D. Connectedness

a topological space X is **disconnected** if there exists disjoint, non-empty, open sets U and V in X such that $X = U \cup V$

if no such sets exist, then X is **connected**

lemma 10:

> a topological space X is connected
>
> $\Leftrightarrow$
>
> the only sets in X that are both closed and open are X and $\varnothing$

Proof: ($\Rightarrow$) let U be an open and closed set in X

set $V = X - U$

note: V is also open and closed

if U **not** X or $\varnothing$, then V and U are non-empty

so X is disconnected, so we must have $U = X$ or $\varnothing$

($\Leftarrow$) essentially the same $\blacksquare$

Thm 11:

> $\mathbb{R}$ is connected

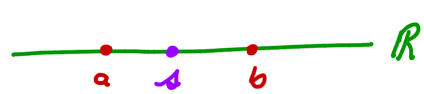
this is proven is analysis, so we skip the proof in class, but I include it here if you want a reminder how this goes.

lecture supplement not given in class

Proof: Suppose $\mathbb{R} = A \cup B$ with A and B open, disjoint, and non-empty

can assume $\exists\, a \in A$ and $b \in B$ with $a < b$

let $S = \{x \in A \text{ s.t. } x < b\}$



note: 1) $S \neq \varnothing$ since $a \in S$

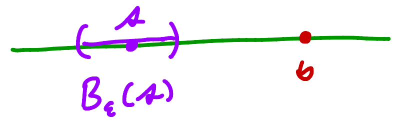
2) S is bounded above by b

so $\exists$ a least upper bound s for S

$s \in \mathbb{R} = A \cup B$ so $s \in A$ or $s \in B$

if $s \in A$, then $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(s) \subset A$ since A open

let $d = \|s - b\|$
and $\delta = \min\{\varepsilon/2, d/2\}$



note $\left.\begin{array}{l} s+\delta \in B_\varepsilon(s) \subset A \\ s+\delta < b \end{array}\right\} \Rightarrow s+\delta \in S$ $\otimes$ s l.u.b.

$\therefore$ s must be in B

so $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(s) \subset B$

by the definition of l.u.b. $\exists$ a sequence $\{s_i\}$ in S
such that $s_i \to s$
so for large i $s_i \in B_\varepsilon(s)$
but $s_i \in A$ so $A \cap \overbrace{(s-\varepsilon, s+\varepsilon)}^{B_\varepsilon(s) \subset B} \neq \emptyset$
this contradicts $A \cap B = \emptyset$

so $s \notin A$ or B $\otimes$

$\therefore$ such A and B don't exist and $\mathbb{R}$ is connected

Th$^{\underline{m}}$ 12:

> a subset of $\mathbb{R}$ is connected
> $\Leftrightarrow$
> it is an interval or $\emptyset$
> (i.e. $(a,b), [a,b), (a,b], [a,b], (-\infty,\infty)$,
> $(-\infty,b), (a,\infty), (-\infty,b], [a,\infty), \emptyset$)

Proof: ($\Leftarrow$) same argument as proof of Th$^{\underline{m}}$ 11

($\Rightarrow$) if A is non-empty and not an interval, then $\exists\, a,b \in A$ and $c \in \mathbb{R} - A$ such that $a < c < b$

now $[(-\infty, c) \cap A]$ and $[(c, \infty) \cap A]$ disconnect A

example: $[0,1]$ and $(0,1)$ are not homeomorphic

note: This seems obvioous but not easy to prove without connectedness!

to prove this note that for any $a \in (0,1)$, $(0,1)-\{a\}$ is not connected

if $[0,1]$ and $(0,1)$ were homeomorphic then $[0,1]$ would have this property too

indeed if $f:[0,1]\to(0,1)$ were a homeomorphism then for any $a\in[0,1]$ we know that $(0,1)-\{f(a)\}$ is disconnected

it's easy to see $f|_{[0,1]-\{a\}}:([0,1]-\{a\})\to((0,1)-\{f(a)\})$ is a homeomorphism, so $[0,1]-\{a\}$ is disconnected

but note $[0,1]-\{0\}=(0,1]$ is connected

Thm 13:

> The image of a connected set under a continuous map is connected

Proof: let X be connected and $f:X\to Y$ continuous

set $Z=f(X)\subset Y$ (with the subspace topology)

Claim: Z is connected

if not, $\exists$ non-empty, open, disjoint sets U and V in Z such that $Z=U\cup V$

we noted earlier that $f:X\to Z$ is continuous

so $X=f^{-1}(Z)=f^{-1}(U\cup V)=f^{-1}(U)\cup f^{-1}(V)$

and $f^{-1}(U), f^{-1}(V)$ are open and non-empty

moreover $f^{-1}(U)\cap f^{-1}(V)=f^{-1}(U\cap V)=f^{-1}(\emptyset)=\emptyset$

so X not connected $\otimes$

a space X is called <u>path connected</u> if for every pair of points $p,q\in X$, there is a continuous map $\gamma:[a,b]\to X$ called a path from p to q

such that $\gamma(a)=p$, $\gamma(b)=q$

Th$^{\underline{m}}$ 14:

$$\boxed{X \text{ path connected} \Rightarrow X \text{ connected}}$$

Proof:

we show that not connected $\Rightarrow$ not path connected

if X not connected, then $\exists$ non-empty, disjoint, open sets U and V s.t. $X = U \cup V$

let $p \in U$ and $q \in V$

if there were a path $\gamma : [a,b] \to X$ from p to q

then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would disconnect $[a,b]$ ✗ Th$^{\underline{m}}$ 12

so X is not path connected ▦

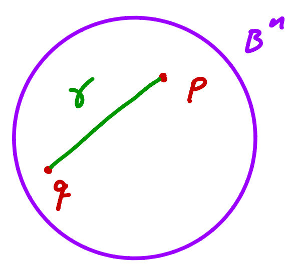
examples:

1) $B^n \subset \mathbb{R}^n$ (and $\mathbb{R}^n$) is connected

since it is path connected

indeed $p,q \in B^n$, then

$$\gamma(t) = (1-t)p + tq \quad \text{is a path } p \text{ to } q$$

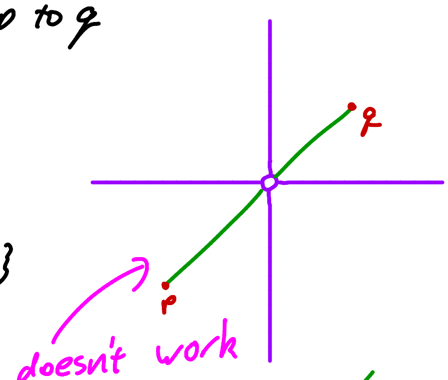


2) $\mathbb{R}^n - \{0\}$ is connected if $n \geq 2$

since it is path connected

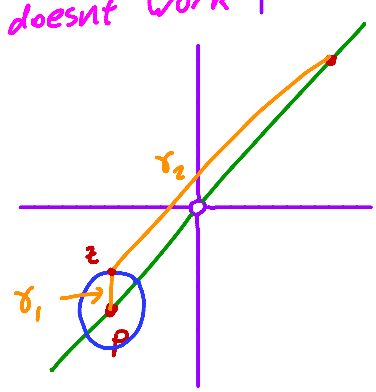
to see this, take any $p,q \in \mathbb{R}^n - \{0\}$

if line ℓ through p,q does not contain the origin, then $\gamma(t) = (1-t)p + tq$ works



if ℓ contains the origin O, then pick $\varepsilon > 0$ s.t. $O \notin B_\varepsilon(p)$

take any $z \in \partial(\overline{B_\varepsilon(p)}) - \ell$



let $\gamma_1(t) = (1-t)p + tz$
$\gamma_2(t) = (1-t)z + tq$ } paths in $\mathbb{R}^n - \{0\}$

then $\gamma(t) = \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t-1) & t \in [1/2, 1] \end{cases}$

is a path p to q (note γ continuous by Th^m 9)

Remark: This shows that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 1$
(since for any $x \in \mathbb{R}^1$, $\mathbb{R}^1 - \{x\}$ disconnected)
is $\mathbb{R}^2 \cong \mathbb{R}^3, \ldots$? no but harder (might do later)

3) $S^{n-1} \subset \mathbb{R}^n$ is connected for $n \geq 2$

by Th^m 13 since $g : (\mathbb{R}^n - \{0\}) \longrightarrow S^n$
$x \longmapsto \frac{x}{\|x\|}$

is continuous

E. Compactness

a collection $\{U_\alpha\}_{\alpha \in J}$ of subsets of X is called a cover of X
if $X = \bigcup_{\alpha \in J} U_\alpha$

a topological space X is called compact if every cover of X
by open sets has a finite subcover

i.e. if $\{U_\alpha\}_{\alpha \in J}$ a cover of X with each U_α open, then
$\exists J_0 \subset J$ a finite subset of J such that
$\{U_\alpha\}_{\alpha \in J_0}$ is a cover of X.

lemma 15:

$\boxed{\text{A closed subset of a compact space is compact}}$

Proof: let C be a closed subset of a compact set X

let $\{U_\alpha\}$ be an open cover of C (U_α open in C)

so $\exists$ sets $\tilde{U}_\alpha$ open in X s.t. $U_\alpha = \tilde{U}_\alpha \cap C$

let $U = X - C$

$\{\tilde{U}_\alpha\} \cup \{U\}$ is an open cover of X

so $\exists \{\tilde{U}_{\alpha_1}, \dots \tilde{U}_{\alpha_n}\} \cup \{U\}$ that also cover X

note $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covers C

lemma 16:

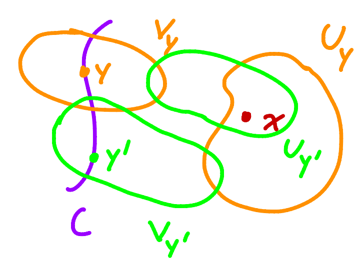
> a compact subset of a Hausdorff space is closed

Proof: let X be a Hausdorff space and $C \subset X$ a compact subspace

We show $X - C$ is open, and hence C is closed, by showing, for each $x \in X - C$, $\exists$ open set U_x such that $x \in U_x \subset X - C$, then (as before) $X - C = \bigcup_{x \in X - C} U_x$ is open

to this end, let $x \in X - C$

$\forall y \in C$, since X is Hausdorff, $\exists$ disjoint open sets V_y and U_y s.t. $x \in U_y$ and $y \in V_y$



Clearly $\{V_y\}_{y \in C}$ is an open cover of C

so $\exists y_1, \dots, y_n$ s.t. $\{V_{y_1}, \dots, V_{y_n}\}$ is a cover of C

let $U_x = U_{y_1} \cap \dots \cap U_{y_n}$

this is an open set and $U_x \cap (V_{y_1} \cup \dots \cup V_{y_n}) = \emptyset$

$\therefore U_x \cap C = \emptyset \Rightarrow x \in U_x \subset X - C$

lemma 17:

> the continuous image of a compact space is compact

Proof: let $f: X \to Y$ be continuous and X compact let $\{U_\alpha\}$ be an open cover of $f(X)$

so $\{f^{-1}(U_\alpha)\}$ an open cover of X

$\therefore \exists$ a finite subcover $\{f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})\}$

so $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a cover of $f(X)$ ▦

Th$^{\underline{m}}$ 18:

> let $f: X \to Y$ be a continuous bijection
> If X is compact and Y is Hausdorff
> then f is a homeomorphism

this th$^{\underline{m}}$ is <u>very</u> helpful with quotient spaces!

Proof: we need to see $f^{-1}: Y \to X$ is continuous

i.e. by Th$^{\underline{m}}$ 7, $\forall$ closed sets C in X we need to see

$$(f^{-1})^{-1}(C) = f(C) \text{ is closed in } Y$$

but C closed in $X \Rightarrow C$ is compact by lemma 15

$\Rightarrow f(C)$ is compact by lemma 17

$\Rightarrow f(C)$ is closed by lemma 16 ▦

Th$^{\underline{m}}$ 19:

> $[0,1]$ is compact

this is proven is analysis, so we skip the proof in class, but I include it here if you want a reminder how this goes.

lecture suppliment not given in class

Proof: let $\{U_\alpha\}$ be an open cover of $[0,1]$

let $C = \{x \in [0,1]$ s.t. $[0,x]$ is contained in a finite subcollection of $\{U_\alpha\}\}$

Clearly $0 \in C$

We show C is open and closed in $[0,1]$

$\therefore$ since C is connected lemma 10 $\Rightarrow C = [0,1]$ and we are done!

<u>C open:</u> if $x \in C$ then let $U_{\alpha_1}, \dots, U_{\alpha_n}$ be sets covering $[0,x]$

$\exists j$ s.t. $x \in U_{\alpha_j}$

U_{α_j} open $\Rightarrow \exists \delta > 0$ s.t. $(x-\delta, x+\delta) \subset U_{\alpha_j}$

so $(x-\delta, x+\delta) \subset C$ ✓

<u>C closed:</u> if x is a limit point of C,

then let U_{α_0} be set containing x.

so $\exists (a,b)$ s.t. $x \in (a,b) \subset U_{\alpha_0}$

since x a limit point of C, we know $((a,b)-\{x\}) \cap C \neq \emptyset$

let $y \in ((a,b)-\{x\}) \cap C$, so $[y,x]$ (or $[x,y]$) $\subset U_{\alpha_0}$

now $y \in C \Rightarrow \exists \{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ s.t. $[0,y] \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$

$\therefore U_{\alpha_0}, \dots, U_{\alpha_n}$ covers $[0,x]$, so $x \in C$ and C closed

<u>exercise:</u> The product of 2 compact spaces is compact

<u>Hint:</u> this is hard. start by interpreting compactness in terms of <u>basic</u> open sets

<u>Th$^{\underline{m}}$ 20 (Heine-Borel):</u>

> a subset of $\mathbb{R}^n$ is compact
> $\Leftrightarrow$
> it is closed and bounded

<u>Proof:</u> ($\Leftarrow$) if $C \subset \mathbb{R}^n$ is closed and bounded

then bounded $\Rightarrow \exists R$ s.t. $C \subset [-R,R]^n$

but $[-R,R]$ is homeomorphic to $[0,1]$ (What's the homeo.?)

so $[-R,R]$ is compact and thus so in $[-R,R]^n$ by exercise

now C closed in a compact set $\Rightarrow C$ compact (lemma 15)

($\Rightarrow$) C a compact set in $\mathbb{R}^n \Rightarrow C$ closed by lemma 16 (since $\mathbb{R}^n$ is Hausdorff by Th$^{\underline{m}}$ 6)

C is bounded because if not, there would be a sequence $\{p_n\}$ in C s.t. $|p_n| > n \quad \forall n$

clearly no subsequence of $\{p_n\}$ can converge

this contradicts the following result

Th^{m} 21:

If X is a 1^{st} countable space, then

X compact $\Rightarrow$ $\underbrace{\text{every sequence in } X \text{ has a convergent subsequence}}$ called sequentially compact

If X a metric space, then $\Leftrightarrow$

this proof is quite involved, we only prove ($\Leftarrow$) for metric spaces

it uses the following lemma that we will need later

lemma 22 (Lebesgue number lemma):

let (X,d) be a sequentially compact metric space

If $\mathcal{C}$ is an open cover of X, then $\exists\, \delta > 0$ (called Lebesgue number)

such that for every set $S \subset X$ with $\operatorname{diam}(S) < \delta$

$\exists$ a set $U \in \mathcal{C}$ such that $S \subset U$

here $\operatorname{diam}(S) = \sup\{d(x,y) \mid x, y \in S\}$

Proof: given (X,d) a metric space and $\mathcal{C}$ an open cover of X

We show that if no such $\delta > 0$ exist, then X is not sequentially cpt.

if no such δ exists then $\forall n > 0$ let C_n be a set with

1) $\operatorname{diam} C_n < \frac{1}{n}$ and

2) C_n not in any open set in $\mathcal{C}$

take a point $x_n \in C_n$ for each n

Claim: $\{x_n\}$ has no convergent subsequence

to see this, suppose $\{x_{n_i}\}$ is a convergent subsequence

and $x_{n_i} \to x$

note $x \in U$ for some $U \in \mathcal{C}$

so $\exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset U$

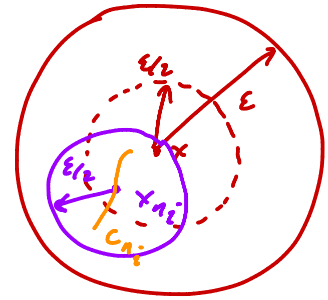
and $\exists I>0$ such that

$$d(x_{n_i}, x) < \varepsilon/2, \text{ and}$$

$$\frac{1}{n_i} < \varepsilon/2 \qquad \forall i \geq I$$

so $C_{n_i} \subset B_{1/n_i}(x_{n_i}) \subset B_\varepsilon(x) \subset U$ ⊗

$\therefore \{x_n\}$ has no convergent subsequence ▦



Proof of Th^m 21 ($\Leftarrow$):

Claim: If X is sequentially compact, then $\forall \varepsilon>0$, X can be covered by finitely many ε-balls

Pf: if not, let $x_1 \in X$ be any point

$B_\varepsilon(x_1)$ does not cover X

let $x_2 \in X - B_\varepsilon(x_1)$

given $x_1, \dots, x_n$ such that $B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$ doesn't cover X

take $x_{n+1} \in X - (B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n))$

note: $d(x_i, x_j) \geq \varepsilon \quad \forall i \neq j$

$\{x_i\}$ can have no convergent subsequence (since all balls of radius $\varepsilon/2$ can have at most one x_i)

$\therefore X$ is not sequentially compact ⊗ ✓

now let $\mathcal{C}$ be an open cover of X

by lemma 22, $\exists$ a Lebesgue number $\delta>0$ for $\mathcal{C}$

find a cover of X by finitely many balls of radius $\delta/3$

each ball has diam $= \frac{2\delta}{3} < \delta$

so each ball in some U_i in $\mathcal{C}$

choose one such U_i for each ball

this is a finite subcover of $\mathcal{C}$ ▦

F. Quotient Spaces

Quotient spaces are a great way to build interesting and complicated spaces, and construct maps between them.

Let X be a topological space,
Y a set, and
$f: X \to Y$ a surjective function

The collection

$$\boxed{\mathcal{T}_f = \{U \subset Y \mid f^{-1}(U) \text{ open in } X\}}$$

is called the quotient topology on Y

exercise: Show $\mathcal{T}_f$ is a topology on Y

Thm 23: Let X and Y be topological spaces, and $f: X \to Y$ a surjective map. Then the quotient topology $\mathcal{T}_f$ on Y agrees with the given topology on Y
$\iff$
(*) U open in Y iff $f^{-1}(U)$ open in X

A surjective map $f: X \to Y$ satisfying (*) is called a quotient map.
Hopefully it is clear a quotient map is continuous.

Proof: ($\Rightarrow$) U open in $Y \overset{\text{hypothesis}}{\iff} U \in \mathcal{T}_f \underset{\text{def}^{\text{n}}\text{ of } \mathcal{T}_f}{\iff} f^{-1}(U)$ open in X

so (*) true

($\Leftarrow$) $U \in \mathcal{T}_f \underset{\text{def}^{\text{n}}\text{ of } \mathcal{T}_f}{\iff} f^{-1}(U)$ open in $X \overset{\text{by (*)}}{\iff} U$ open in Y ▦

lemma 24:

> let $f: X \to Y$ be a continuous surjection
> If f is a closed map or an open map, then
> f is a quotient map.

here f being closed / open means that for any closed/open set A in X, $f(A)$ is closed/open in Y

Proof: Assume f is an open map

If U open in Y, then $f^{-1}(U)$ open in X since X is continuous.

If U any set in Y and $f^{-1}(U)$ open in X, then

$$f(f^{-1}(U)) = U \text{ is open in } Y \text{ since } f \text{ is an open map}$$

since f is surjective!

so $f^{-1}(U)$ open in $X \Leftrightarrow U$ open in Y.

thus f is a quotient map

similar argument for f a closed map (exercise) ∎

example:

let $X = [0,1]$

$Y = S^1$ = unit circle in $\mathbb{R}^2$ with the subspace topology

$f: X \to Y: t \longmapsto (\cos 2\pi t, \sin 2\pi t)$

f is continuous (we know cos, sin are continuous from calculus now done by discussion of continuous maps to products)

f is clearly surjective

Claim: f is a quotient map

to see this we show f is a closed map

note: X is compact (Thm 19)

Y is Hausdorff (since it is a metric space, Thm 6)

A closed in $X \Rightarrow A$ compact (lemma 15)

$\Rightarrow f(A)$ compact (by lemma 17)

$\Rightarrow f(A)$ closed (by lemma 16)

Intuition: given $f: X \to Y$ with Y having the quotient topology we think of Y as "constructed" from X by identifying points

this is clear in this example but quotient maps make this rigorous

Thm 25:

Given a quotient map $f: X \to Y$ and another space Z

Then

$$g: Y \to Z \text{ is continuous}$$
$$\Leftrightarrow$$
$$g \circ f : X \to Z \text{ is continuous}$$

More Intuition: If $f: X \to Y$ is a quotient map, then studying continuous functions on Y is equivalent to studying continuous functions on X that are constant on the preimage of points in Y

example: continuous functions on S^1 are the same as continuous functions on $[0,1]$ that map 0 and 1 to same point! — we use this all the time!

note: $[0,1]$ is a "simpler" space than S^1 so quotient maps allow us to study continuous functions on S^1 by looking at such functions on a "simpler" space

Proof: $(\Rightarrow)$ composition of continuous functions is continuous

($\Leftarrow$) If U is open in Z, then $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X

by definition of the quotient topology, $g^{-1}(U)$ open in Y

so g is continuous ▦

We now make precise the idea of "gluing spaces together from simple pieces"

let X be a topological space

a decomposition $\mathcal{D}$ of X is a collection of disjoint subsets of X whose union is X

let $p : X \to \mathcal{D} : x \mapsto$ set in $\mathcal{D}$ containing x

this is clearly a surjective map

so we give $\mathcal{D}$ the quotient topology $\mathcal{T}_p$

(i.e. $\mathcal{S} \subset \mathcal{D}$ is open $\Leftrightarrow \bigcup_{s\in\mathcal{S}} s$ is open in X)

$\mathcal{D}$ with this topology is called a decomposition space, or quotient space, of X

you should think of $\mathcal{D}$ as X where all the sets $S \in \mathcal{D}$ have been collapsed to points

example:

let $X = [0,1]$

$\mathcal{D} = \{ \{x\} \mid x \in (0,1)\} \cup \{\{0,1\}\}$

each point on the interior of $[0,1]$ is in its own set in $\mathcal{D}$

the only set in $\mathcal{D}$ with more than one point is $\{0,1\}$

so $\mathcal{D}$ is $[0,1]$ with $0,1$ identified to a single point

let $p : [0,1] \to \mathcal{D}$ be the quotient map

not surprisingly $\mathcal{D}$ is homeomorphic to S^1

Proof:

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "p"'] & S^1 \\ \mathcal{D} \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$f(t) = (\cos 2\pi t, \sin 2\pi t)$

clearly $\exists\, \bar{f} : \mathcal{D} \to S^1$

by Th^{m} 25 $\bar{f}$ continuous

also clearly $\bar{f}$ a bijection

now $\bar{f}$ a homeomorphism by Th$^{\underline{m}}$ 18

(since S^1 Hausdorff and $\mathcal{D}$ is compact since it is the continuous image of $[0,1]$)

so we have rigorously seen S^1 is just $[0,1]$ with 0 and 1 "glued together"

generalizing this we have

Th$^{\underline{m}}$ 26:

let $f: X \to Z$ be a continuous surjection

set $\mathcal{D} = \{ f^{-1}(a) : a \in Z \}$

give $\mathcal{D}$ the quotient topology

The map f induces a continuous bijection $g: \mathcal{D} \to Z$

Moreover, g is a homeomorphism

$\Leftrightarrow$

f is a quotient map

Proof: Clearly f induces a bijection $g: \mathcal{D} \to Z$

(for any $x \in p^{-1}(s)$, set $g(s) = f(x)$)

let $p: X \to \mathcal{D}$ be the quotient map from above

by Th$^{\underline{m}}$ 25 g is continuous since $g \circ p = f$ is

($\Rightarrow$) if g a homeomorphism we know

U open in $Z \Leftrightarrow g^{-1}(U)$ open in $\mathcal{D}$

$\Leftrightarrow f^{-1}(U) = p^{-1}(g^{-1}(U))$ open in X

($\Leftarrow$) assuming f is a quotient map we see

$\mathcal{S} \subset \mathcal{D}$ open in $\mathcal{D} \Leftrightarrow p^{-1}(\mathcal{S})$ open in X

but $p^{-1}(\mathcal{S}) = f^{-1}(g(\mathcal{S}))$

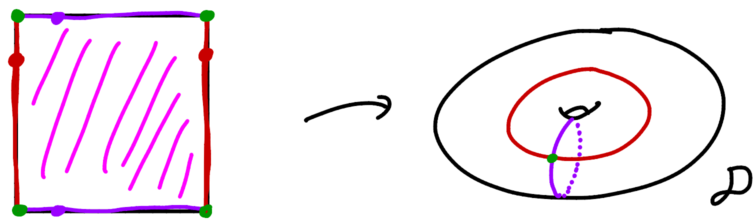
so $\mathcal{S}$ open in $\mathcal{D} \Leftrightarrow g(\mathcal{S})$ open in Z

thus g^{-1} is continuous and hence g is a homeomorphism ▦

examples:

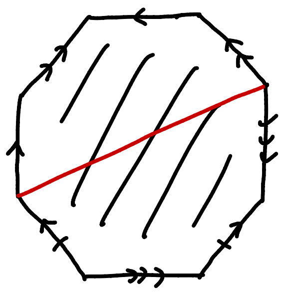
1) $X = [0,1]\times[0,1]$

$\mathcal{D} = \{ \{(x,y)\} \mid 0<x,y<1\} \cup \{ \{(1,y),(0,y)\} \mid 0<y<1\}$
$\cup \{ \{(x,1),(x,0)\} \mid 0<x<1\} \cup \{ \{(0,0),(0,1),(1,0),(1,1)\}\}$

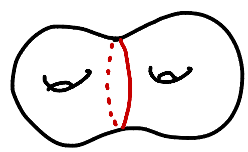


exercise: $\mathcal{D} \cong S^1\times S^1$
homeomorphic

2) $X =$

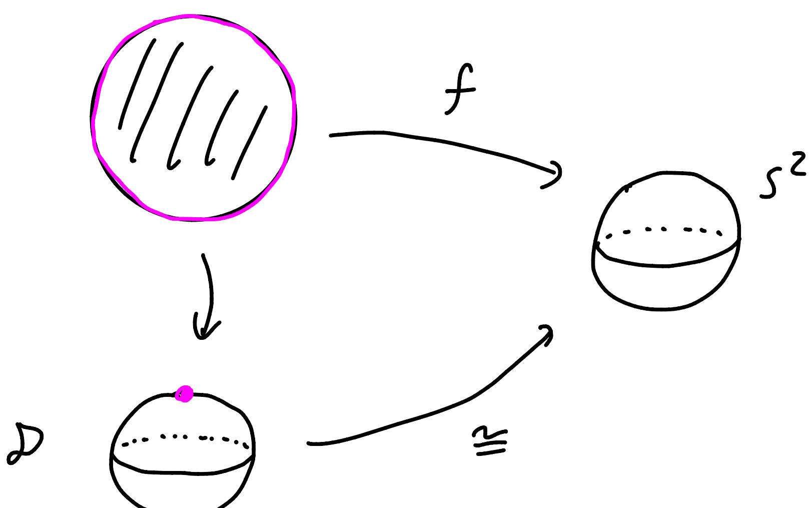


$\mathcal{D} \cong$



3) let $X = D^2 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\} = \{(r,\theta)\in\mathbb{R}^2 : r\le 1\}$

$\mathcal{D} = \{ \{(x,y)\} : x^2+y^2<1\} \cup \{ \underbrace{\{(x,y) : x^2+y^2=1\}}_{S^1} \}$



So S^2 is just D^2 with boundary crushed to a point!

unit sphere

Proof: define $f : X \to S^2 : (r,\theta) \mapsto (\sin\pi r\cos\theta, \sin\pi r\sin\theta, \cos\pi r)$
can check f is a continuous closed surjection

so f is a quotient map and f induces $\mathcal{D}$

similarly check S^n is D^n with ∂D^n collapsed to point.

4) $X = S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + |z_1|^2 + \dots + |z_n|^2 = 1\}$

we say $z, w \in S^{2n+1}$ are equivalent if $\exists \lambda \in S^1$ such that $\lambda z = w$

unit S^1 in $\mathbb{C}$

(i.e. $(\lambda z_0, \dots, \lambda z_n) = (w_0, \dots, w_n)$)

let $\mathcal{D} = \{$equivalence classes of points in $S^{2n+1}\}$

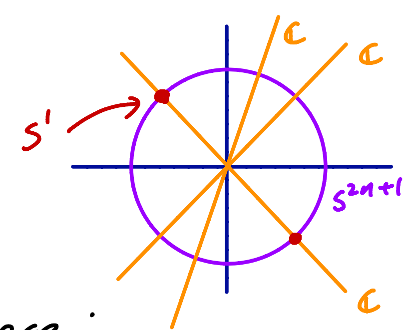
denote this by S^{2n+1}/S^1 and give it the quotient topology

another way to think of S^{2n+1}/S^1

let $\mathbb{C}P^n = \{$complex lines in $\mathbb{C}^{n+1}\}$

one dimensional linear subspaces

<u>note</u>: each complex line intersects S^{2n+1} in an S^1



<u>exercise</u>: Show there is a one-to-one correspondence between S^{2n+1}/S^1 and $\mathbb{C}P^n$

(we can use this to put a topology on $\mathbb{C}P^n$)

we call $\mathbb{C}P^n$ <u>complex projective space</u>

<u>hard exercise</u>: $\mathbb{C}P^1$ is homeomorphic to S^2

<u>Hint</u>: consider map $S^3 \xrightarrow{h} S^2$

$$(z_0, z_1) \longmapsto (2 z_0 \bar{z}_1, |z_0|^2 - |z_1|^2)$$

here $S^2 \subset \mathbb{C} \times \mathbb{R}$

h is called the <u>Hopf map</u> and is a famous "fibration"

5) given spaces Y and Z,

a subspace A of Y, and

a continuous map $f: A \to Z$

consider the following decomposition of $Y \cup Z$:

the non-trivial elements of $\mathcal{D}$ are

sets that contain more than one point

$$\{\{a, f(a)\} \mid a \in A\}$$

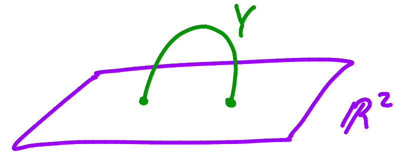
we say that $\mathcal{D}$ is the space obtained by <u>gluing Y to Z along A</u> (<u>via f</u>)

denote it by $Y \cup_A Z$, or better $Y \cup_f Z$

e.g. a) $Y = [0,1]$ $\quad A = \{0,1\}$ $\quad f(0) = (0,0)$ $\quad f(1) = (1,0)$

$Z = \mathbb{R}^2$

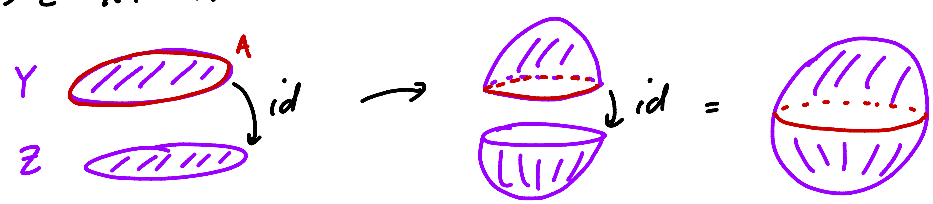
$Y \cup_f Z$



b) $Y = Z = D^2$

$A = S^1 = \text{boundary } D^2$

$f: A \to Z: x \mapsto x$



<u>exercise</u>: $Y \cup_f Z$ homeomorphic to S^2

c) $Y = D^4 = \text{unit disk in } \mathbb{R}^4 = \mathbb{C}^2$

$A = \text{boundary of } D^4 = S^3$

$Z = S^2$

$f: A \to Z$ the Hopf map from above

<u>hard exercise</u>: $Y \cup_f Z$ homeomorphic to $\mathbb{C}P^2$