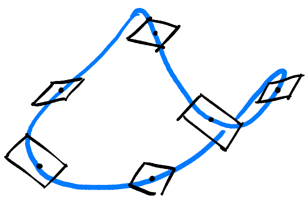
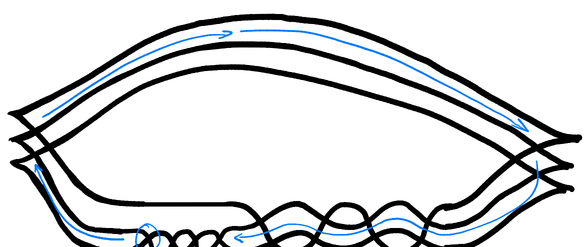
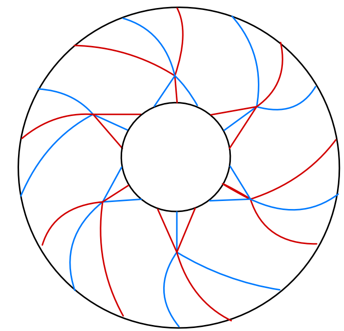
A Nielsen-Thurston classification of Legendrian Loops

James Hughes (Duke)

@Tech Topology 2023



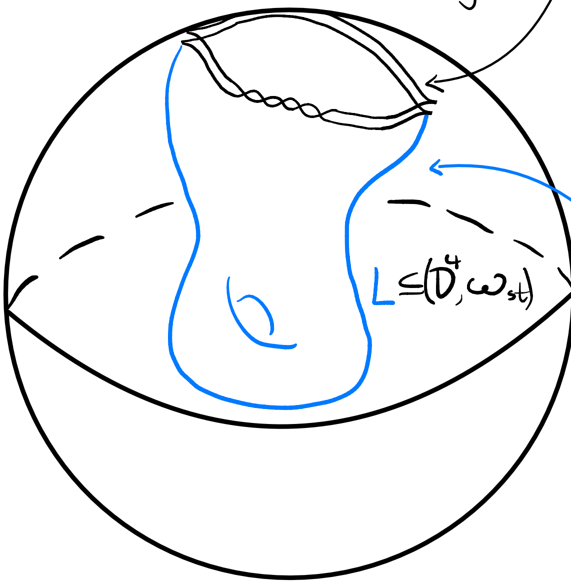




Legendrian Links + Lagrangian Fillings

Legendrian link $\Lambda \subseteq \partial(D^4, \omega_{st}) \cong (S^3, \xi_{st})$

$$T_x \Lambda \subseteq \xi_{st} := \ker(\underbrace{dz - y\,dx}_{\alpha_{st}})$$

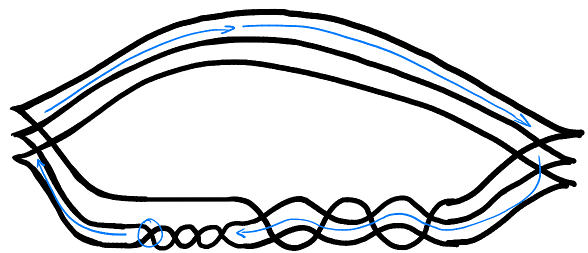


Exact Lagrangian filling L:

- $\partial L = \Lambda$
- $\omega_{st}|_L \equiv 0$
- $\alpha_{st}|_L$ is exact

Legendrian Loops

Def: A <u>Legendrian Loop</u> of Λ is a Legendrian isotopy fixing Λ pointwise at time 1.



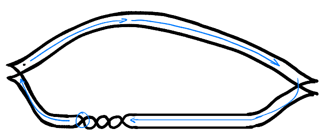
Legendrian loops act on the set of exact Lagrangian fillings by concatenation.

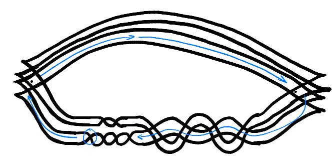
Contact geometry		(Cluster) algebraic invariants
• Legendrian link Λ (braid positive)	$\longrightarrow$	Algebraic variety $M_1(\Lambda)$
• Exact Lagrangian filling L of Λ	$\longrightarrow$	Toric chart $(\mathbb{C}^\times)^{b_1(L)} \subseteq M_1(\Lambda)$
• Legendrian loop	$\longrightarrow$	(cluster) automorphism of $M_1(\Lambda)$

Nielsen Thurston Classification

Def: A Legendrian Loop φ of Λ is:

- periodic if $\tilde{\varphi}^n = id$ for some $n \in \mathbb{N}$.
- reducible if $\tilde{\varphi}$ fixes some set of cluster coordinates in $\mathcal{M}_1(\Lambda)$
- pseudo-Anosov if $\tilde{\varphi}^n$ is neither periodic nor reducible.





Fixed Points

Thm (H. 23): The induced action of any periodic Legendrian loop has a fixed point in $\mathcal{M}_1(\Lambda)_{>0}$.

Ex: $\Lambda((21)^9) \cong \Lambda(3,6)$

$$a_1 = \frac{a_2 + a_3 + a_1 a_4}{a_2}$$

$$a_2 = a_4$$

$$a_3 = \frac{a_2 + a_3}{a_1}$$

$$a_4 = \frac{a_3 a_6 + a_4 a_7}{a_5}$$

$$a_5 = \frac{a_1 a_3 a_6 + (a_1 a_4 + (a_2 + a_3) a_5) a_7}{a_1 a_3 a_5}$$

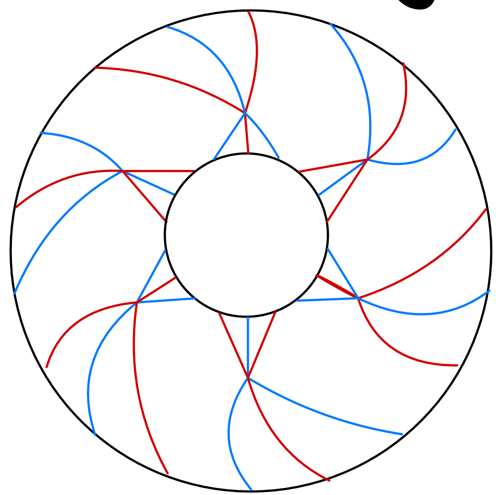
$$a_7 = a_1 a_3 a_5 a_6 a_8 + (a_1 a_3 a_6^2 + ((a_1 a_4 + (a_2 + a_3) a_5) a_6) a_7) a_9 + (a_1 a_3 a_6 a_7 + (a_1 a_4 + (a_2 + a_3) a_5) a_7^2) a_{10} \Big/ (a_1 a_3 a_5 a_7 a_8)$$

$$a_8 = a_{10}$$

$$a_9 = \frac{a_1 a_3 a_5 a_8 + (a_1 a_3 a_6 + (a_1 a_4 + (a_2 + a_3) a_5) a_7) a_9}{a_1 a_3 a_5 a_7}$$

$$a_{10} = a_9$$

Thank you!



Fundamental Groups of nontrivial genus-2 Lefschetz Fibrations

Sierra Knavel

Tech Topology Conference 2023

Georgia Tech, Advised by John Etnyre

Lefschetz Fibrations: Why do we care?

- Lefschetz fibrations* $\longleftrightarrow$ symplectic 4-mfds

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Lefschetz Fibrations: Why do we care?

- Lefschetz fibrations* $\longleftrightarrow$ symplectic 4-mfds

Question:

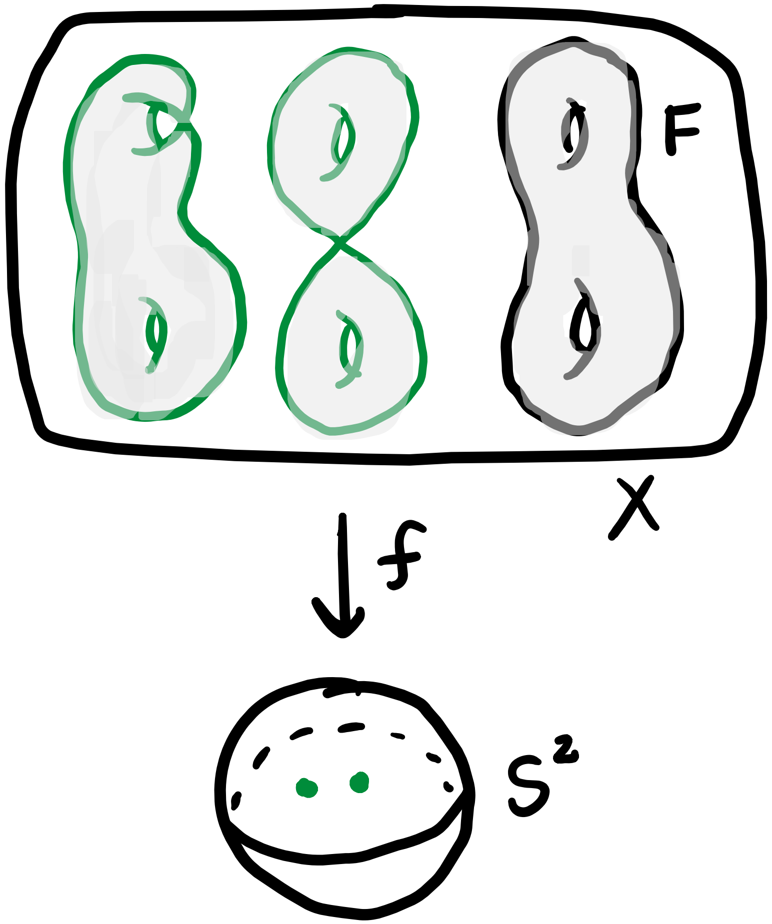
What are possible π_1's of genus-g LF?

Question for today:

What are possible π_1's of genus 2 LFs over S^2?

Lefschetz Fibrations: definition

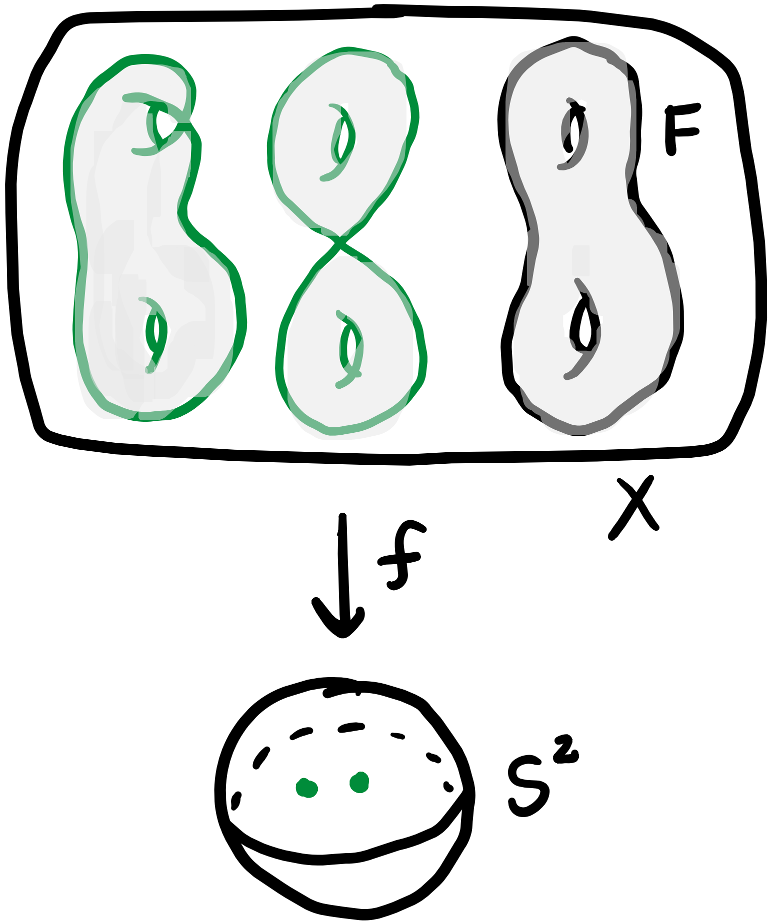
Lefschetz fibration:



Lefschetz Fibrations: definition

Lefschetz fibration:

- surjection $f : X^4 \longrightarrow \Sigma_h^2$



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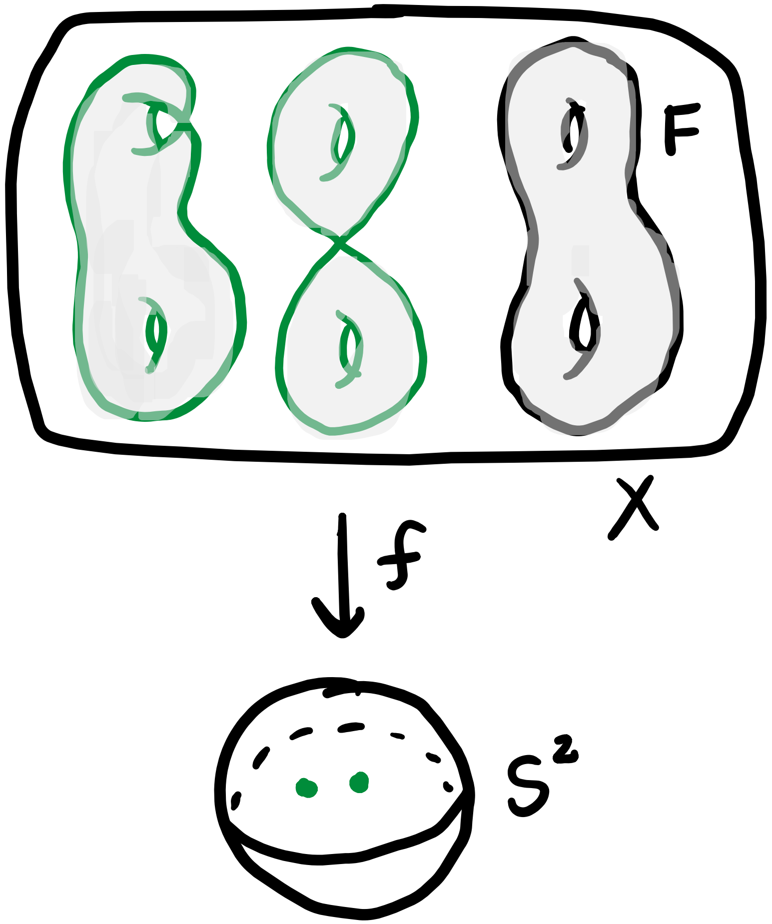
- singular points $f(z,w) = zw$



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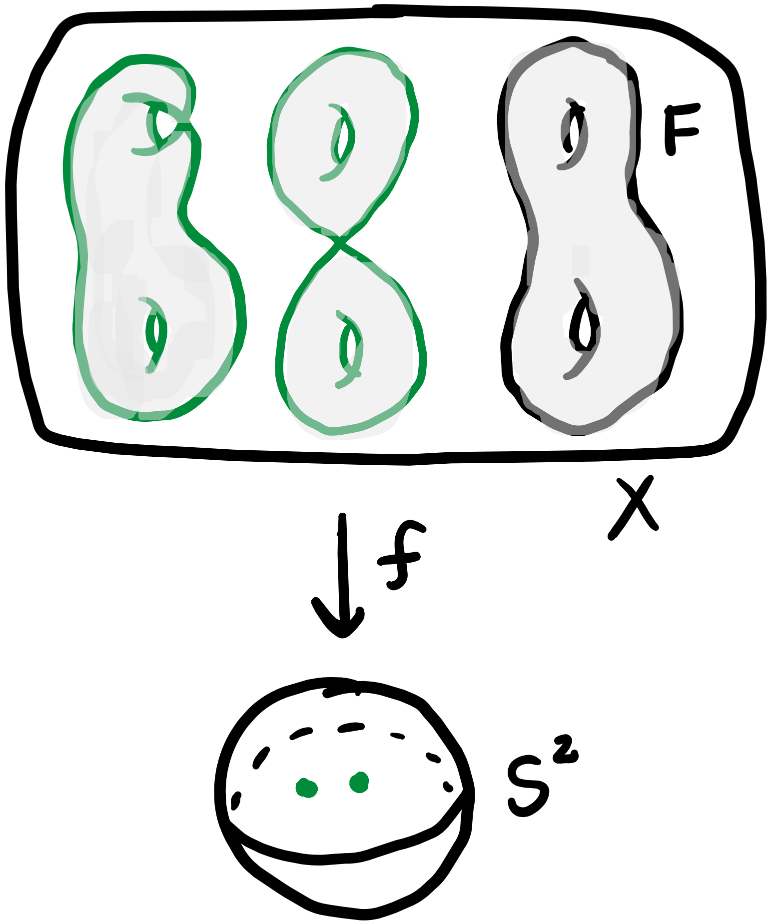
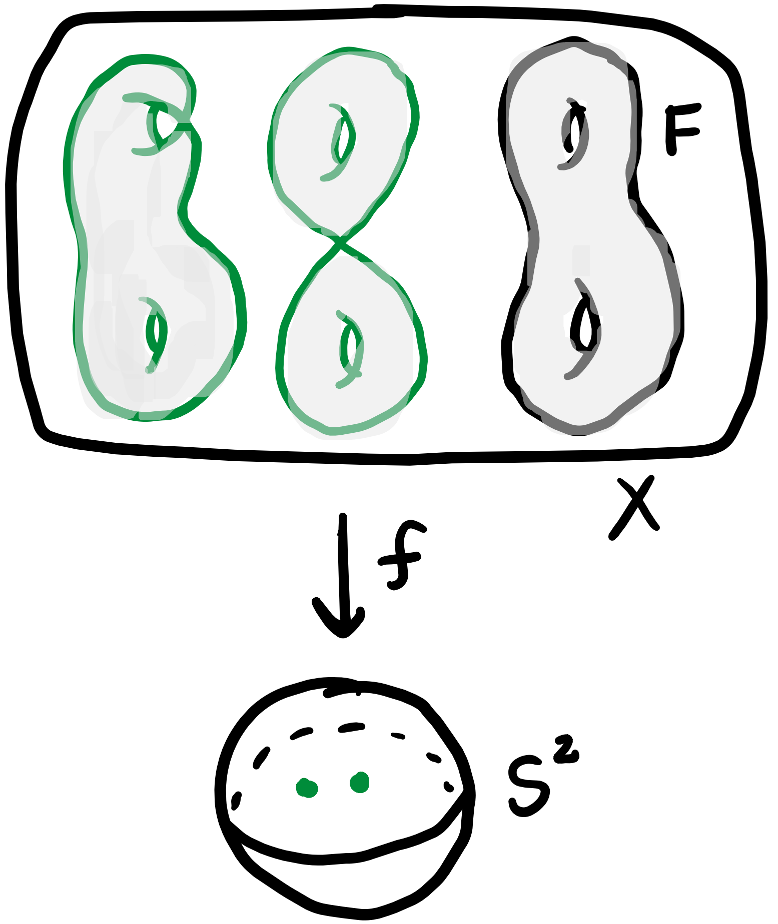


Lefschetz Fibrations: definition

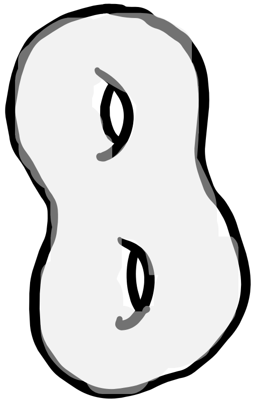
Lefschetz fibration:

- surjection $f: X^4 \longrightarrow \Sigma_h^2$
- singular points $f(z,w) = zw$
  - ↳ finitely many
- genus of LF = genus of fiber



Lefschetz Fibrations: fibers

regular fiber

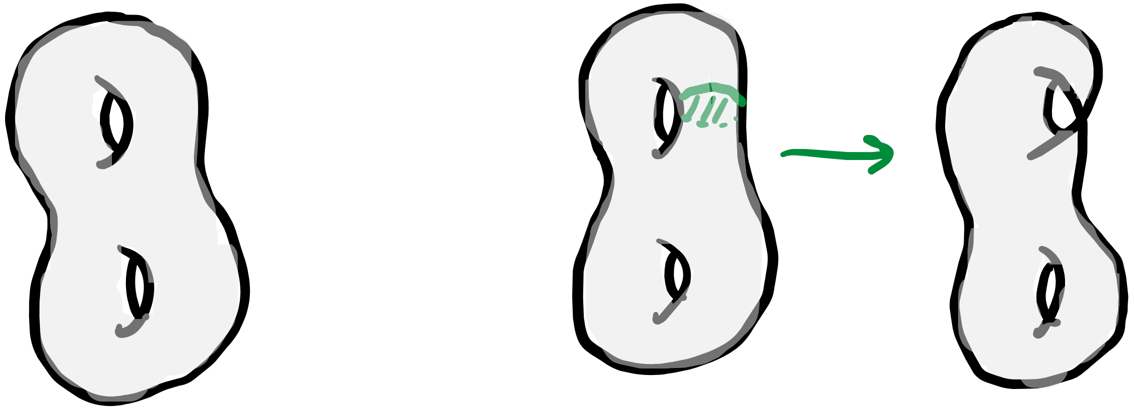


genus g

Lefschetz Fibrations: fibers

regular fiber

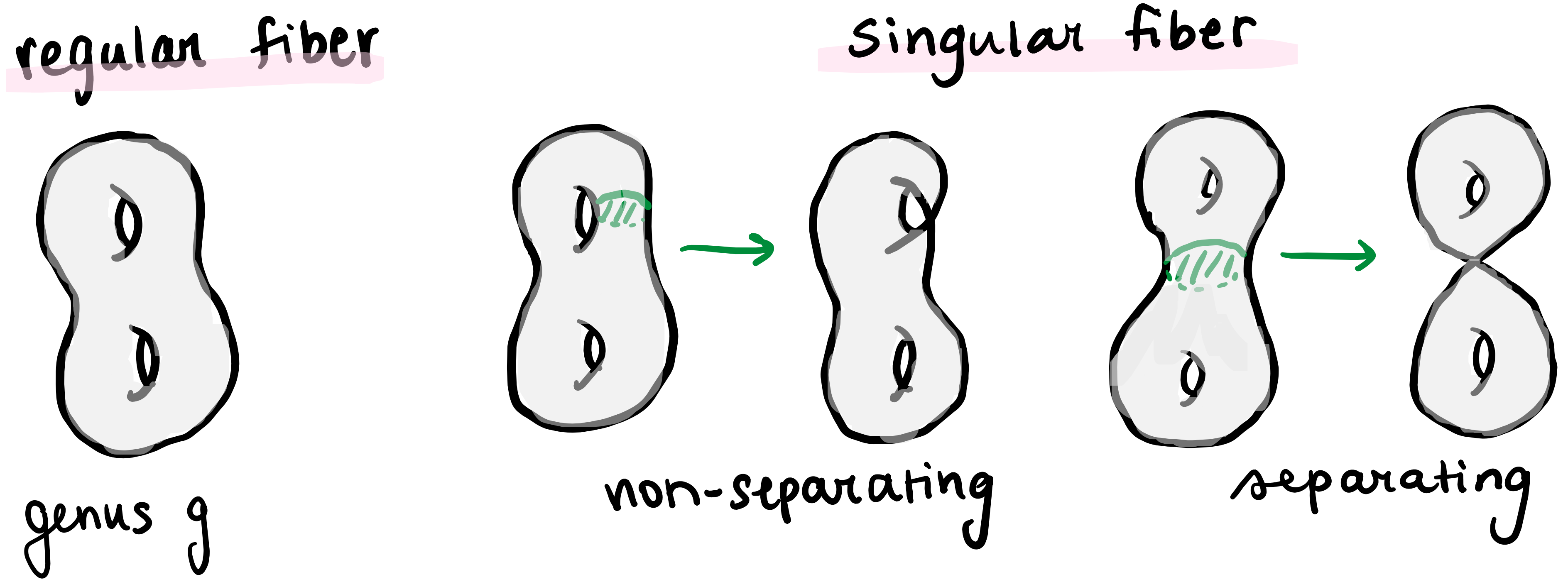
singular fiber



genus g

non-separating

Lefschetz Fibrations: fibers



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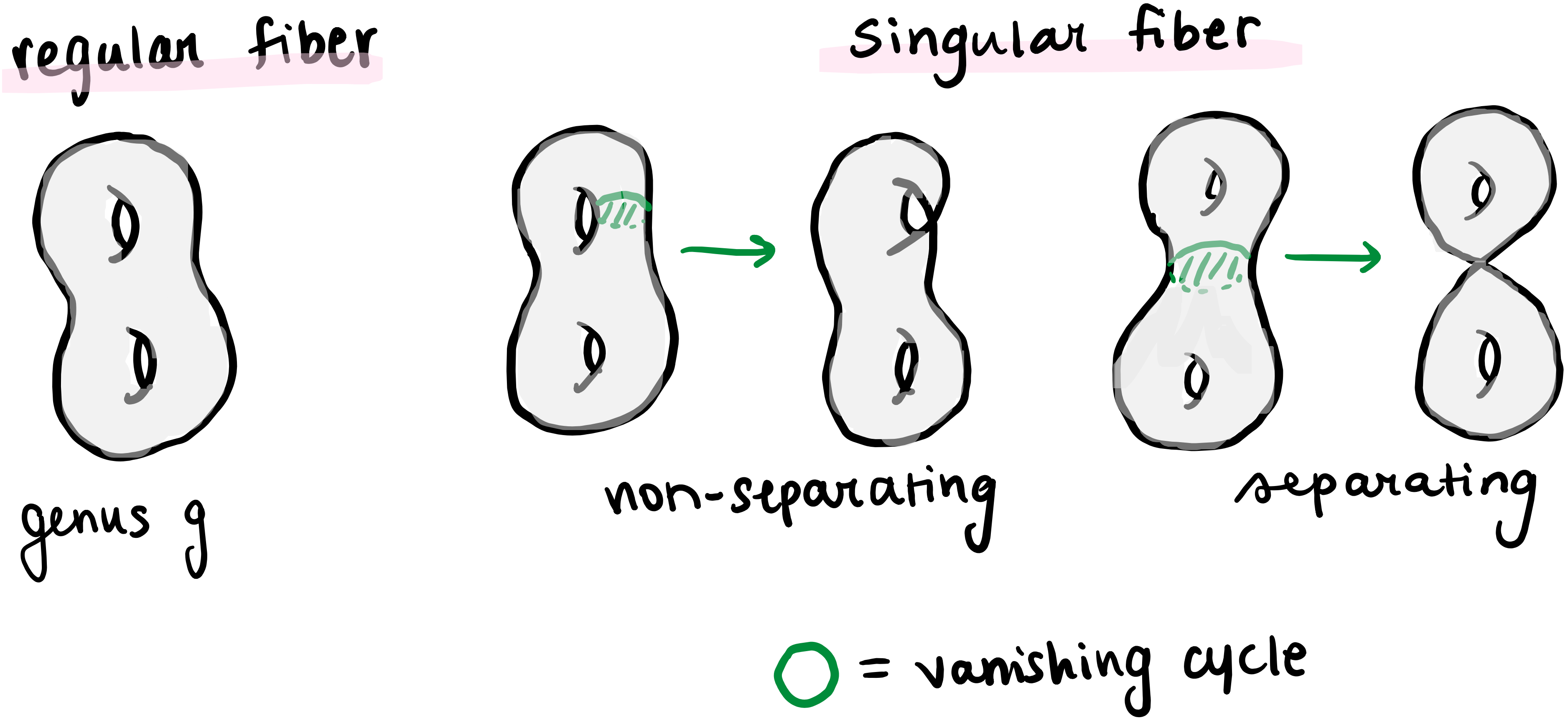
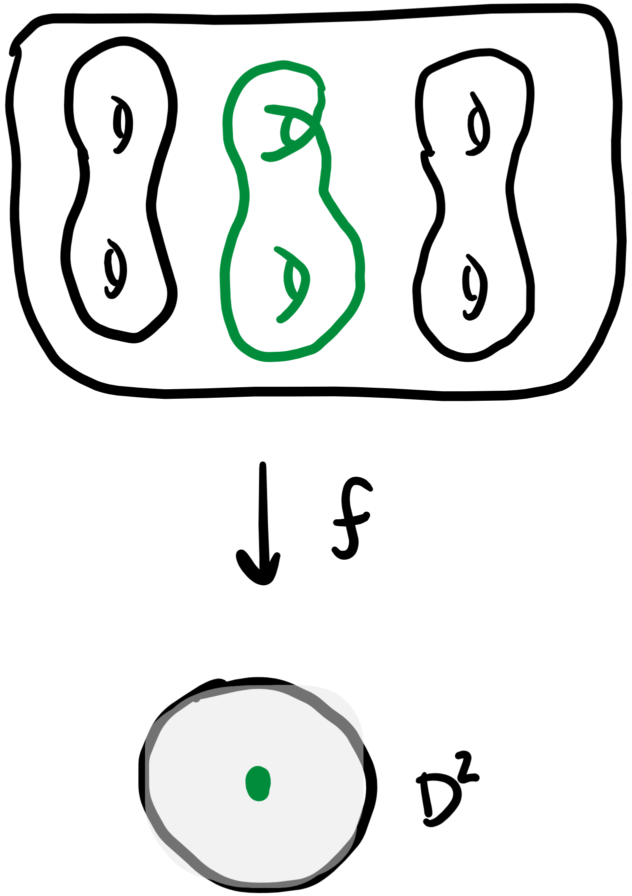


Lefschetz Fibrations: monodromy



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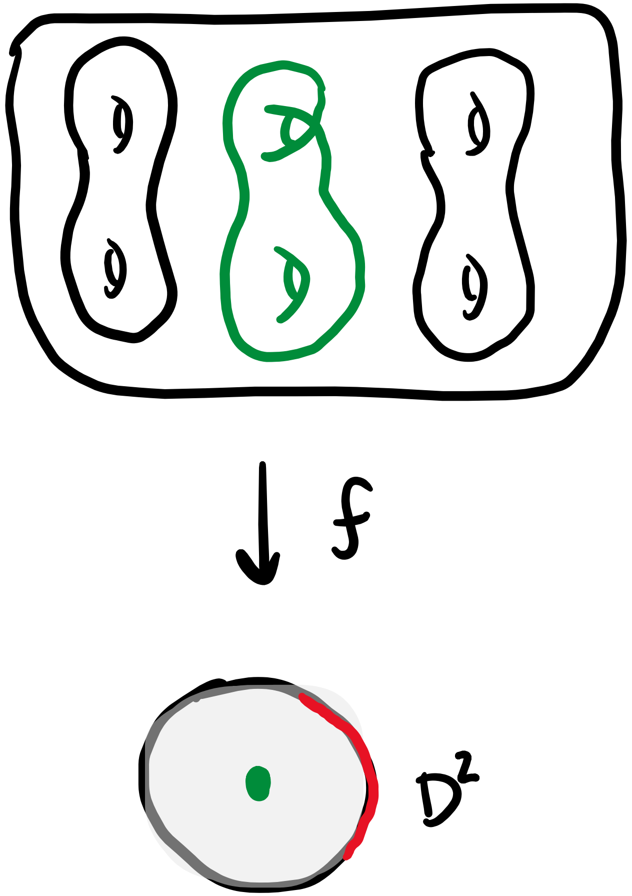

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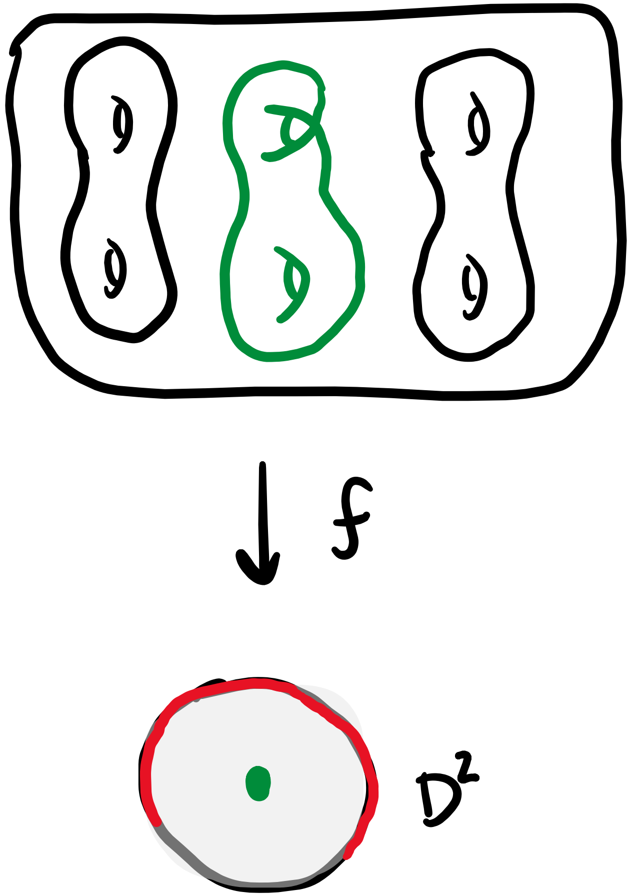

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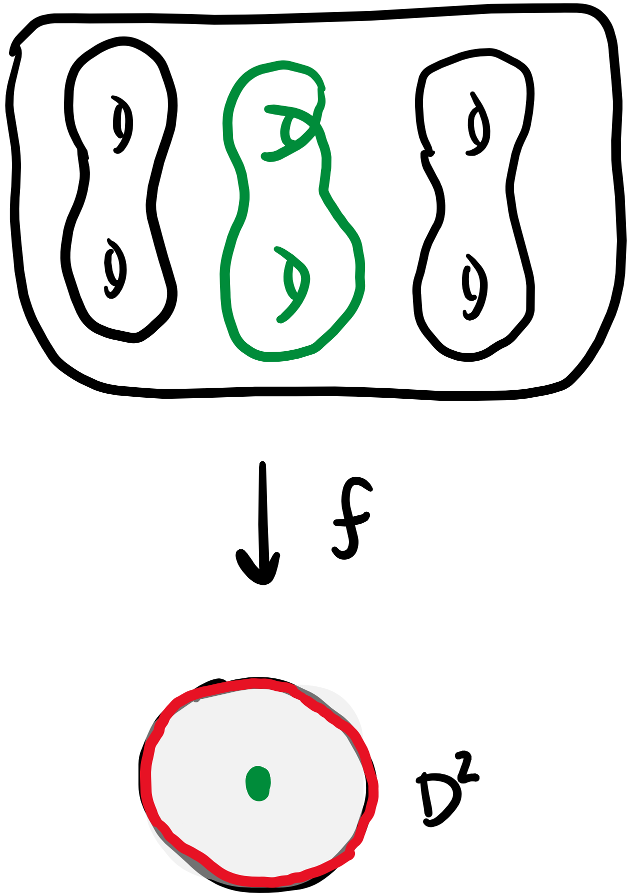
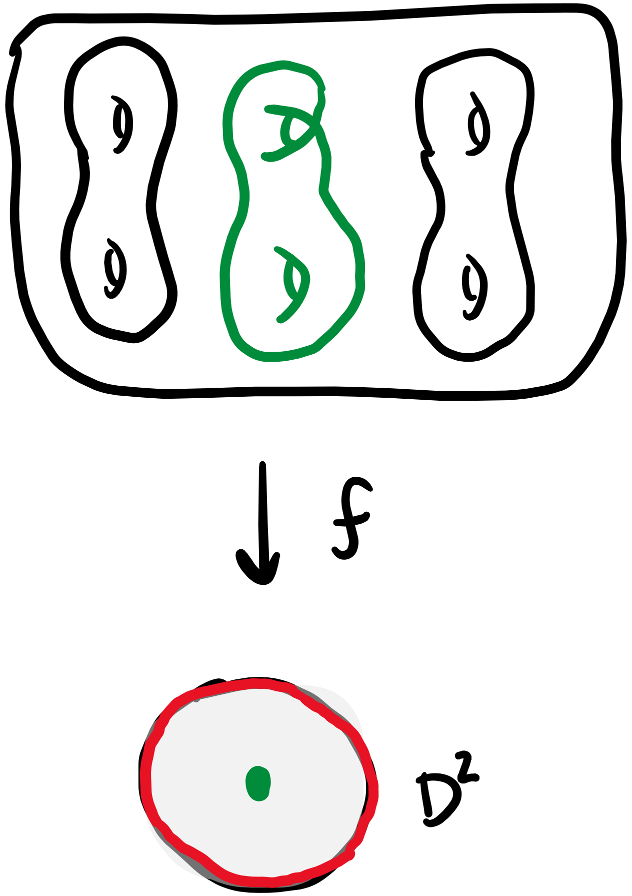


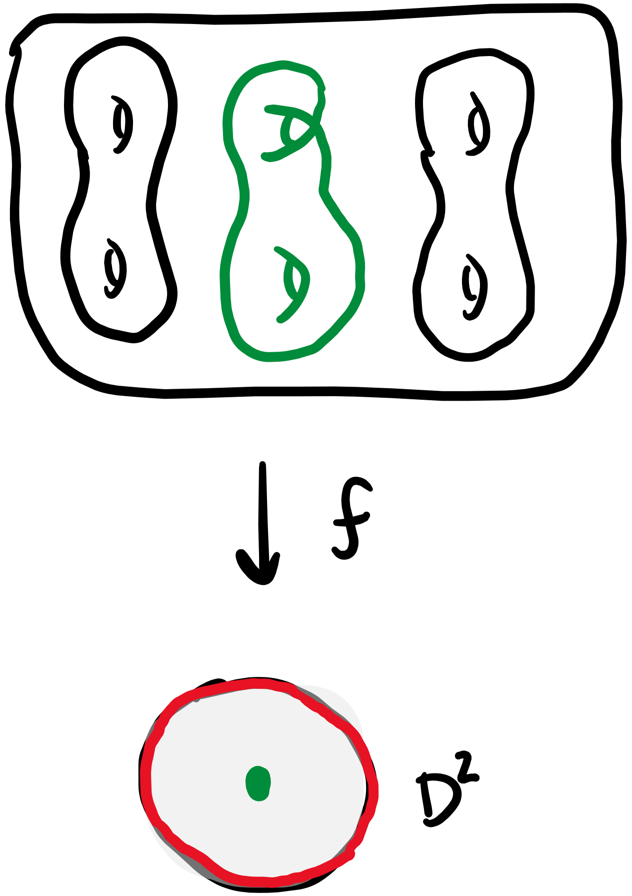
- tracing disks in base space give F-bundle/S^1

Lefschetz Fibrations: monodromy



- tracing disks in base space give F-bundle/S^1
- monodromy $\phi : F \to F$ is a Dehn twist about v.c.

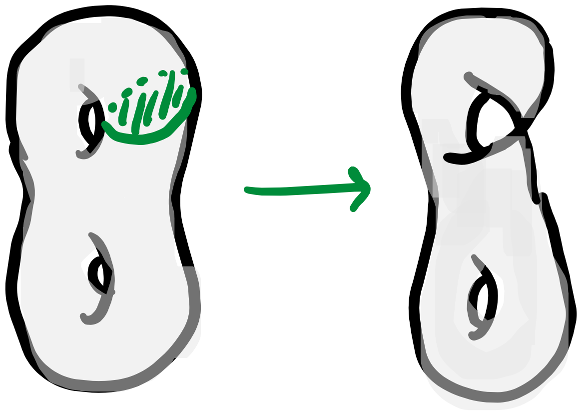
Lefschetz Fibrations: monodromy



- tracing disks in base space give F-bundle/S^1
- monodromy $\phi : F \to F$ is a Dehn twist about v.c.
- going around all singular values gives factorization of id $\in$ Mod(F)

Lefschetz Fibrations: vc's as 2-handles

Takeaway:

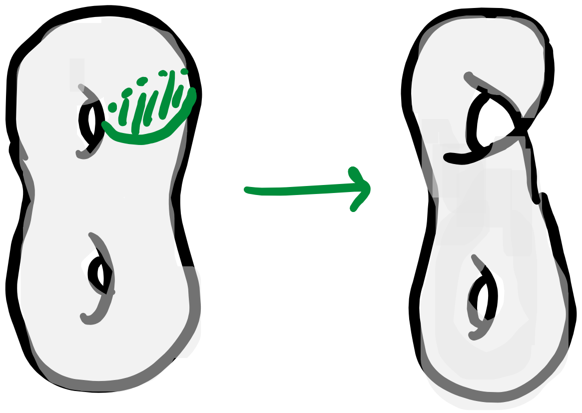


= vanishing cycle

Lefschetz Fibrations: vc's as 2-handles

Takeaway:

- vanishing cycles on singular fibers = important



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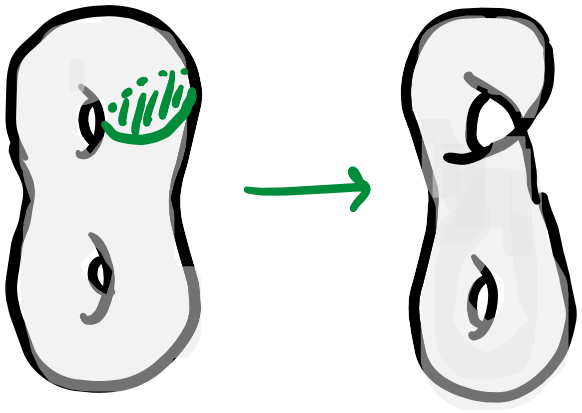

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Fundamental Group:

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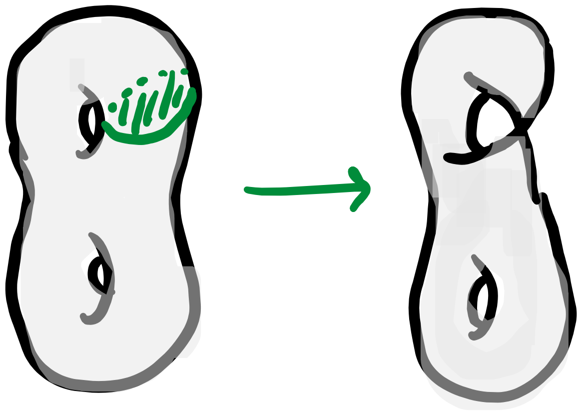

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Fundamental Group:

- generators = 1-handles
- relations = 2-handles

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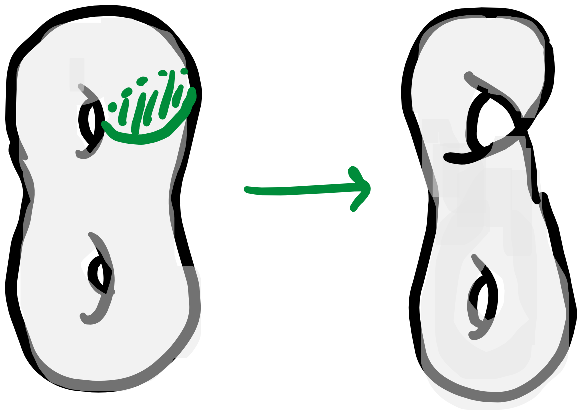

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Fundamental Group:

- generators = 1-handles
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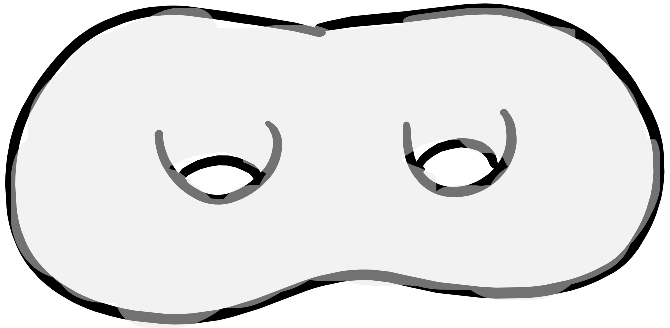
∴ disks glued along v.c's give relations

Simply Connected Genus-2 Lefschetz Fibrations



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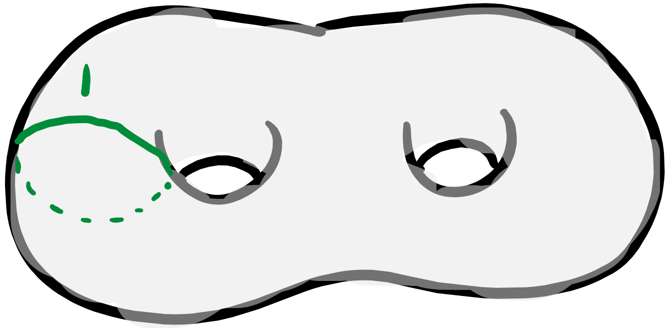

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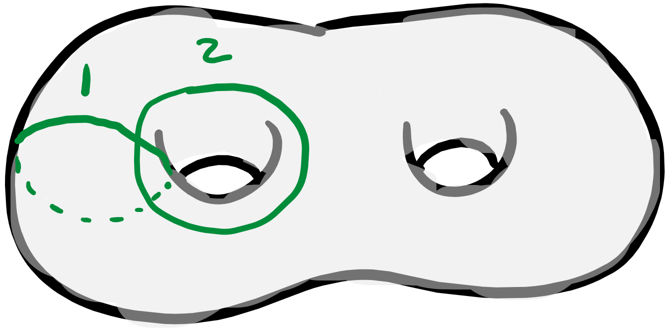

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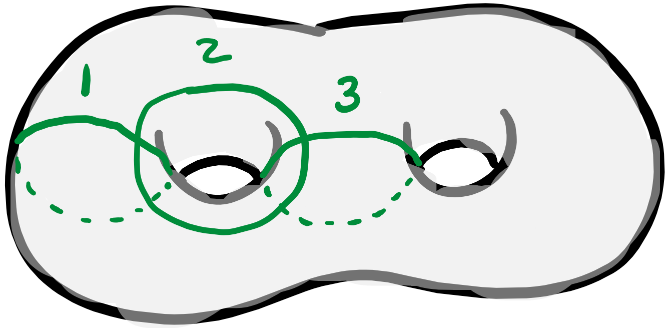

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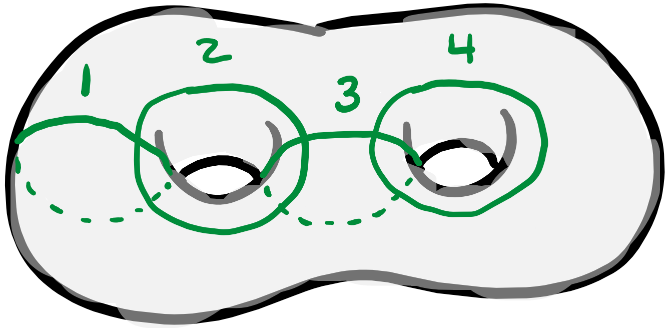

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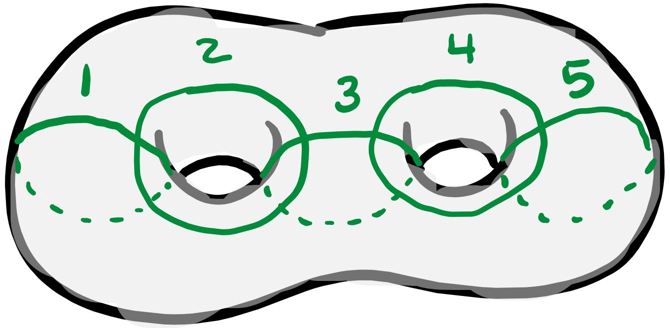

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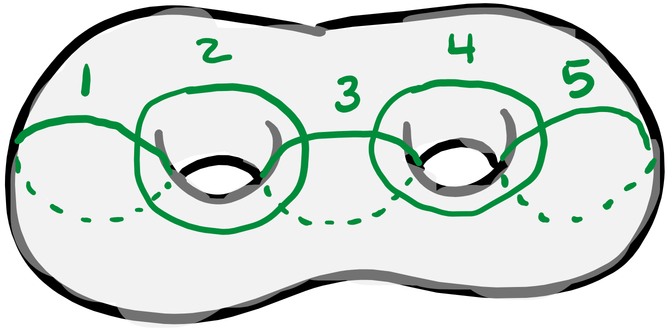
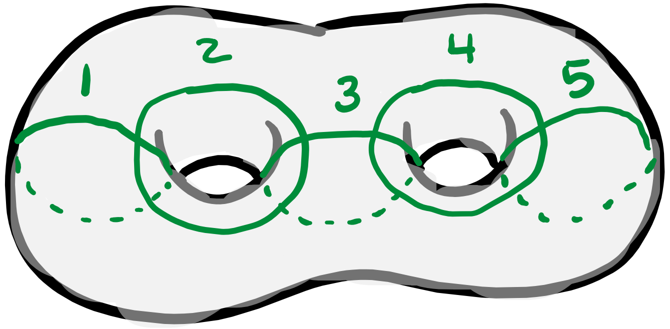


Chakiris '83

Every holomorphic genus 2 LF with no separating vc's is a fiber sum of α

Simply Connected Genus-2 Lefschetz Fibrations

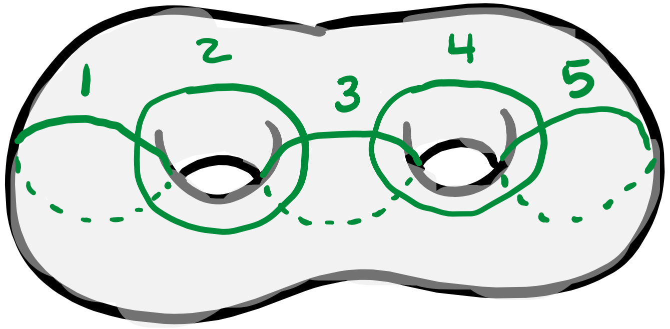


Chakiris '83

Every holomorphic genus 2 LF with no separating vc's is a fiber sum of α

$$\alpha : (t_1 t_2 t_3 t_4 t_5 t_5 t_4 t_3 t_2 t_1)^2$$

Simply Connected Genus-2 Lefschetz Fibrations



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Every holomorphic genus 2 LF with no separating vc's is a fiber sum of α, β

α : $(t_1 t_2 t_3 t_4 t_5 t_5 t_4 t_3 t_2 t_1)^2$

β : $(t_1 t_2 t_3 t_4 t_5)^6$

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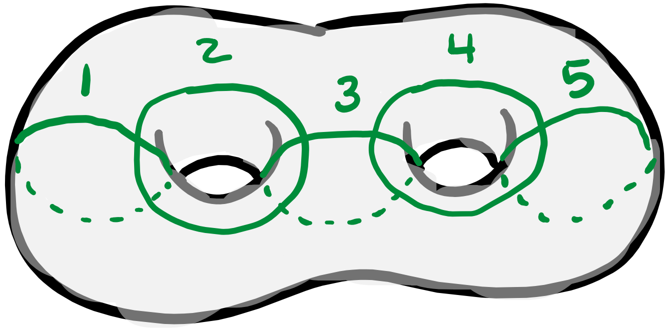

Chakiris '83

Every holomorphic genus 2 LF with no separating vc's is a fiber sum of α, β, and γ

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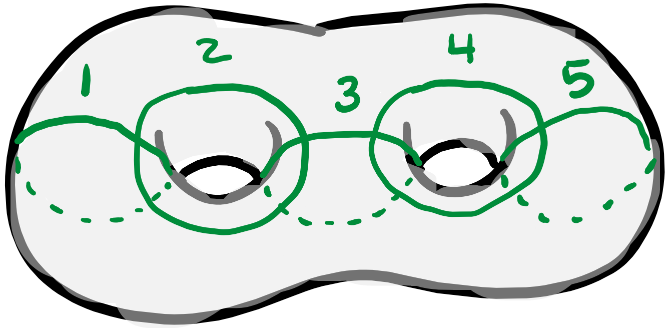
β: $(t_1 t_2 t_3 t_4 t_5)^6$

γ: $(t_1 t_2 t_3 t_4)^{10}$

Simply Connected Genus-2 Lefschetz Fibrations



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$\beta: (t_1 t_2 t_3 t_4 t_5)^6$

$\gamma: (t_1 t_2 t_3 t_4)^{10}$

Siebert-Tian '03:

- no separating
- transitive monodromy

Results with separating vanishing cycles

all non-separating:

- $\pi_1(X) = 0$
- except for case of technical condition

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future directions:

- Always Abelian?
- At most 2 generators?

thanks for listening! ◡̈

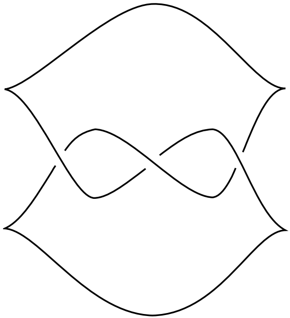
The Homotopy Cardinality of the the Representation Category

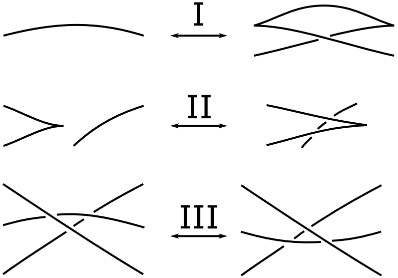
Justin Murray

Louisiana State University

Tech Topology
December 8, 2023

For us, $\Lambda \subset (\mathbb{R}^3, \ker(dz - ydx))$ is a connected Legendrian with $r(\Lambda) = 0$.





Cooking up Invariants

Given Λ one can form a differential graded algebra (DGA), $(\mathcal{A}_\Lambda, \partial_\Lambda)$ such that $H_*((\mathcal{A}_\Lambda, \partial_\Lambda))$ is invariant under Legendrian isotopy.

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Cooking up Invariants

Given Λ one can form a differential graded algebra (DGA), $(\mathcal{A}_\Lambda, \partial_\Lambda)$ such that $H_*((\mathcal{A}_\Lambda, \partial_\Lambda))$ is invariant under Legendrian isotopy. BUT $H_*((\mathcal{A}_\Lambda, \partial_\Lambda))$ is hard to compute in general! Instead we can look at DGA

maps

$$\varepsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \to (\mathbb{F}, 0) \qquad \text{called augmentations}$$

or

$$\rho : (\mathcal{A}_\Lambda, \partial_\Lambda) \to (Mat_n(\mathbb{F}), 0) \qquad \text{called representations}$$

Counting

If $\mathbb{F} = \mathbb{F}_q$, then you can count these maps

Count all maps and renormalize	$Aug(\Lambda, \mathbb{F}_q)$	$Rep_n(\Lambda, \mathbb{F}_q)$
Count isomorphism classes of maps	$\#\pi_{\geq 0}\mathcal{A}ug_+(\Lambda, \mathbb{F}_q)^*$	$\#\pi_{\geq 0}\mathcal{R}ep_n^+(\Lambda, \mathbb{F}_q)^*$

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Theorem (Pan, Capovilla-Searle-Legout-Limouzineau-Murphy-Pan-Traynor)

If there is an exact Lagrangian cobordism from Λ_- to Λ_+ then

$$\#\pi_{\geq 0}\mathcal{A}ug_+(\Lambda_-, \mathbb{F}_q)^* \leq \#\pi_{\geq 0}\mathcal{A}ug_+(\Lambda_+, \mathbb{F}_q)^*$$

Some Results

Theorem (M'23)

Two representations in the representation category are isomorphic $\iff$ they are conjugate up to DGA homotopy.

Theorem (M'23)

The homotopy cardinality can be computed via colored ruling polynomials:

$$\#\pi_{\geq 0}\mathcal{R}ep_n^+(\Lambda, \mathbb{F}_q)^* = q^{n^2 tb(\Lambda)/2} R_{n,\Lambda}(q)$$

Corollary

If there is an exact Lagrangian cobordism from Λ_- to Λ_+ then

$$\#\pi_{\geq 0}\mathcal{R}ep_n^+(\Lambda_-, \mathbb{F}_q)^* \leq \#\pi_{\geq 0}\mathcal{R}ep_n^+(\Lambda_+, \mathbb{F}_q)^*$$

Conjectures

Conjecture A

There exists a Legendrian Λ_n that has no augmentations but a higher n-dimensional (0-graded) representation.

Conjecture B

The obstruction to reversing Lagrangian concordance using representations is strictly stronger than that for augmentations (would follow from Conjecture A).

References

Preprint:

Legendrian Knot Atlas:

(Where you might find Λ_n, still under construction)

Negative contact surgery on Legendrian non-simple knots

(Joint with Hugo Zhou)

Shunyu Wan
University of Virginia

Tech Topology Conference Lightning Talk

Contact 3-manifolds and Legendrian knots

- A contact 3-manifold (Y, ξ) is a smooth 3-manifold Y together with a 2-plane field distribution ξ such that for any one form α with $\ker(\alpha) = \xi$, $\alpha \wedge d\alpha > 0$.
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Classical invariants associated to a Legendrian knot L

- $\text{tb}(L)$ (Thurston-Bennequin number)
- $\text{rot}(L)$ (rotation number)

A knot is called Legendrian non-simple if it has two Legendrian representatives with same *tb* and *rot* that are not Legendrian isotopic to each other.

Contact surgery on non-simple knots

An oriented Legendrian knot L in a contact 3-manifold (Y, ξ) admits a canonical contact framing, and we can perform r-surgery with respect to the contact framing. Moreover, we can put a contact structure $\xi_r(L)$ on the surgery manifold $Y_r(L)$.

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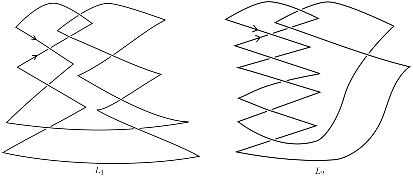
Question: If K is a Legendrian non-simple knot, and we let L_1 and L_2 be two Legendrian non isotopic representatives of K in (Y, ξ), then what can we say about the contact manifolds $(Y_r(L_1), \xi_1)$, and $(Y_r(L_2), \xi_2)$?

Specific example

We focus on the following two Legendrian non-isotopic representatives L_1 and L_2 of the twist knot E_5 in (S^3, ξ_{std}). Both L_1 and L_2 have $\text{tb} = 1$ and $\text{rot} = 0$.



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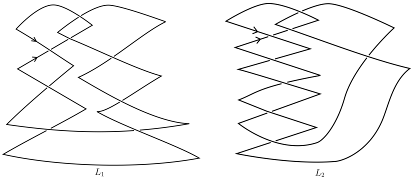

Theorem 1 (Etnyre, 2006)

$(S^3_{+1}(L_1), \xi_1)$, and $(S^3_{+1}(L_2), \xi_2)$ are contactomorphic.

Theorem 2 (Bourgeois-Ekholm-Eliashberg, 2009)

$(S^3_{-1}(L_1), \xi_1)$, and $(S^3_{-1}(L_2), \xi_2)$ are not contactomorphic.

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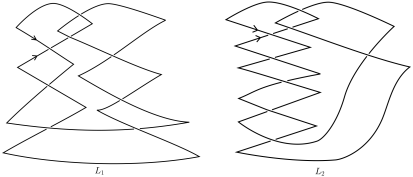

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$(S^3_{-1}(L_1), \xi_1)$, and $(S^3_{-1}(L_2), \xi_2)$ are not contactomorphic.

Theorem 3 (W, Zhou)

$(S^3_r(L_1), \xi_1)$, and $(S^3_r(L_2), \xi_2)$ are not contact isotopic for all $r < 0$.

Contact invariant and LOSS invariant

Ozsváth-Szabó and later Honda-Kazez-Matić showed that (Y, ξ) determines a distinguished element $c(\xi) \in \widehat{HF}(-Y)$, called the Heegaard Floer "contact invariant". Subsequently, for a Legendrian knot L in (Y, ξ), Lisca-Ozsváth-Stipsicz-Szabó defined the "LOSS invariant" $\mathfrak{L}(L) \in HFK^{-}(-Y, L)$.

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Ozsváth and Stipsicz proved these two Legendrian representatives of E_5, L_1 and L_2 have different LOSS invariants.

Relation between contact invariant and LOSS invariant

Lemma 4 (Lisca-Ozsváth-Stipsicz-Szabó)

For any 3-manifold Y and a knot K in Y there is a natural chain map

$$g : \mathrm{CFK}^-(Y, K, \mathfrak{t}) \to \widehat{\mathrm{CF}}(Y, \mathfrak{t}).$$

Moreover let L be a null-homologous Legendrian knot in a contact 3-manifold (Y, ξ), then the map on homology induced by g

$$G : \mathrm{HFK}^-(-Y, L, \mathfrak{t}) \to \widehat{HF}(-Y, \mathfrak{t}) \tag{1.1}$$

has the property that

$$G(\mathfrak{L}(L)) = c(\xi).$$

Contact -2 surgery on L_1 and L_2

Theorem 5 (Wan, Zhou)

Contact -2 *surgery on* L_1 *and* L_2 *give different contact manifolds with different contact invariants.*

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Proof.

1. Let P_i be the Legendrian push-offs of L_i, P'_i be the induced Legendrian knots of P_i in $S^3_{-2}(L_i)$.

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3. Calculate $HFK^-(-S^3_{-2}(L_i), P_i')$, and show the map G is injective on the LOSS invariants. (Using Hedden-Levine mapping cone formula for duel knot.)

□

Thank You for Your Attention!

Thickening finite complexes into manifolds

– Arka Banerjee

Tech Topology conference, 2023

Definition: The thickening dimension of a simplicial complex K, denoted by $\mathrm{thkdim}(K)$, is the minimum dimension of a manifold (M, ∂) that is homotopy equivalent to K.

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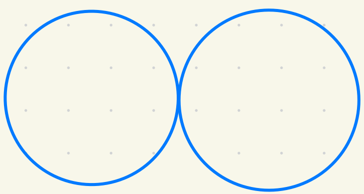
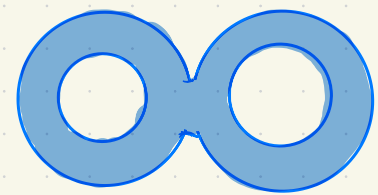
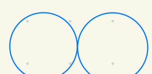
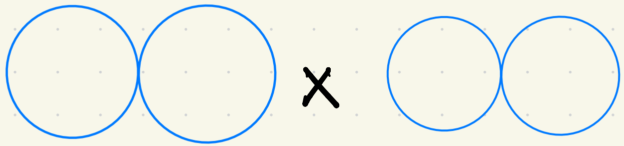
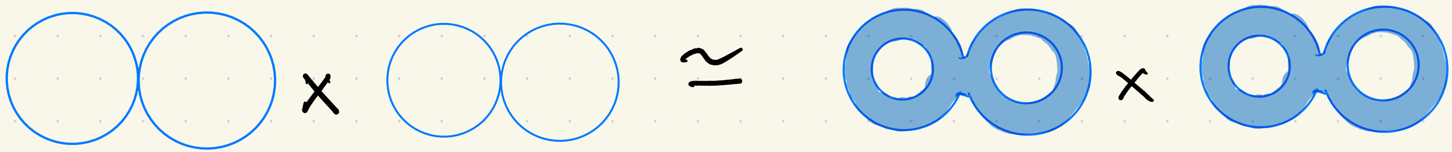


$\simeq$

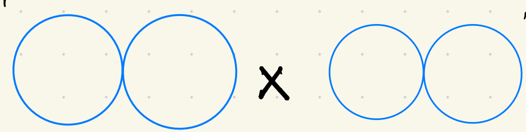


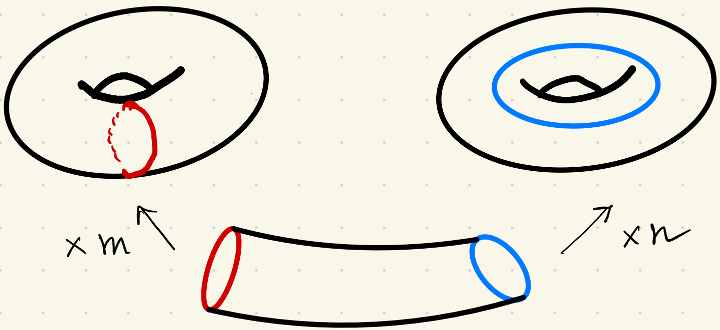
- thkdim () $= 2$


x

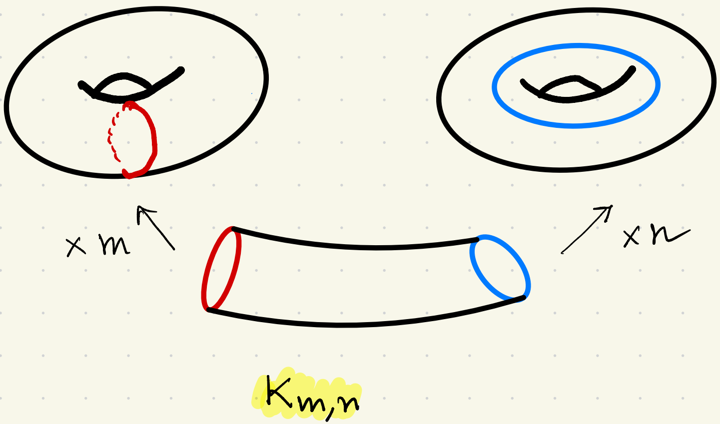


Thm (Bestvina–Kapovich–Kleiner, 2002):

thkdim() $= 4$



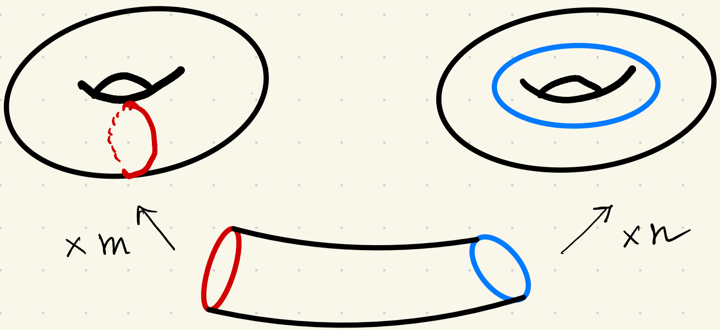
$K_{m,n}$



- $\text{thkdim}(K) \leq 2\dim(K)$ (Stallings)

$$\Downarrow$$

$$\text{thkdim}(K_{m,n}) \leq 4$$



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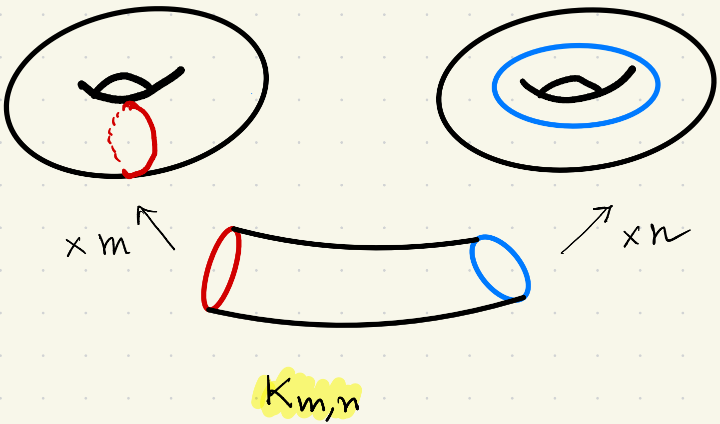

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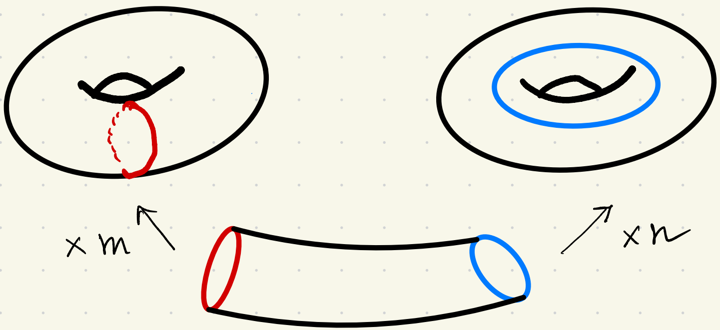
$\text{thkdim}(K_{m,n} \times K_{m,n})$



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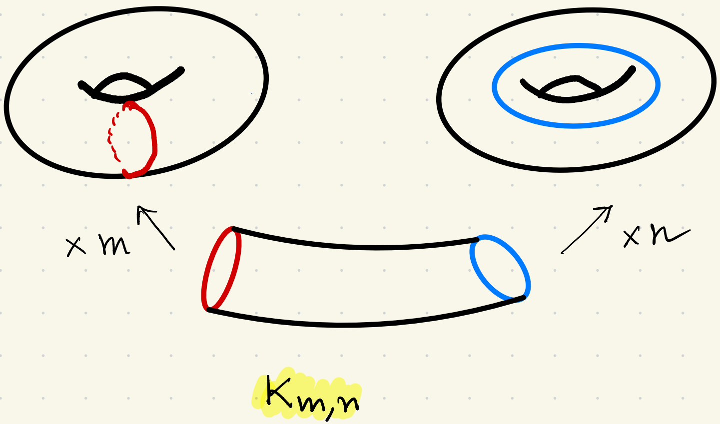
$\text{thkdim}(K_{m,n} \times K_{m,n}) \leq 8$



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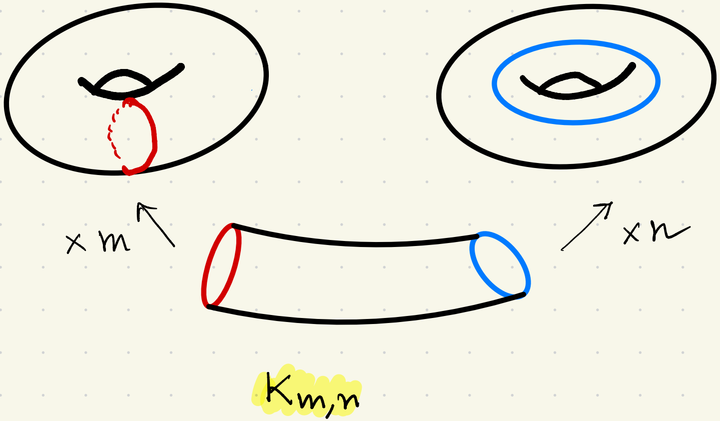
- $7 \leq \text{thkdim}(K_{m,n} \times K_{m,n}) \leq 8$ if $m | 4$ or $n | 4$ (Schreve, '19)



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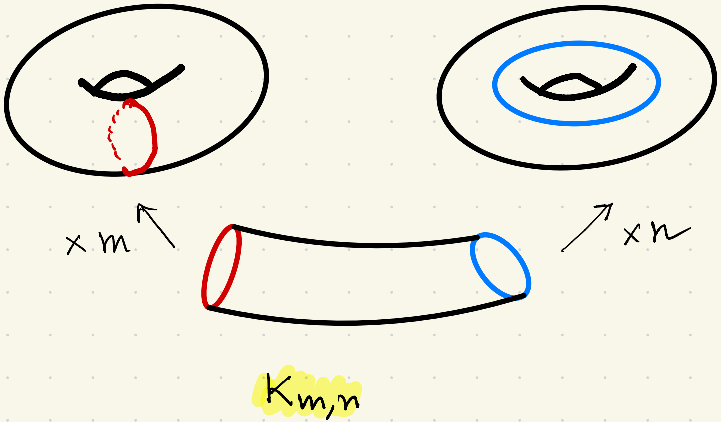
- $7 \leq \mathrm{thkdim}(K_{m,n} \times K_{m,n}) \leq 8$ if $m \geq 3$ or $n \geq 3$ (B., in progress)



Question:

$$\text{thkdim}(K_{m,n} \times K_{m,n}) = ?$$

Thank you

Towards a count of holomorphic sections of Lefschetz fibrations over the disc

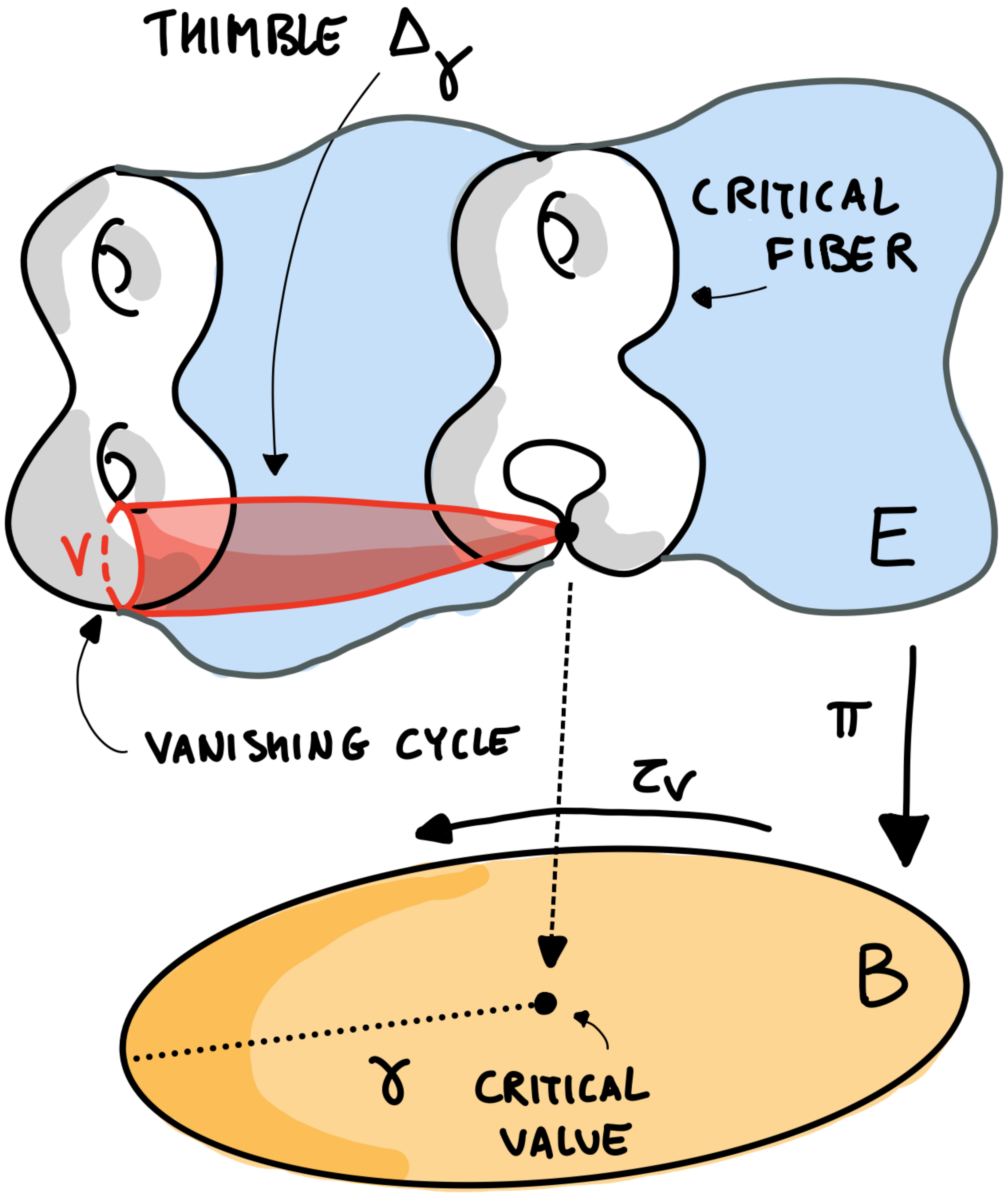
2023 Tech Topology Conference - Lightning Talk

Riccardo Pedrotti - UT Austin

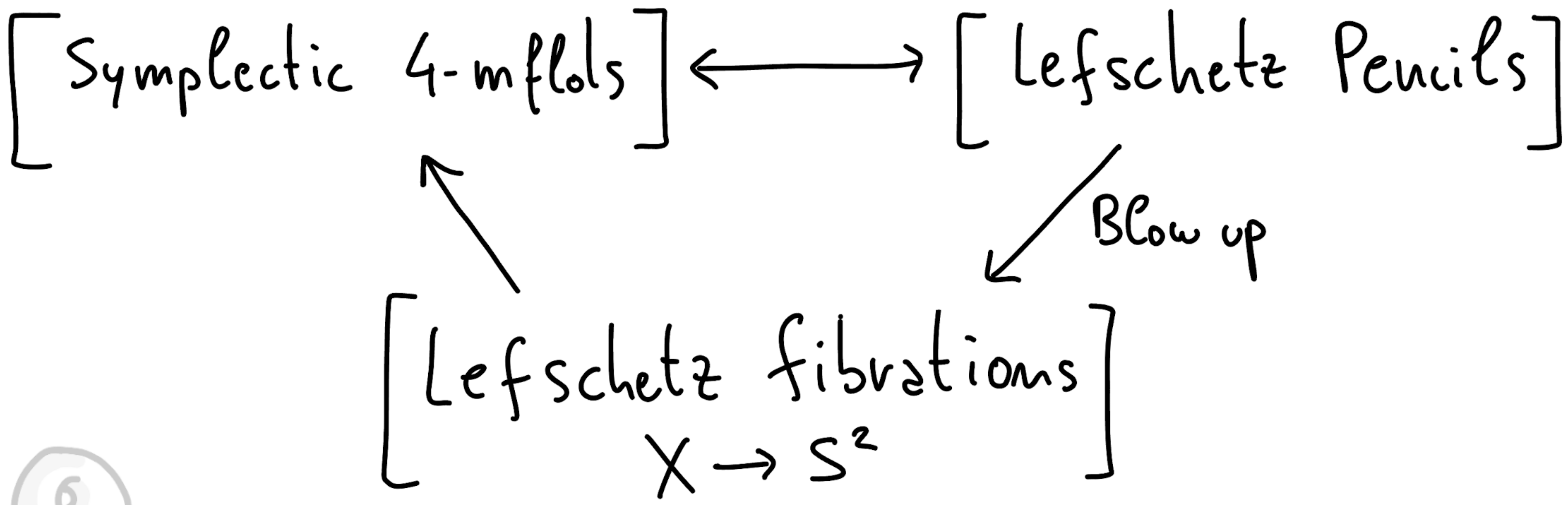
(Work in progress w/ T. Perutz)

Lefschetz fibration

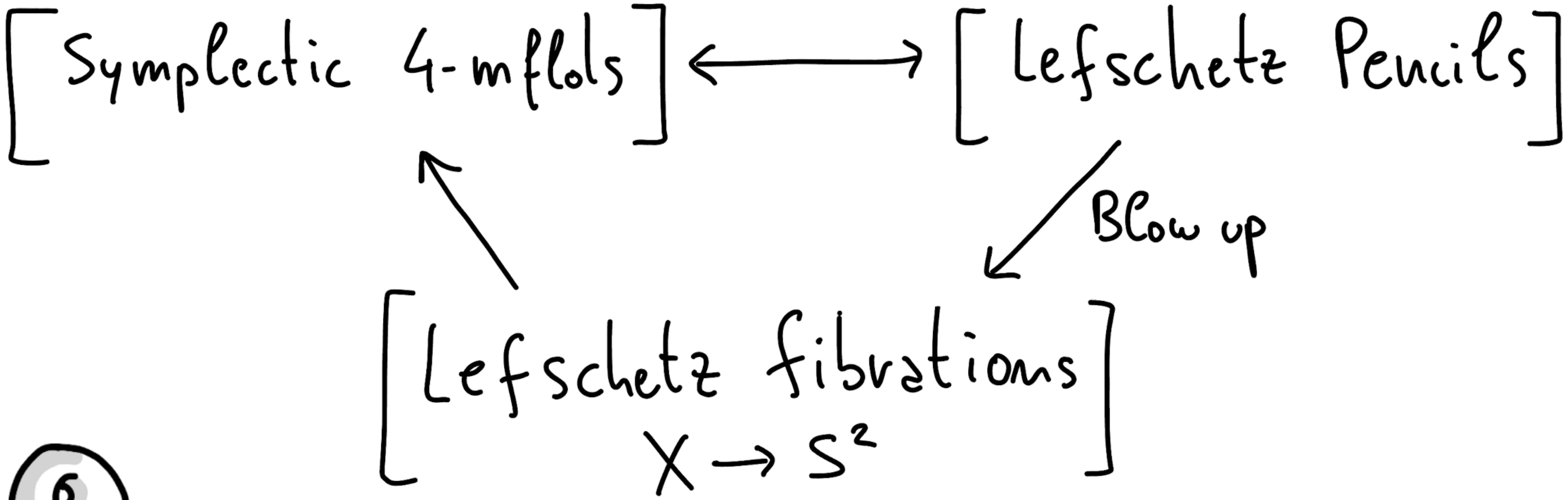
- $\pi : E^4 \to B^2$ (smooth, proper)
- $\partial E = \pi^{-1}(\partial B)$
- Standard neighbourhood around critical points of π



- Donaldson and Gompf:



- “Bijection” between positive factorization of the identity in $MCG(\Sigma)$ and Lefschetz fibrations over S^2



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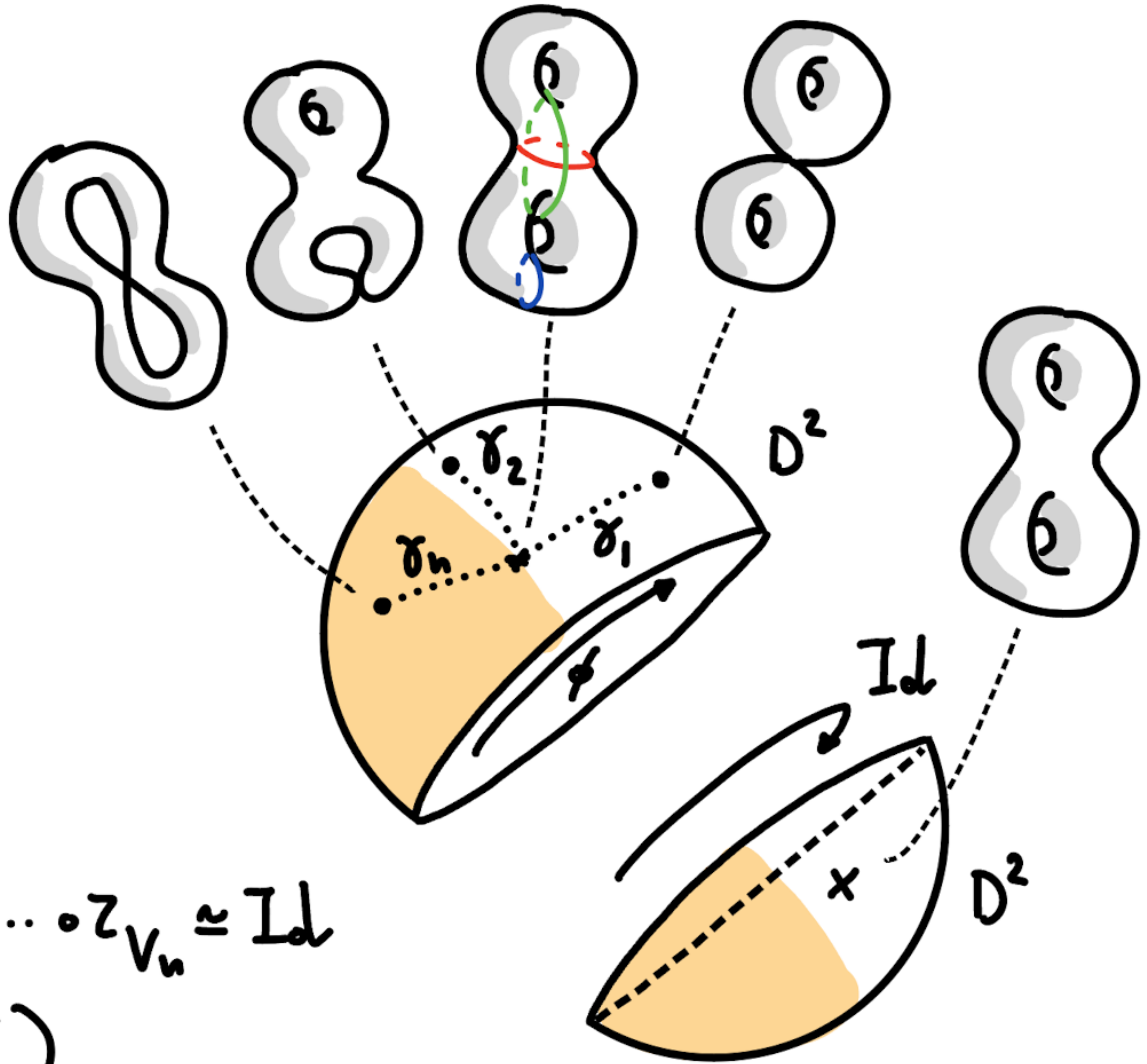


$\phi = \tau_{V_1} \circ \tau_{V_2} \circ \cdots \circ \tau_{V_n} \simeq Id$
in $MCG(\Sigma)$

- "Bijection" between positive factorization of the identity in $MCG(\Sigma)$ and Lefschetz fibrations over S^2

Can we use this combinatorial description of X^4 to compute its SW invariants?

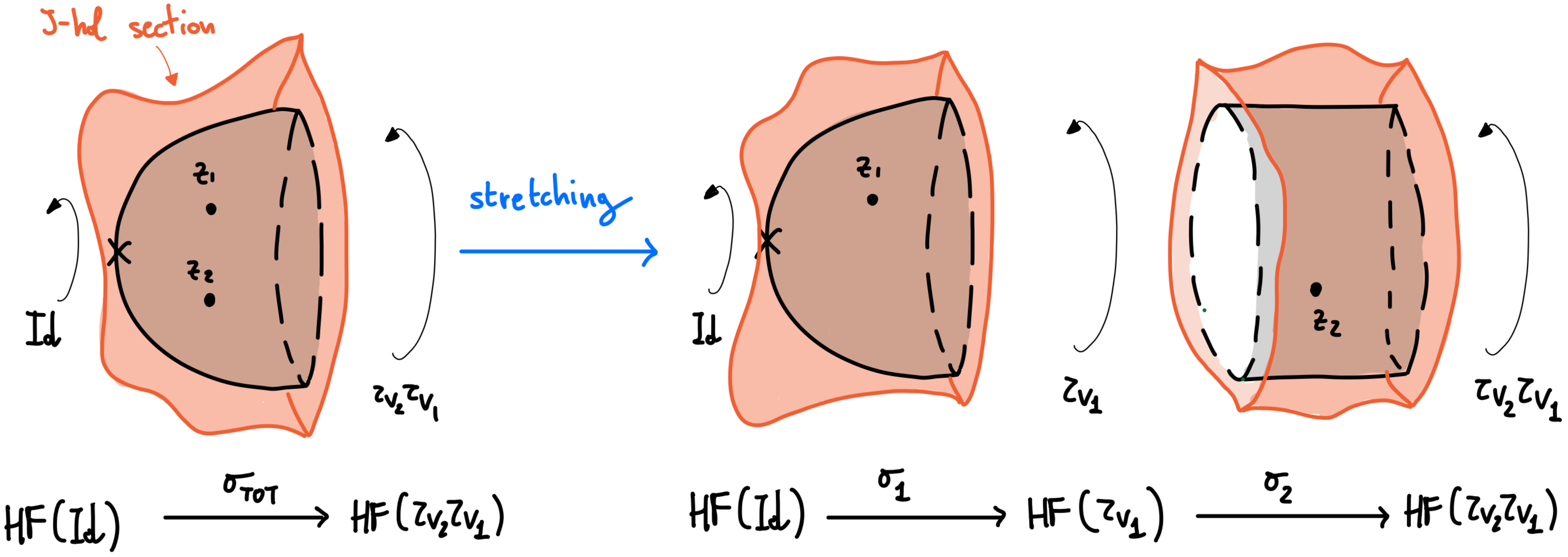
- We want to count pseudo-holomorphic sections of $\pi : X^4 \to D^2$ by keeping track of their (relative) homology class

- We can get insights into SW invariants of the (capped-off) symplectic manifold X^4

Counting sections

$$\cdots \to HF_*(\phi) \xrightarrow{\sigma_i} HF_*(\tau_{V_1}\phi) \to HF^{-*}(\phi V_i, V_i) \to \cdots$$

Counting sections

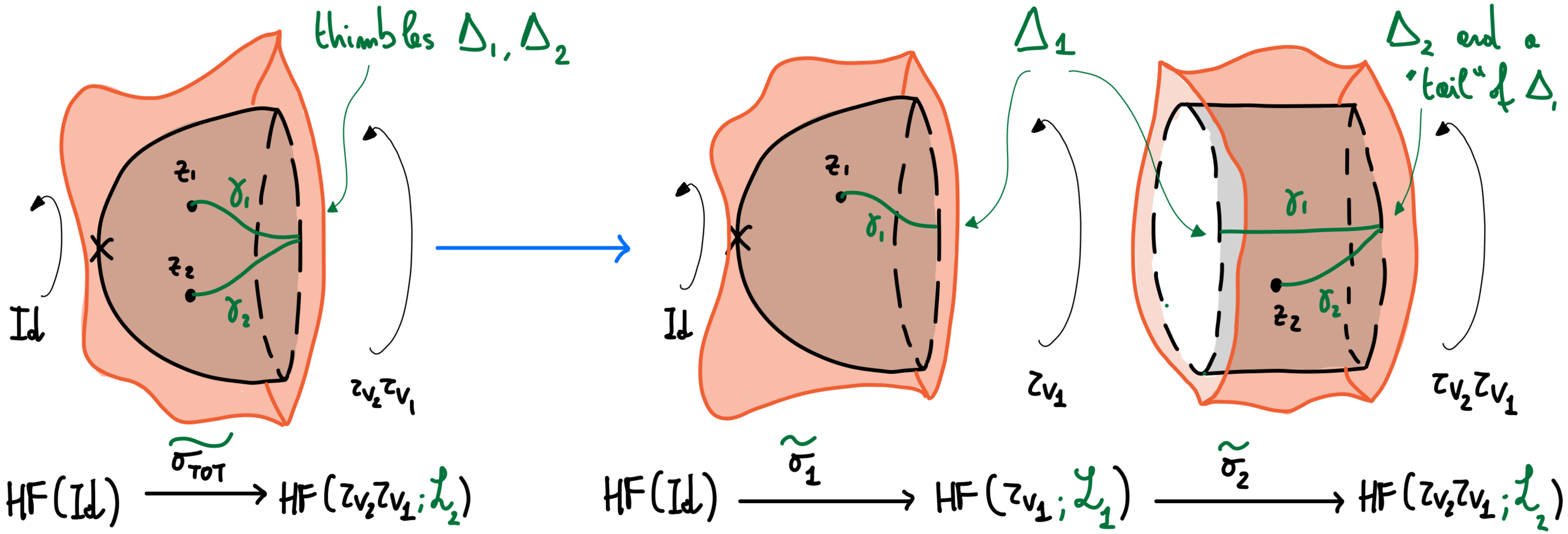


$$\cdots \to HF_*(\phi) \xrightarrow{\sigma_i} HF_*(\tau_{V_1}\phi) \to HF^{-*}(\phi V_i, V_i) \to \cdots$$

Counting sections keeping track of their homology class

$$\cdots \to HF_*(\phi; \mathscr{L}_i) \xrightarrow{\overline{\sigma}_i} HF_*(\tau_{V_1}\phi; \mathscr{L}_i) \to HF^{-*}(\phi V_i, V_i; \mathscr{L}_i) \to \cdots$$

Counting sections keeping track of their homology class



$$\cdots \to HF_*(\phi;\mathscr{L}_i) \xrightarrow{\overline{\sigma}_i} HF_*(\tau_{V_1}\phi;\mathscr{L}_i) \to HF^{-*}(\phi V_i, V_i;\mathscr{L}_i) \to \cdots$$

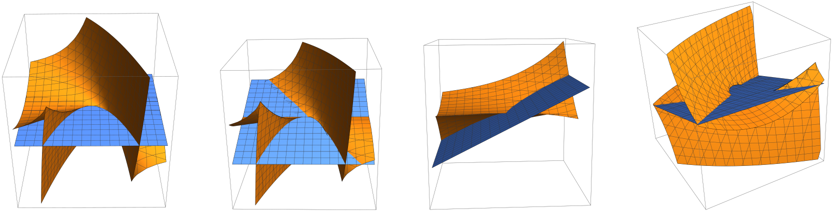
State of the project

- Using the mapping cone, we have a combinatorial formula for $\widetilde{\sigma_{tot}}$ in the Lagrangian and Fixed Point case (more complicate)
  - (Lagrangian) it involves counting triangles and heart-shaped domains in the regular fiber, with appropriate weights.
  - By iterating the mapping cone, we have formula for composition of twists
- We want to compare it with SW invariants (GW=SW)
- Extend to multi-sections (via relative Hilbert schemes?)

THANKS

Geometric Structures and Foliations Associated to $PSL_4\mathbb{R}$ Hitchin Representations

Alex Nolte
Rice University / Georgia Tech



This material is based upon work supported by the National Science Foundation under Grant No. 1842494.

$\mathrm{PSL}_n\mathbb{R}$ Hitchin components

$\mathrm{Hit}_n(S)$:

- Special component of $\mathrm{Hom}(\pi_1 S, \mathrm{PSL}_n\mathbb{R})/\mathrm{PSL}_n\mathbb{R}$
- Analogues of Teichmüller spaces

Question (Hitchin '92)

What geometric content does $\rho \in \mathrm{Hit}_n(S)$ have?

Guichard-Wienhard's work ('08, '11)

- Analogues of hyperbolic structures exist. Non-qualitative.

- Qualitative $n = 4$ theory:
  - $\rho \in \mathrm{Hit}_4(S)$ acts on $\Omega_\rho \subset \mathbb{RP}^3 \rightsquigarrow$ projective structure on T^1S
  - Ω_ρ has invariant foliations $\mathcal{F}, \mathcal{G}$ by convex sets in $\mathbb{RP}^2, \mathbb{RP}^1$
  - "Decorates" projective structure on T^1S
  - Characterizes Hitchin condition

Motivating Question

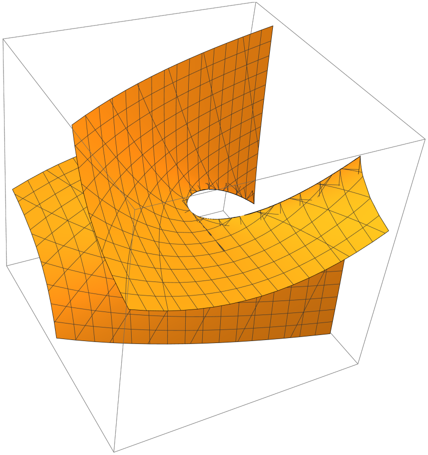
How rigid are the “decorations” of these projective structures?

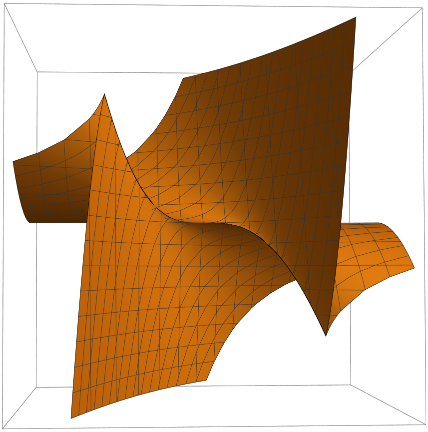
- Going the “other way” of Guichard-Wienhard’s ’08

Results (N)

- Classification of similar "decorations":
  - There are 2. (1 new). Analogue for other connected component
- Foliations of Ω_ρ by properly embedded properly convex domains:
  - In $\mathbb{RP}^1$s: exactly 2 group-invariant foliations (central theorem)
  - In $\mathbb{RP}^2$s: unique foliation
- Detailed basic structure of Ω_ρ
- Projective equivalences of Guichard-Wienhard's structures automatically preserve decorations
  - Answers question in Guichard-Wienhard '08

Fuchsian domain

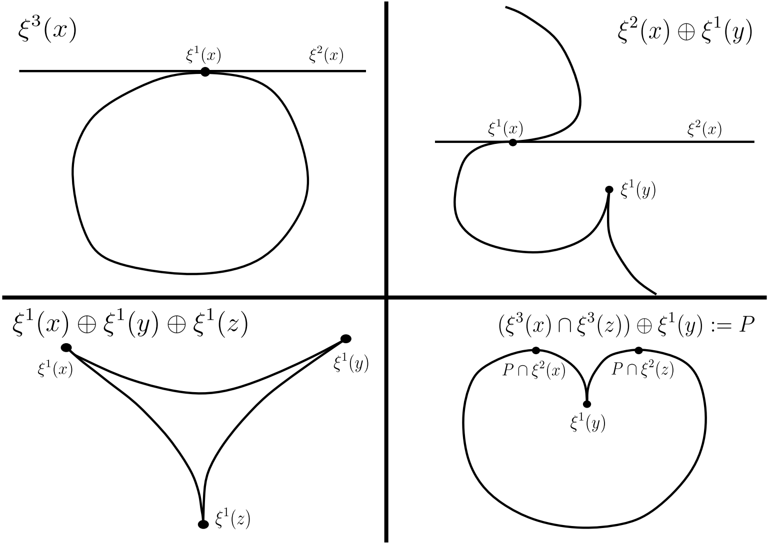




Not like $SL(3, \mathbb{R})$, where domain is convex!

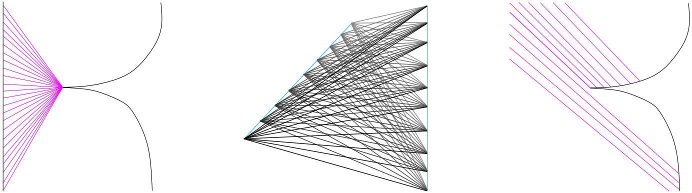
Sample Basic Structure Theorem (N)

$\rho \in \mathsf{Hit}_4(S)$. Frenét curve (ξ^1, ξ^2, ξ^3). Projective planes in $\mathbb{RP}^3$ and their qualitative intersections with $\partial\Omega_\rho$ have 4 forms:

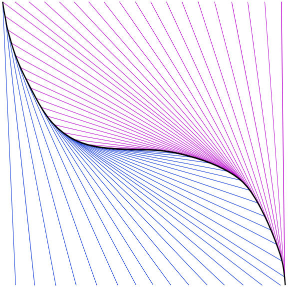


Geometry in Pf. of Only 2 Foliations by Segments

- Invariant foliation $\mathcal{F}$. Arrange for a leaf to stare straight at cusp
- Control these with qualitative geometry:



- Conclude from ruling's structure in what the staring leaf sees:



Crossing Number of Cable Knots (Joint with E. Kalfagianni)

Rob McConkey

Binghamton University

November 11th, 2023

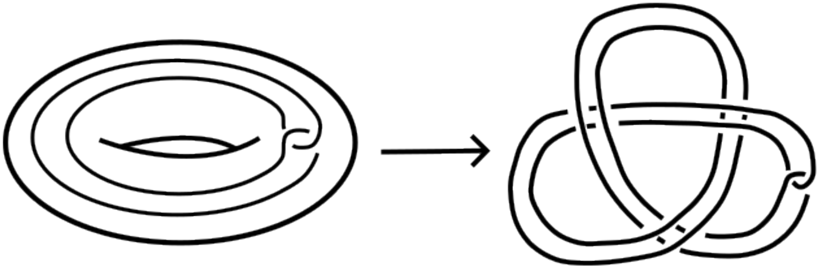
MICHIGAN STATE UNIVERSITY

Crossing Number

- Knot Theory is the study of knots and links.
- We study invariants of links to differentiate links, but also other topological objects which arise.
- One such invariant is the **crossing number**, which is the minimum number of crossing for a knot across all diagrams.
- We will refer to the crossing number of a knot K as $c(K)$.
- Despite being easy to define, the crossing number is notoriously intractable.

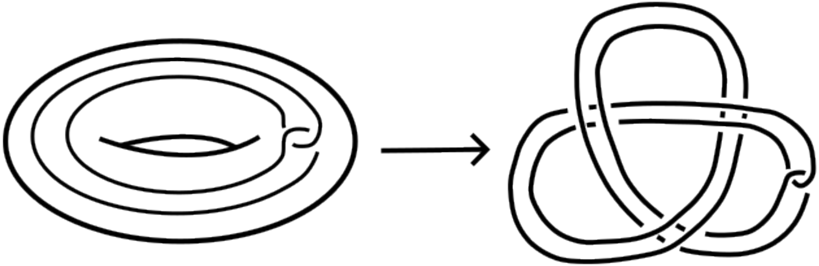
Satellite Knots

- To construct a **satellite knot** K start with a non-trivial knot K' inside of a torus T, then given a non-trivial knot C in S^3 we map T to a neighborhood of C.
- We will refer to C as the *companion knot* for K.



Satellite Knots

- Crossing number is not well understood for satellite and connect sums of knots.
- Remains an open conjecture whether or not $c(K) \geq c(C)$ where C is the companion knot for a satellite knot K.



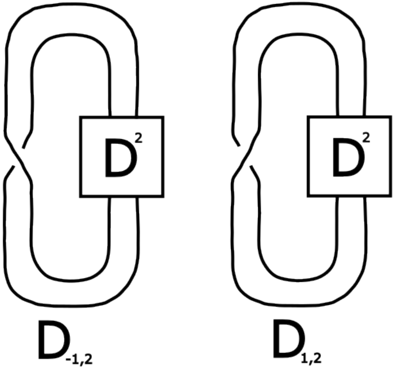
History

Theorem (Kalfagianni and Lee)

Let $W(K)$ be the untwisted whitehead double of a knot K. If K is adequate with writhe number zero, then $c(W(K)) = 4c(K) + 2$.

Satellite Knots

- We consider the satellite knots $K_{p,q}$ which is the (p,q)-cabling operation on a knot K.



Results

Theorem (Kalfagianni and M.)

For any adequate knot K with crossing number $c(K)$, and any coprime integers p, q, we have $c(K_{p,q}) \geq q^2 \cdot c(K) + 1$.

Corollary (Kalfagianni and M.)

Let K be an adequate knot with crossing number $c(K)$ and writhe number $w(K)$. If $p = 2w(K) \pm 1$, the $K_{p,2}$ is non-adequate and $c(K_{p,2}) = 4c(K) + 1$.

Thank You!

MICHIGAN STATE
UNIVERSITY

On the Hofer Zehnder Conjecture for Semipositive Symplectic Manifolds

joint work with Marcelo Atallah

Han Lou

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Tech Topology Conference

Georgia Institute of Technology

Dec 8 – Dec 10, 2023

Main Result (Atallah – L.) (M, ω) closed semipositive symplectic manifold

for definitions of Hamiltonian Floer homology and Gromov–Witten invariants.

$H_{ev}(M, \mathbb{Q}) \otimes \Lambda_{\mathbb{Q}, univ}$ is semisimple

$\sum a_i T^{\lambda_i}$, $a_i \in \mathbb{Q}$, $\lambda_i \in \mathbb{R}$ — generated by idempotents

Hamiltonian diffeomorphism ϕ has finitely many contractible 1-periodic orbits.

$\phi^t(x)$ where $\phi(x) = x$, $\phi^1 = \phi$

$\#\{\text{contractible 1-periodic orbits}\} > \dim_{\mathbb{Q}} H(M, \mathbb{Q})$

$\Longrightarrow$ ϕ has infinitely many contractible periodic orbits.

key: The coefficient field of Hamiltonian Floer homology has characteristic p.
There is an upper bound of Usher's boundary depth that is independent of p for sufficiently large p.

Basis of chain complex $\{\zeta_1, \cdots, \zeta_K, \eta_1, \cdots, \eta_B, \xi_1, \cdots, \xi_B\}$ such that $d(\zeta_i) = 0$, $d(\xi_i) = \eta_i$

$d(\zeta_i)$: $(-\infty, \ell(\zeta_i)]$

$d(\xi_i) = \eta_i$: $[\ell(\eta_i), \ell(\xi_i)]$ "longest finite bar length"

$\ell(\cdot)$: filtration.

Thank you for listening!