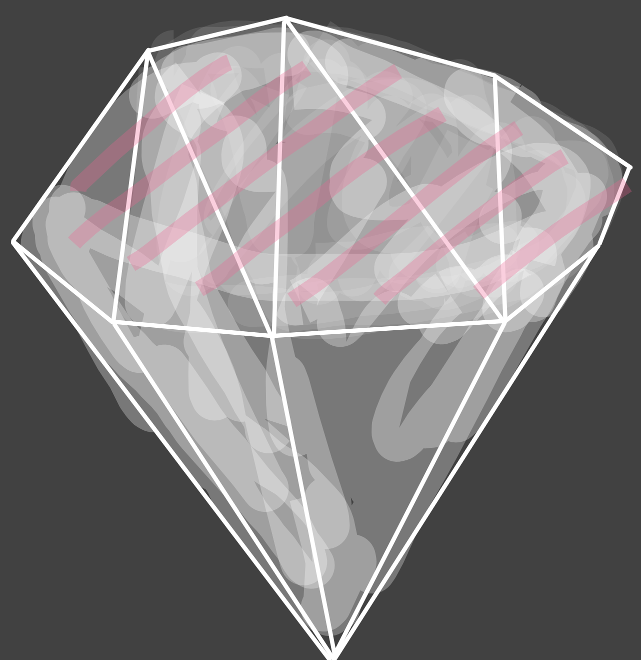
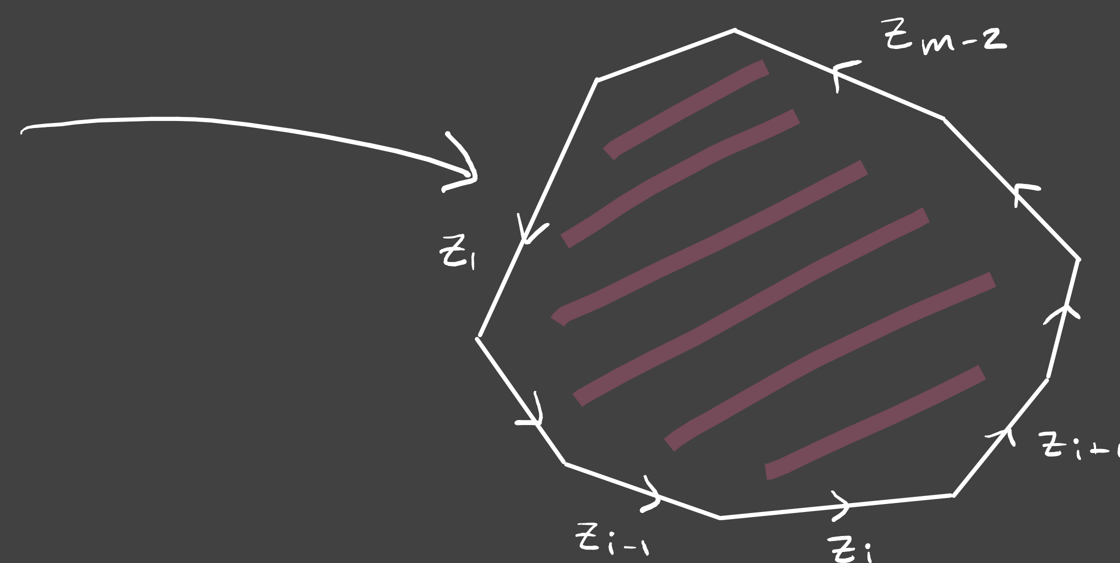
Remark: these normal subgroups are all infinite index in B_4.

Main Theorem

Proof method uses Thurston's moduli space of flat cone spheres





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Example $\ker \beta_6 \leq \langle\langle \sigma_i^4 \rangle\rangle$

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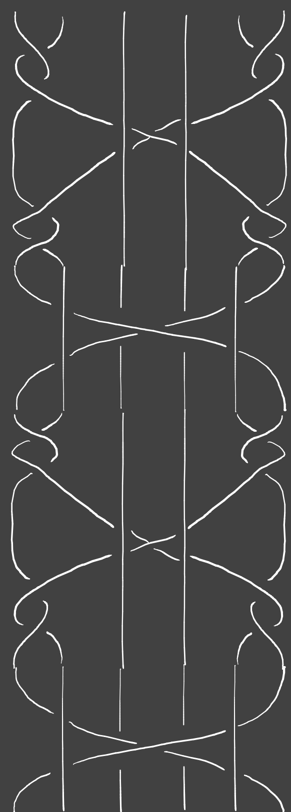

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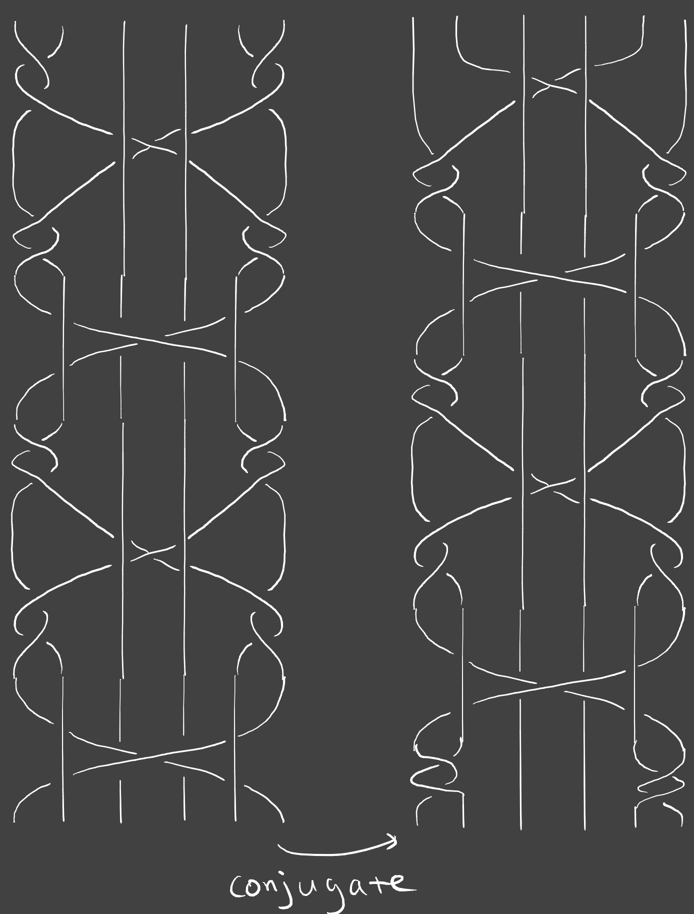

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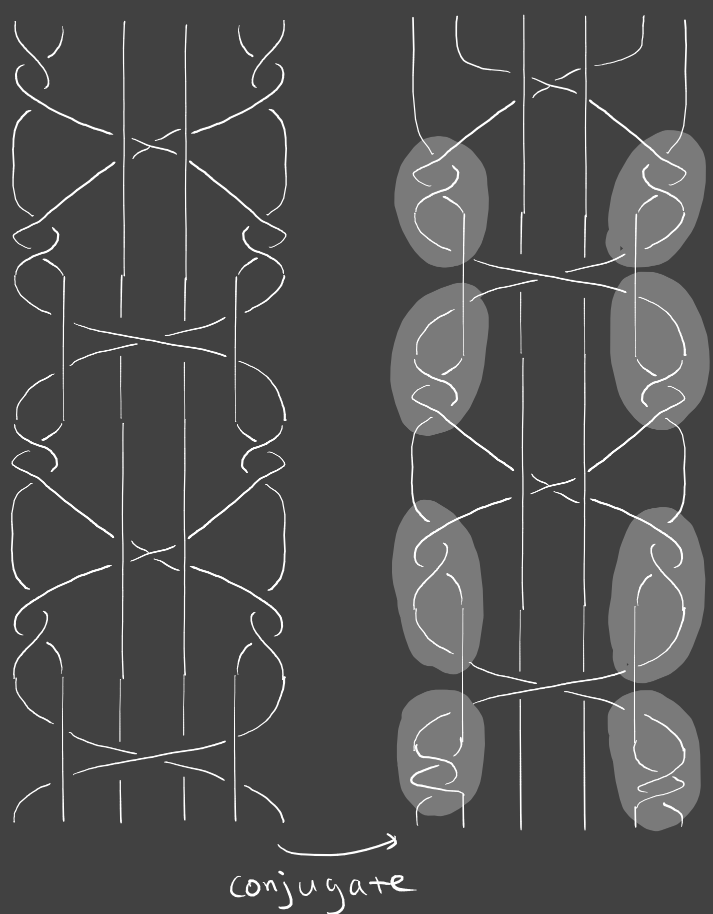

Main Theorem

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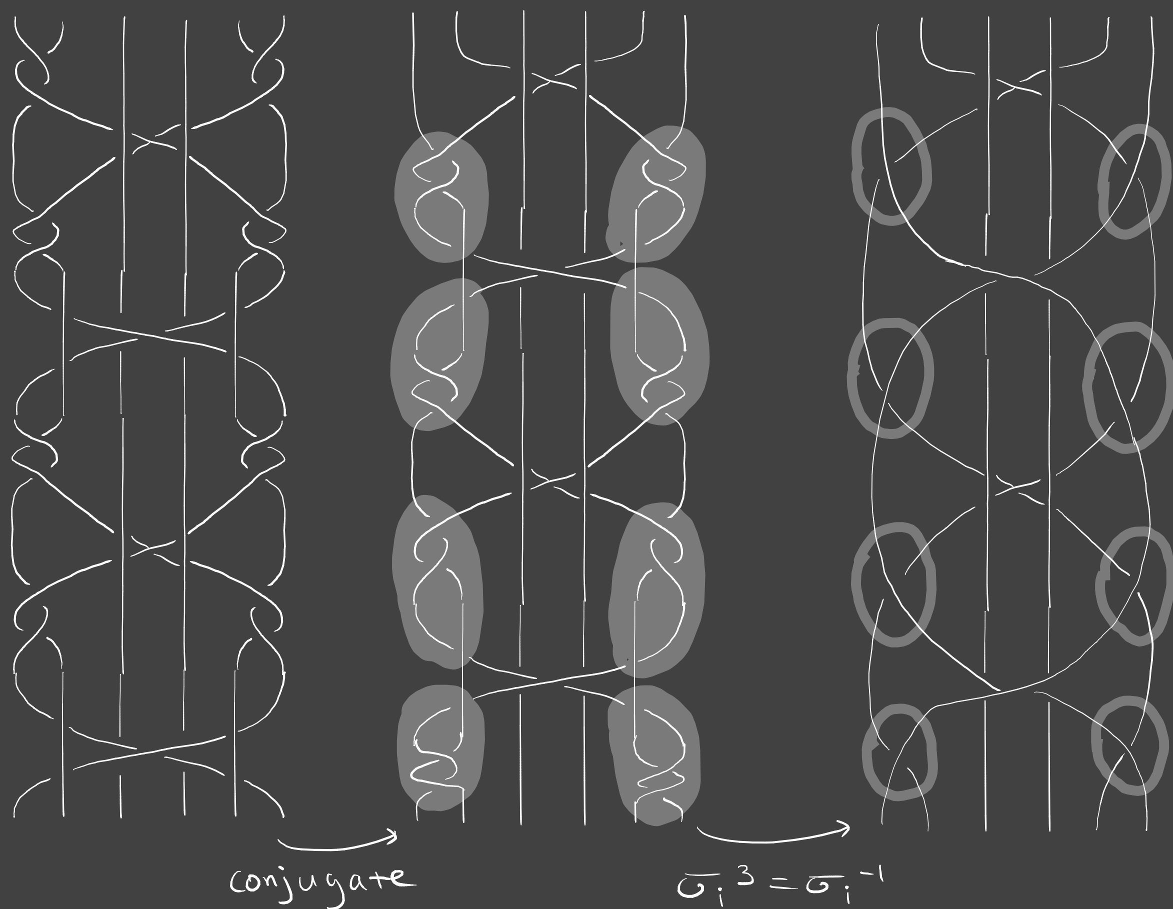

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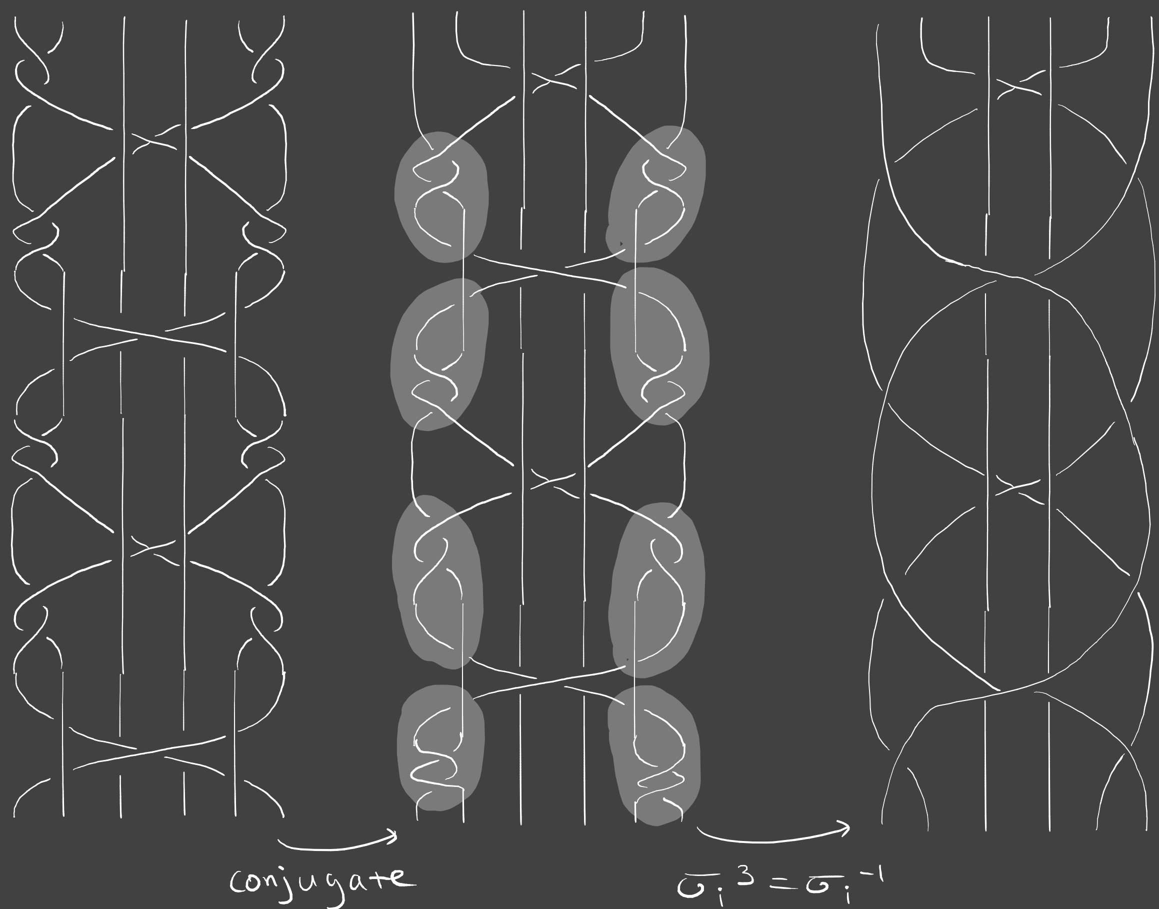

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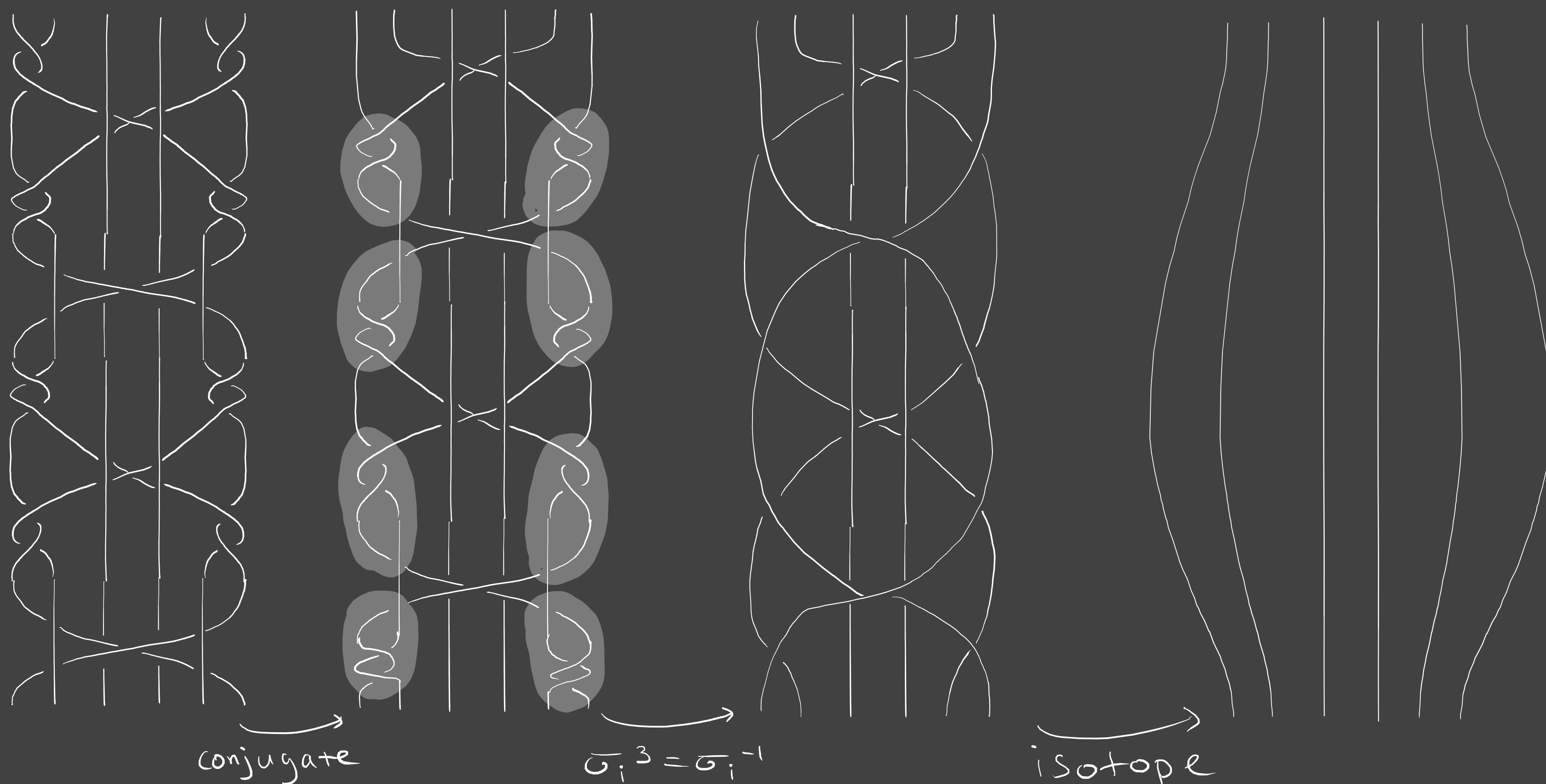

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Thanks!!!

Milnor's invariants for knots and links in closed orientable 3-manifolds

Ryan Stees (Indiana U.)

2022 Tech Topology Conference

Milnor's invariants for knots and links in closed orientable 3-manifolds

Milnor's invariants for knots and links in closed orientable 3-manifolds

<u>Milnor ('57):</u>

Milnor's invariants for knots and links

in closed orientable 3-manifolds

Milnor ('57):

$L \subset S^3 \rightsquigarrow \bar{\mu}(\alpha) \in \mathbb{Z}$

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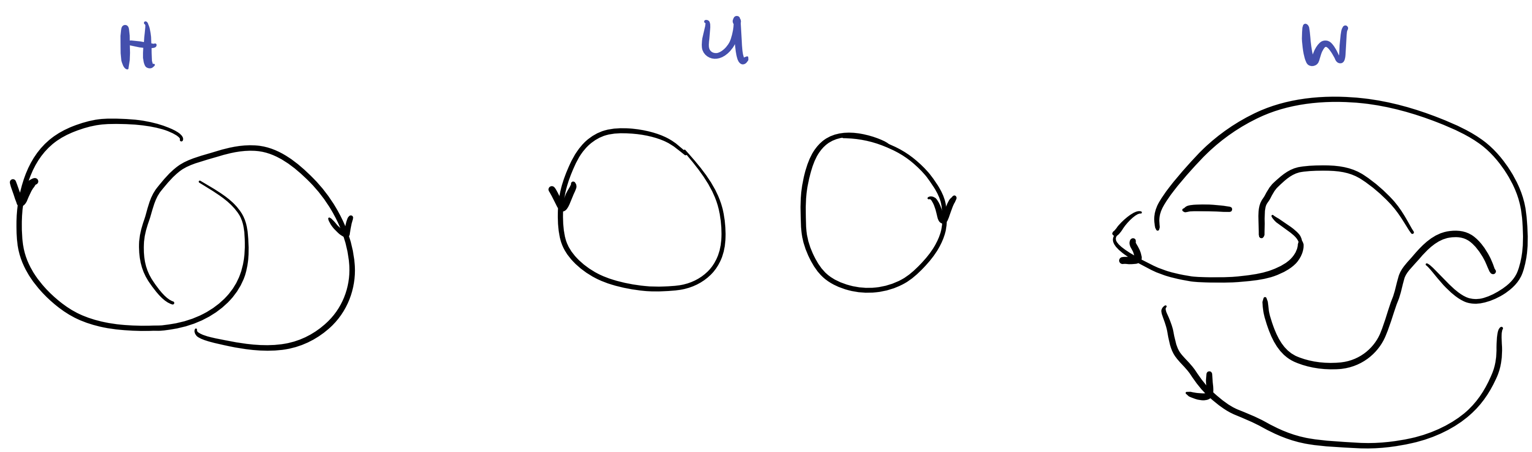

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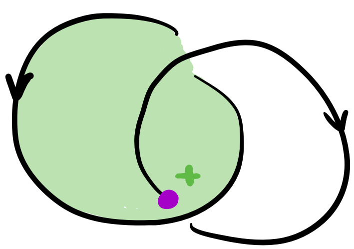
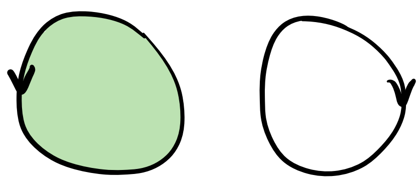
$\mathcal{H}$



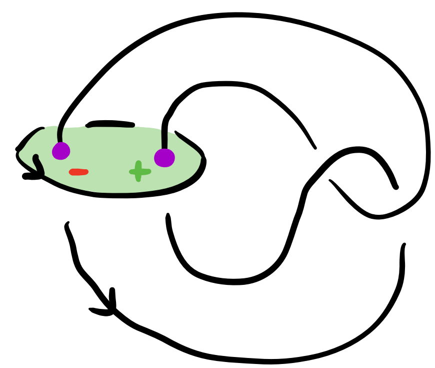
$lk(H_1, H_2) = 1$

$\mathcal{U}$



$lk(U_1, U_2) = 0$

$\mathcal{W}$



$lk(W_1, W_2) = 0$

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Milnor's invariants for knots and links in closed orientable 3-manifolds

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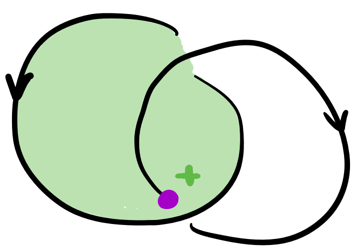
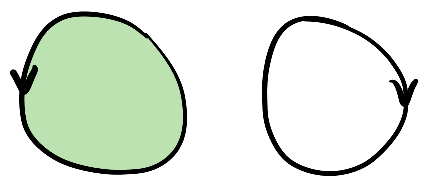


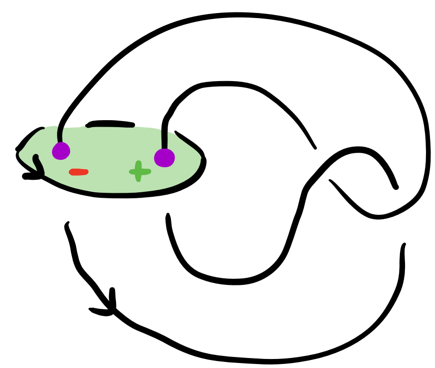
$\bar{\mu}(12) = 1$

U



$\bar{\mu}(12) = 0$

W



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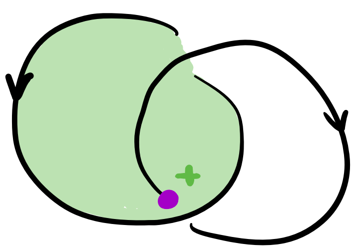
H



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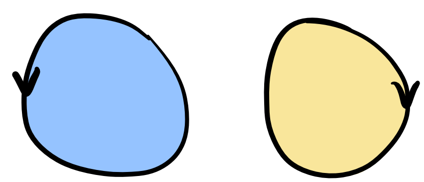
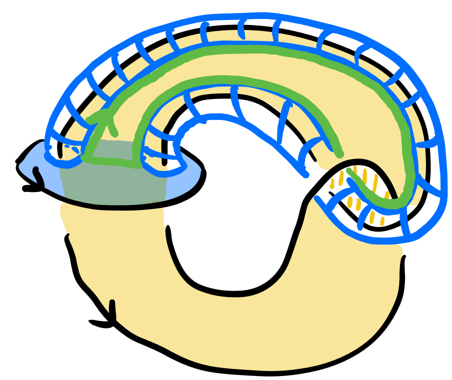
U



$\bar{\mu}(12) = 0$

$\bar{\mu}(1122) = 0$

W



$\bar{\mu}(12) = 0$

$\bar{\mu}(1122) = -1$

Milnor ('57):

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"Higher-order linking numbers"

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Inductively compare $\pi_1(E_L)/\pi_1(E_L)_n$ and $\pi_1(E_U)/\pi_1(E_U)_n$.

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THM. (Milnor, '57) Let $L \subset S^3$, and suppose there is an isomorphism $\phi : \pi_1(E_L)/\pi_1(E_L)_n \xrightarrow{\cong} \pi_1(E_U)/\pi_1(E_U)_n$. Then the following three statements are equivalent:

1. $\bar{\mu}(\text{length } n) = 0$.
2. $\pi_1(E_L)/\pi_1(E_L)_{n+1} \cong \pi_1(E_U)/\pi_1(E_U)_{n+1}$
3. $\bar{\mu}(\text{length } n+1)$ are well-defined.

Milnor's invariants for knots and links in closed orientable 3-manifolds

Milnor's invariants for knots and links in closed orientable 3-manifolds

D. Miller ('95)
Heck ('11)
Kuzbary ('19)
Cha–Orr ('20)

Milnor's invariants for knots and links in closed orientable 3-manifolds

THM. (Milnor, '57) Let $L \subset S^3$, and suppose there is an isomorphism

$$\phi : \pi_1(E_L)/\pi_1(E_L)_n \xrightarrow{\cong} \pi_1(E_U)/\pi_1(E_U)_n .$$

Then TFAE:

1. $\bar{\mu}$(length n) $= 0$.
2. $\pi_1(E_L)/\pi_1(E_L)_{n+1} \cong \pi_1(E_U)/\pi_1(E_U)_{n+1}$
3. $\bar{\mu}$(length $n+1$) are well-defined.

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① $\bar{\mu}(\text{length } n) = 0$.

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③ $\bar{\mu}(\text{length } n+1)$ are well-defined.

THM. (S., '22) Fix $L \subset M$. Let $L' \subset M$, and suppose there is an isomorphism $\phi : \pi_1(E_{L'})/\Gamma(L')_n \xrightarrow{\cong} \pi_1(E_L)/\Gamma(L)_n$. Then there exist invariants $\bar{\mu}_n$ such that TFAE:

① $\bar{\mu}_n(L') = \bar{\mu}_n(L)$.

② $\pi_1(E_{L'})/\Gamma(L')_{n+1} \xrightarrow{\cong} \pi_1(E_L)/\Gamma(L)_{n+1}$

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Features:

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Features:

- Recovers Milnor's classical $\bar{\mu}$-invariants for $L \subset S^3$ (Orr '89)

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- Can be nontrivial for knots in $M \neq S^3$

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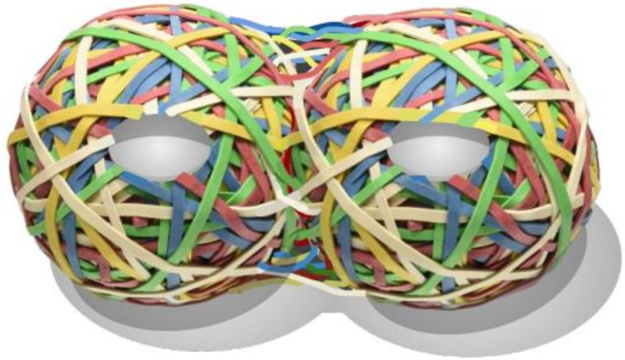
Features:

- Recovers Milnor's classical $\bar{\mu}$-invariants for $L \subset S^3$ (Orr '89)
- Can be nontrivial for knots in $M \neq S^3$
- Can be defined for empty links

$\rightsquigarrow$ Invariant of H_*-cob. of 3-mfld. (Cha-Orr '20)

Thank you!

Automorphisms of the fine 1-curve graph



Roberta Shapiro

Joint with K. W. Booth & D. Minahan

Tech Topology Conference 2022

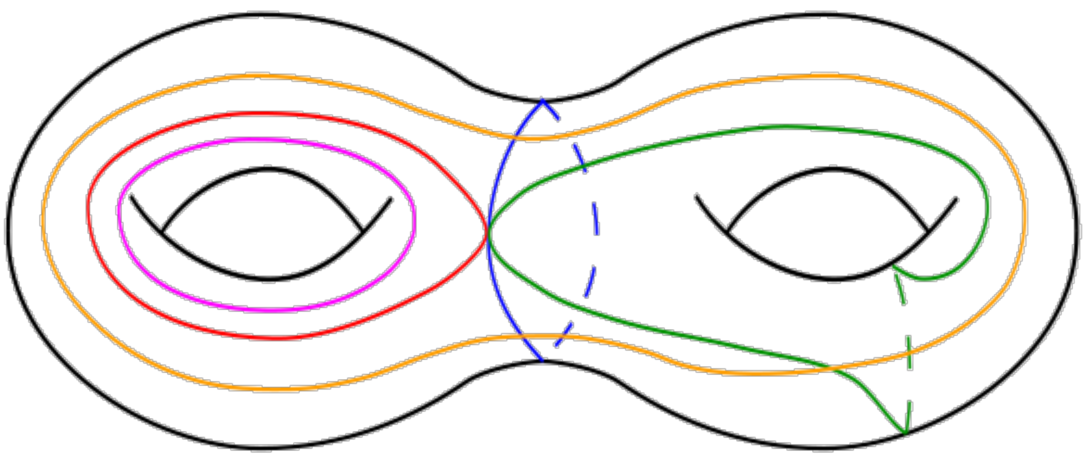
Goal:

Surface $\Leftrightarrow$ graph

Main Theorem:

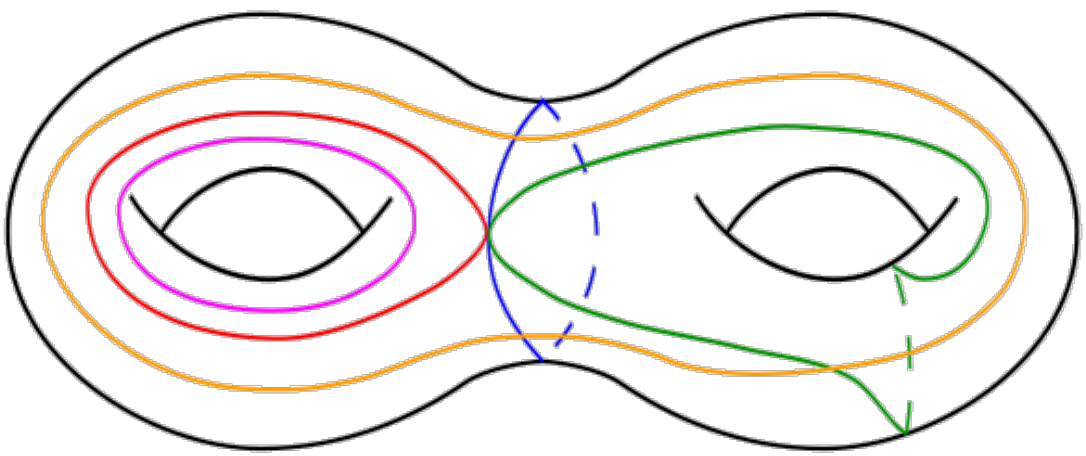
Homeo(Surface) ⇔ Aut(graph)

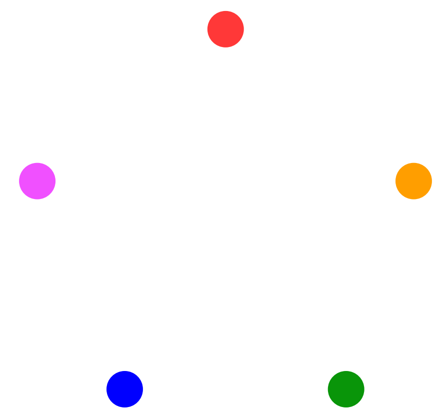
Fine curve graph: $\mathcal{C}^\dagger(S)$



Fine curve graph: $\mathcal{C}^\dagger(S)$

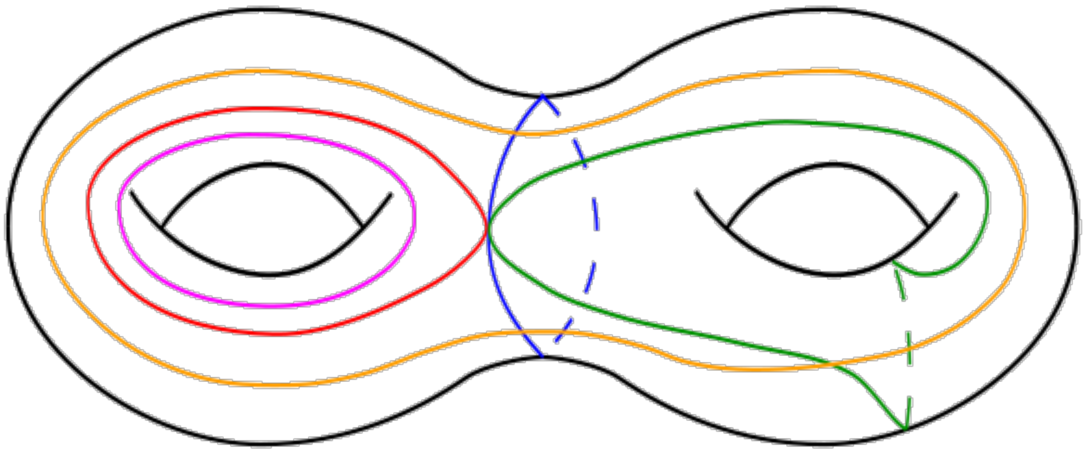
Vertices: curves

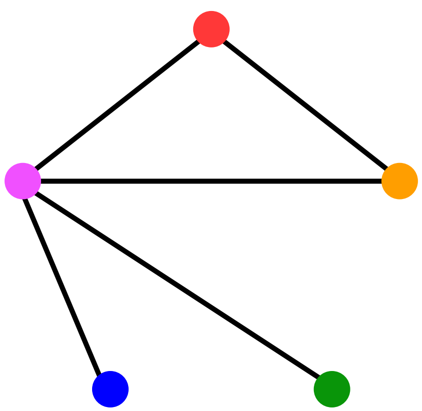




Fine curve graph: $\mathcal{C}^\dagger(S)$

Vertices: curves
Edges: disjointness



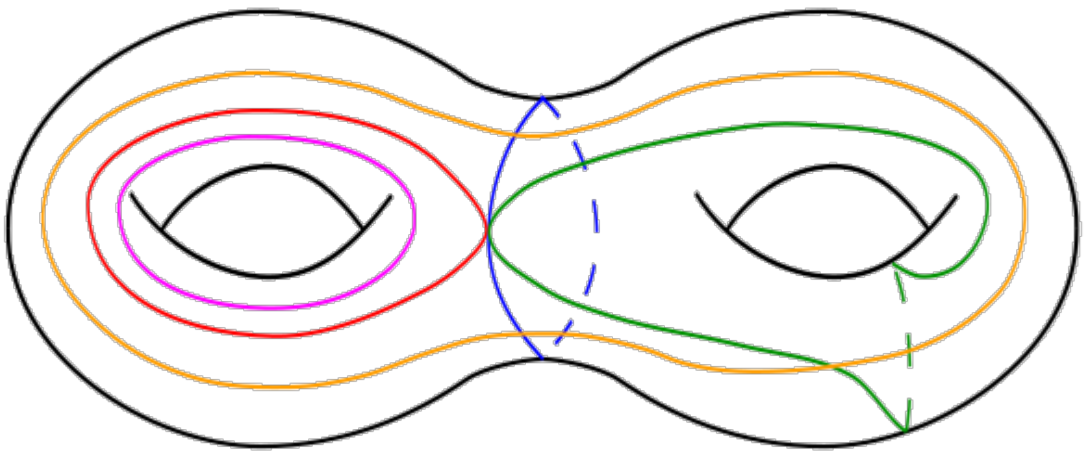


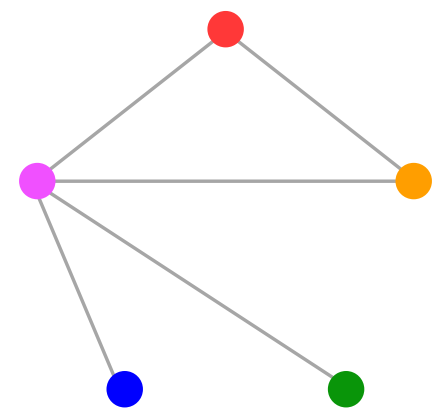
(Bowden-Hensel-Webb)

Fine 1-curve graph: $\mathcal{C}_1^{\dagger}(S)$

Vertices: curves

Edges: disjointness or intersect once





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Vertices: curves

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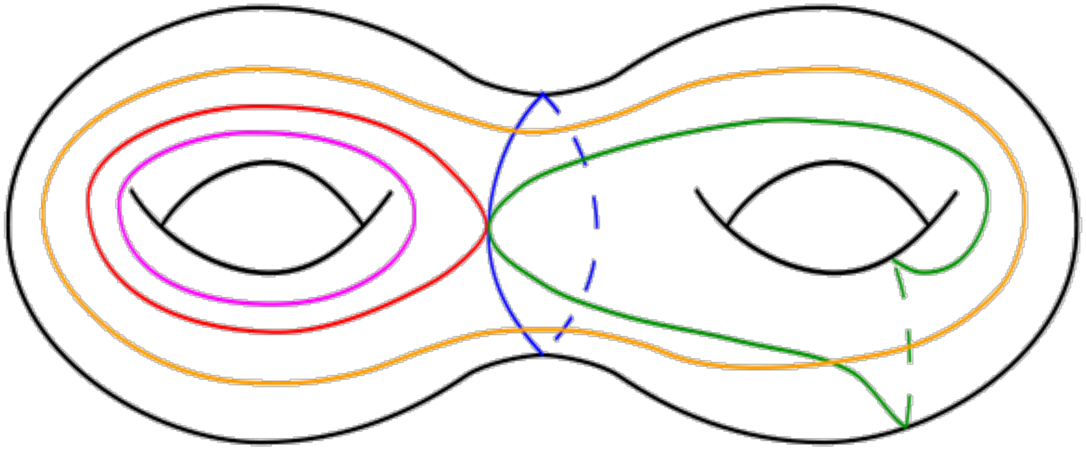

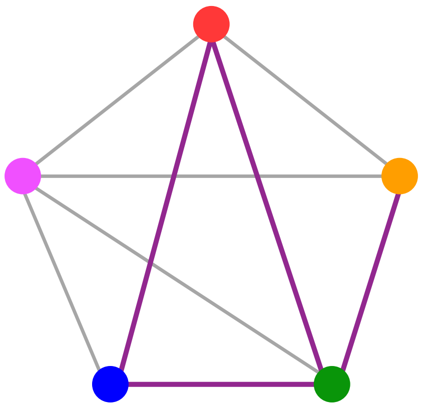

Natural map:

$$\mathbf{Homeo}(\boldsymbol{S}) \longrightarrow \mathbf{Aut}\left(\mathcal{C}_{\mathbf{1}}^{\dagger}(\boldsymbol{S})\right)$$

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$$\mathbf{Homeo}(\boldsymbol{S}) \xleftarrow{?} \mathbf{Aut}\left(\mathcal{C}_1^{\dagger}(\boldsymbol{S})\right)$$

Main Theorem (Booth-Minahan-S.)

S = closed, oriented surface, genus ≥ 1.
Then, the natural map

$$\mathbf{Homeo}(\boldsymbol{S}) \xrightarrow{\cong} \mathbf{Aut}\left(\mathcal{C}_{\mathbf{1}}^{\dagger}(\boldsymbol{S})\right)$$

is an isomorphism.

Similar theorem: Le Roux-Wolff

Proof outline, $g > 1$

Long-Margalit-Pham-
Verberne-Yao

$$\mathrm{Aut}\left(\mathcal{C}^{\dagger}(S)\right) \xrightarrow{\cong} \mathrm{Homeo}(S)$$

Proof outline, $g > 1$

Long-Margalit-Pham-Verberne-Yao

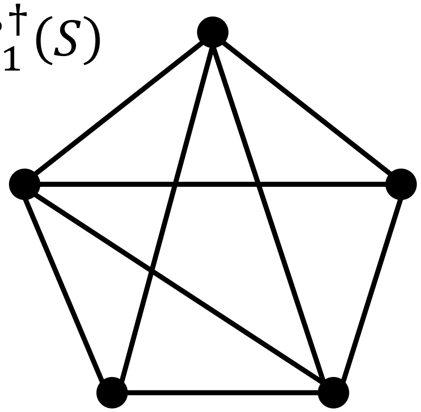
$$\mathrm{Aut}\left(\mathcal{C}_1^\dagger(S)\right) \longrightarrow \mathrm{Aut}\left(\mathcal{C}^\dagger(S)\right) \xrightarrow{\cong} \mathrm{Homeo}(S)$$

Proof outline, $g > 1$

Long-Margalit-Pham-Verberne-Yao

$$\mathrm{Aut}\left(\mathcal{C}_1^\dagger(S)\right) \longrightarrow \mathrm{Aut}\left(\mathcal{C}^\dagger(S)\right) \xrightarrow{\cong} \mathrm{Homeo}(S)$$

$\mathcal{C}_1^\dagger(S)$



Proof outline, $g > 1$

Long-Margalit-Pham-Verberne-Yao

$$\mathrm{Aut}\left(\mathcal{C}_1^\dagger(S)\right) \longrightarrow \mathrm{Aut}\left(\mathcal{C}^\dagger(S)\right) \xrightarrow{\cong} \mathrm{Homeo}(S)$$

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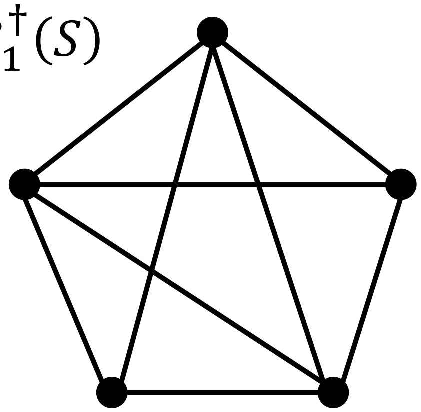
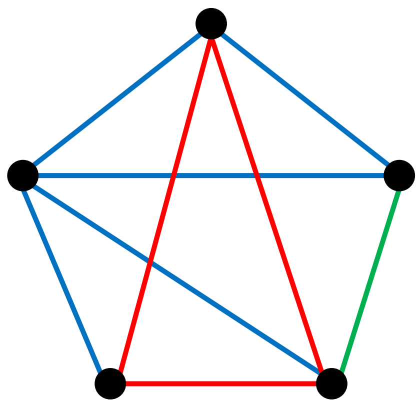


Edges:
Disjoint
Touching
Crossing



Proof outline, $g > 1$

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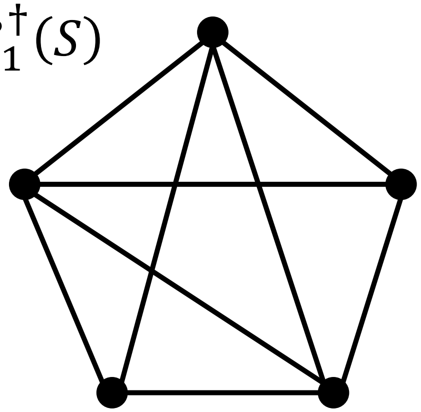

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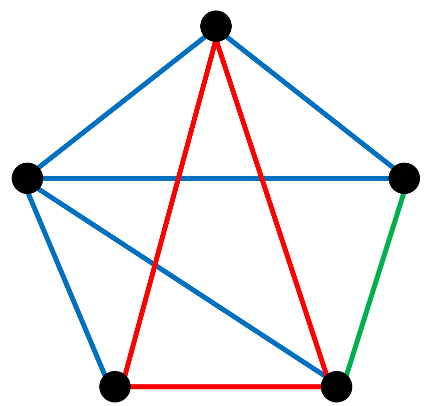
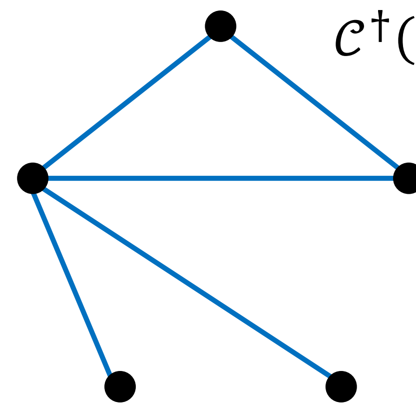
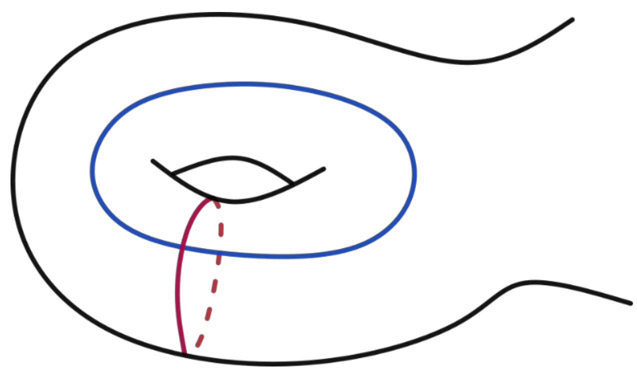




$\mathcal{C}^\dagger(S)$

Proposition: automorphisms preserve <u>crossing curves</u>

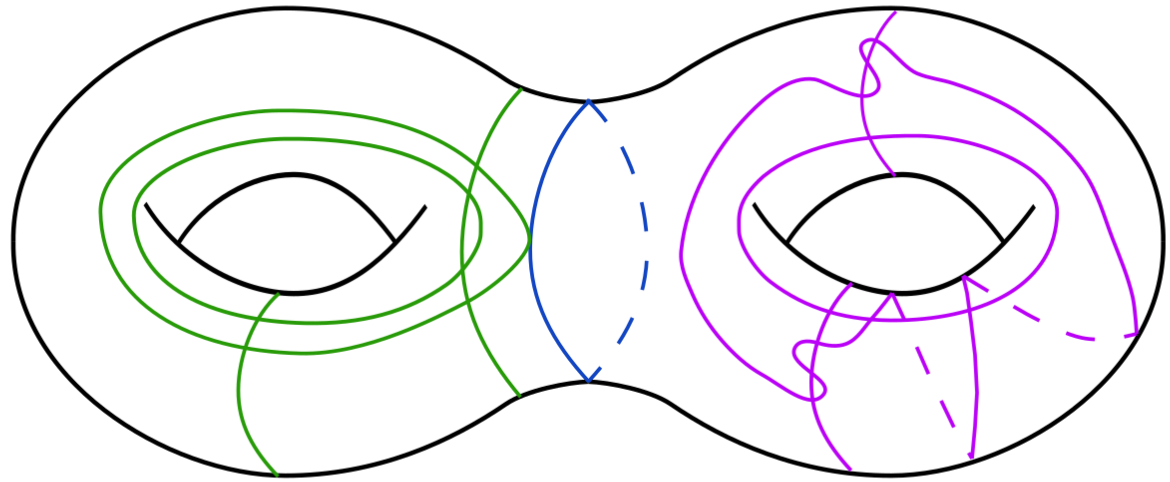


Main idea: look at graph structures surrounding the pair of curves

Proposition: automorphisms preserve crossing curves

Steps: distinguish…

1. **separating curves**



Proposition: automorphisms preserve crossing curves

Steps: distinguish…

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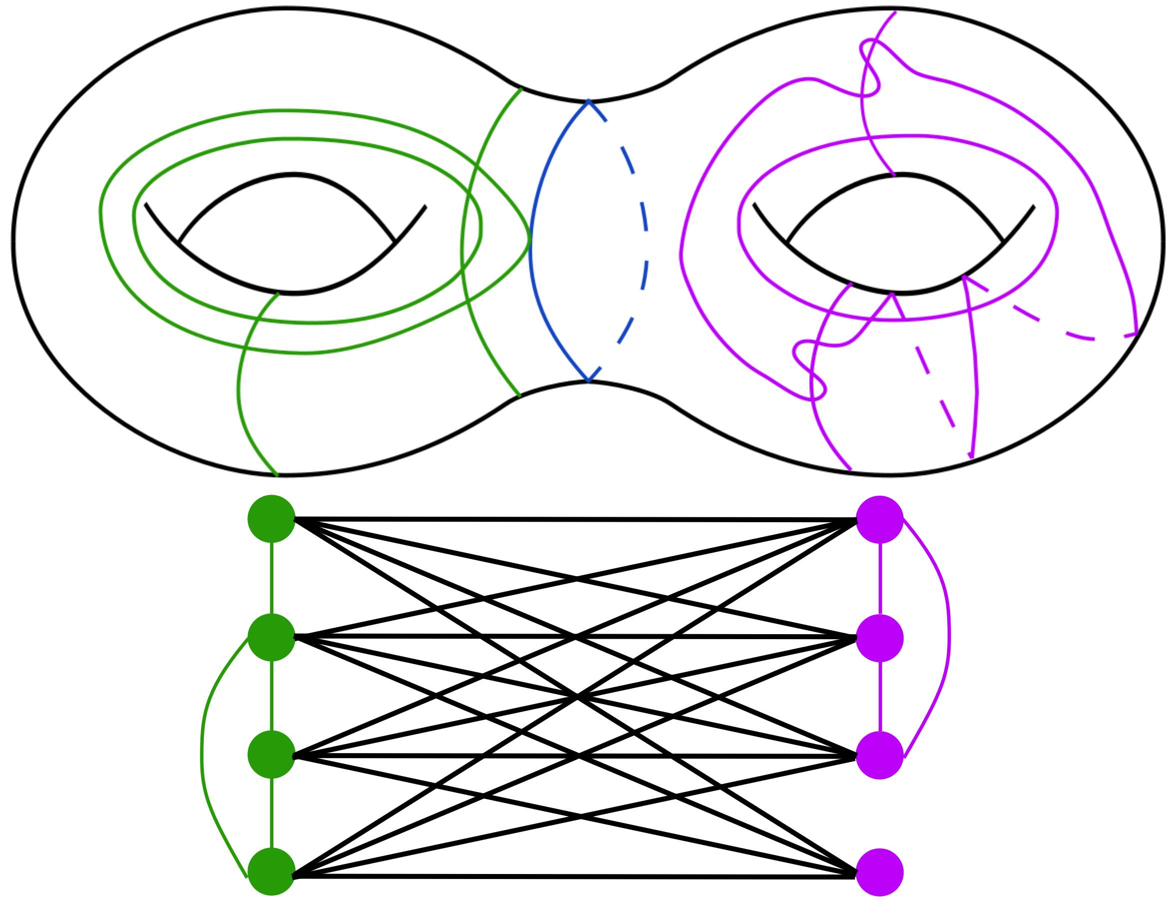
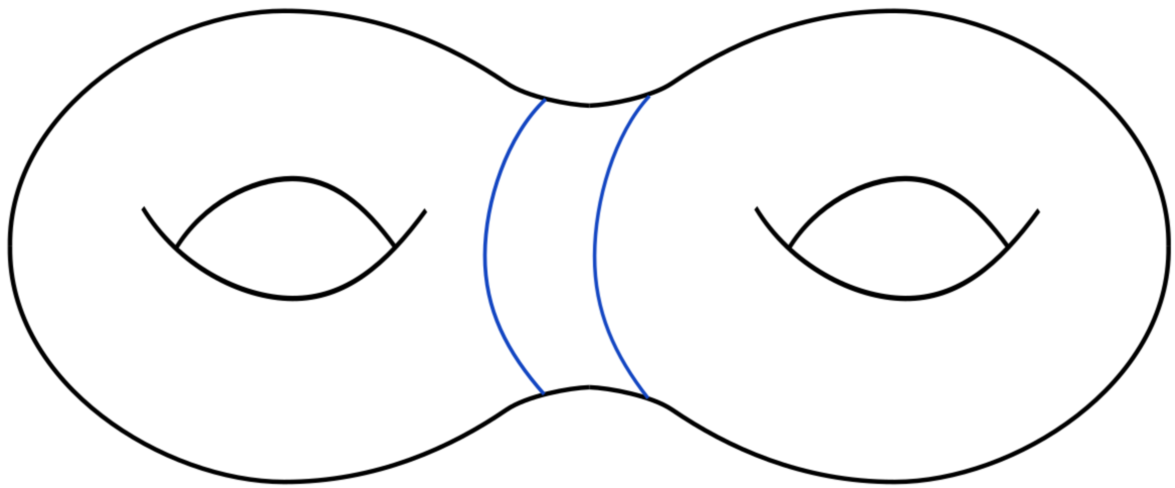


Proposition: automorphisms preserve crossing curves

Steps: distinguish…

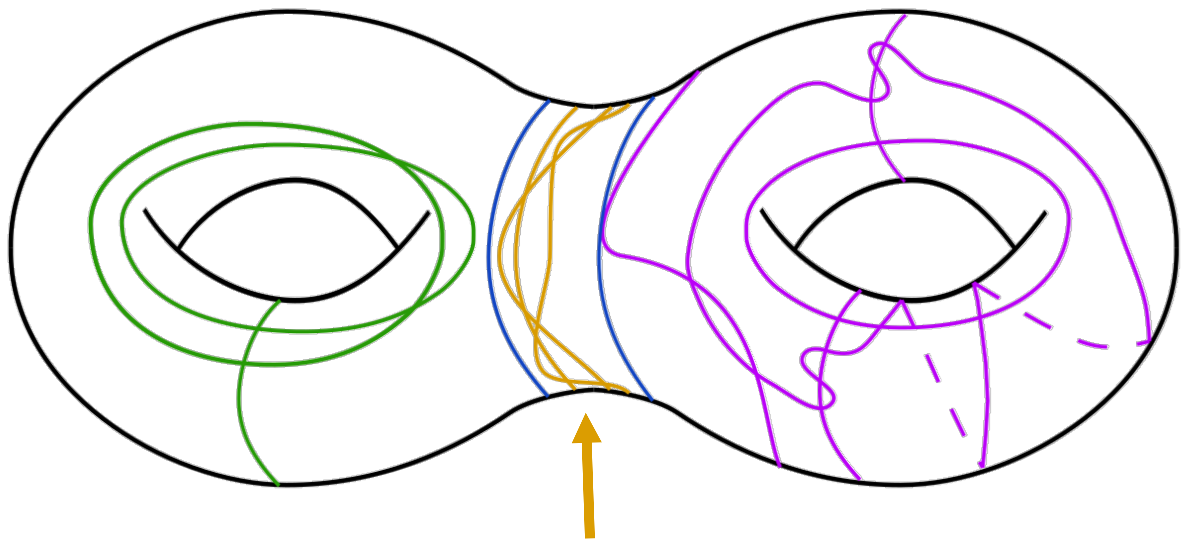
1. separating curves
2. **isotopy classes of separating curves**



Proposition: automorphisms preserve crossing curves

Steps: distinguish…

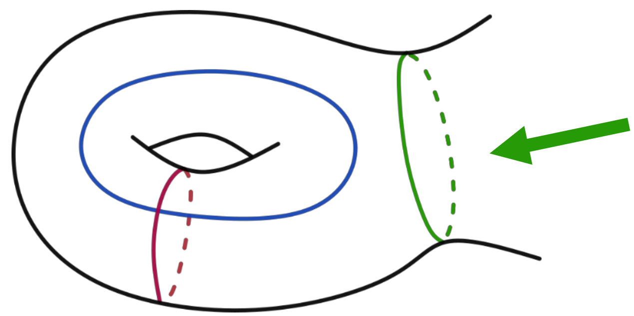
1. separating curves
2. **isotopy classes of separating curves**

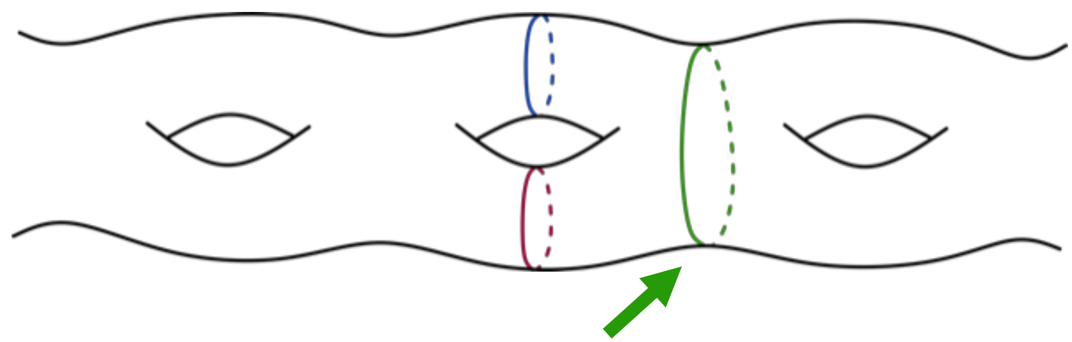


Proposition: automorphisms preserve crossing curves

Steps: distinguish…

1. separating curves
2. isotopy classes of separating curves
3. **crossing curves**

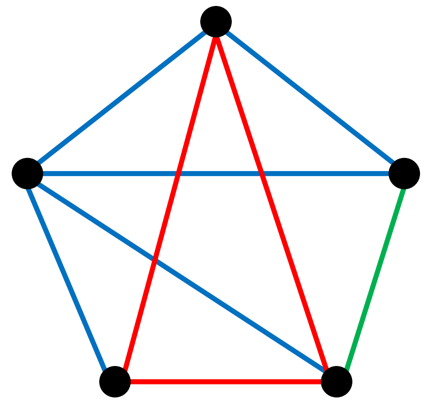




Proof outline, $g > 1$

$$\mathrm{Aut}\left(\mathcal{C}_1^\dagger(S)\right) \longrightarrow \mathrm{Aut}\left(\mathcal{C}^\dagger(S)\right) \xrightarrow{\cong} \mathrm{Homeo}(S)$$

<u>Edges</u>:
Disjoint
Touching
✓Crossing



Thank you!