Morse lemma

we will show

lemma:

given a critical point $p \in W^n$ of $f: W^n \to \mathbb{R}$ of index k,
$\exists$ coordinates $\phi: U \to V$ about p s.t. $\phi(0) = p$ and

$$f \circ \phi(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

so upto change of coordinates, functions with such critical points all look the same

warmup 1D:

If $f: \mathbb{R} \to \mathbb{R}$ has a critical point at 0, and ℓ is first pos. integer s.t. $\frac{\partial f}{\partial x^\ell}(0) \neq 0$, then
$\exists$ "coordinates" $\phi: U \to V$ for 0 in $\mathbb{R}$
s.t. $\phi(0) = 0$ and $f \circ \phi(y) = f(0) \pm y^\ell$

so in 1D "all" critical points have "normal form"

Proof:

note $f(x) - f(0) = \int_0^1 \frac{df}{\partial t_1}(xt_1)\, dt_1$

$$= \int_0^1 \frac{df}{dx}(xt_1)\, x\, dt_1 = x \int_0^1 \frac{df}{dx}(xt_1) - \frac{df}{dx}(0)\, dt_1$$

$$= x \int_0^1 \int_0^1 \frac{d^2 f}{dt_2 dx}(xt_1 t_2)\, dt_2 dt_1$$

$$= x^2 \int_0^1 \int_0^1 \frac{d^2 f}{dx^2}(xt_1 t_2)\, dt_2 dt_1$$

$$\vdots$$

$$= x^\ell \underbrace{\int_0^1 \cdots \int_0^1 \frac{d^\ell f}{dx^\ell}(xt_\ell \cdots t_1)\, dt_\ell \cdots dt_1}_{g(x)}$$

$$= x^\ell g(x)$$

and $g(0) = \frac{1}{\ell!} f^{(\ell)}(0) \neq 0$

let ε be sign of $g(0)$

so $\varepsilon g(x) > 0$ for small x

$\therefore$ define $\psi(x) = \varepsilon x \,(g(x))^{1/\ell}$

<u>now</u>: 1) $\psi'(0) = \left(\varepsilon (g(x))^{1/\ell} + \varepsilon x \frac{1}{\ell} (g(x))^{1-\ell} g'(x)\right)\Big|_{x=0}$

$= \varepsilon\, g(0)^{1/\ell} > 0$

$\therefore$ ψ is a local diffeo from, say, $\underset{x}{V} \to \underset{y}{U}$

2) $\psi(0) = 0$

3) $\psi(x)^\ell = \varepsilon^\ell x^\ell g(x) = \varepsilon^\ell \left(f(x) - f(0)\right)$

so set $\phi = \psi^{-1}$ and notice that

$$f \circ \phi(y) = \varepsilon^\ell \psi(\phi(y))^\ell + f(0)$$

$$= f(0) + \varepsilon^\ell y^\ell \quad \square$$

? think abt sign!

<u>exercise</u>: Proove that for a non-critical point p of $f: W \to \mathbb{R}$
$\exists$ a coordinate chart $\phi: U \to V$ around p s.t.

$$f \circ \phi(x_1, \dots, x_n) = x_1$$

In general, higher order critical points can't (?) always be put in normal form, but for non-degenerate ones we can!

<u>Proof of Morse lemma 1</u>:

take any coordinate chart ψ sending 0 to p, writing f in these coordinants (i.e. $f \circ \psi$ but we just write f)

$$\left(\frac{\partial^2}{\partial x_i \partial x_j} f\Big|_0\right)$$

can be diagonalized by an appropriate choice of basis.

If $L:\mathbb{R}^n \to \mathbb{R}^n$ is this change of basis map then replace ψ by $\psi \circ B$ (still call it ψ), then f in these coords

$$A = \left(\frac{\partial^2}{\partial x_i \partial x_j} f\Big|_0\right) = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & +1 & & \\ & & & & \ddots & \\ & & & & & +1 \end{pmatrix}$$

to further normalize $f \circ \psi$ we write in is a special way

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx)\, dt \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, x_i \, dt \\ &= \sum_{i,j=1}^n \underbrace{\int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(st\, x)\, dt\, ds}_{b_{ij}(x)}\; x_i x_j \\ &= \sum_{i,j=1}^n b_{ij}(x)\, x_i x_j \quad = x^T B_x x \end{aligned}$$

where $B_x = (b_{ij}(x))$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

by replacing b_{ij} with $\frac{1}{2}(b_{ij} + b_{ji})$ we can assume B_x is symmetric

<u>note</u>: $B_0 = A$

<u>Claim</u>: we can find an invertable matrix Q_x depending on x such that $Q_x^T B_x Q_x = A$

(e.g. $Q_0 = I$)

then set $\psi(x) = Q_x^{-1} x$ and note

$$d\psi_0 = Q_0^{-1} = I$$

so ψ is a local diffeo (coord. chart!)

$$\psi : V \to U$$

and

$$\begin{aligned} f(x) = f(0) + x^t B_x x &= f(0) + x^t (Q_x^{-1})^t Q_x^t B_x Q_x Q_x^{-1} x \\ &= f(0) + (Q_x^{-1} x)^t A (Q_x^{-1} x) = f(0) + \psi(x)^t A\, \psi(x) \end{aligned}$$

$\therefore$ if we set $\phi = \psi^{-1}$, then

$$
\begin{aligned}
f \circ \phi(y) &= f(0) + \psi(\phi(y))^t A \, \psi(\phi(y)) \\
&= f(x) + y^t A y \\
&= f(x) - y_1^2 \cdots - y_h^2 + y_{h+1}^2 + \cdots y_n
\end{aligned}
$$

Proof of Claim: We show, given $A = \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix}$

$\exists$ a nbhd N of A in space of $n \times n$ matricies

and a smooth map $P : N \to GL(n, \mathbb{R})$

s.t. $P(A) = I$ and $P(B)^t B \, P(B) = A \quad \forall B \in N$

to see this suppose B is close enough to A so that

$b_{11} \neq 0$ and has the same sign as a_{11}

consider the map $T : \mathbb{R}^n \to \mathbb{R}^n$, $y \longmapsto x$ given by

$$
T = \sqrt{\frac{|a_{11}|}{|b_{11}|}} \begin{pmatrix} 1 & -b_{12}/b_{11} & \cdots & -b_{1n}/b_{11} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}
$$

note:

$$
T^t B T = \sqrt{\frac{|a_{11}|}{|b_{11}|}} \, T^t \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{12} & & & \\ \vdots & & B' & \\ b_{1n} & & & \end{pmatrix}
$$

$$
= \frac{|a_{11}|}{|b_{11}|} \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B'' & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B''' & \\ 0 & & & \end{pmatrix}
$$

and if $b_{11} \sim \pm 1$, $b_{1i} \sim 0$ then B''' close

to A_{11} so we can induct

minor

Proof of Morse lemma 2:

example of a "Moser trick" let critical point be 0

let $A = \text{Hess}\, f_0$ and $Q = \frac{1}{2} x^t A x$ (note $\text{Hess}\, Q_0 = A$)

consider $f_t = (1-t)\, Q + t f$

we want to find a family of diffeomorphisms ϕ_t (defined near 0) so that

$$\phi_0 = id \quad \text{and}$$

$$(*) \quad \phi_t^* f_t = Q$$

if we find the f_t, then f_1 is the desired coordinate change

we look for f_t as the flow of a vector field v_t

that is $\begin{cases} \frac{d}{dt}(\phi_t(x)) = v_t(\phi_t(x)) \\ \phi_t = id \end{cases}$

differentiating $(*)$ w.r.t. t gives

$$0 = \phi_t^*\left(\mathcal{L}_{v_t} f_t + \frac{\partial f_t}{\partial t}\right) = \phi_t^*\left(df_t(v_t) + f - Q\right)$$

so we want v_t to satisfy:

$$(**) \quad df_t(v_t) = \underbrace{Q - f}_{\text{denote by } g}$$

(note: Q, f, df_t are given)

note $g(0) = Q(0) - f(0) = 0$ and

$$dg_0 = dQ_0 - df_0 = 0$$

as before we can write $g(x)$ as

$$g(x) = x^T G_x x \quad \text{for some matrix } G_x \text{ depending on } x$$

note: $G_0 = \text{Hess}\, f_0$ so G_0 invertable as is G_x for small x

now consider

$$
\begin{aligned}
(df_t)_x(v_t) &= (df_t)_x(v_t) - (df_t)_0(v_t) \\
&= \int_0^1 \frac{d}{ds} (df_t)_{sx}(v_t)\, ds \\
&= \int_0^1 \frac{d}{ds} \left(\sum (v_t)_i \frac{\partial f_t}{\partial x_i}(sx) \right) ds \\
&\overset{\text{as above}}{=} \sum_{i,j} (v_t)_i\, x_j \underbrace{\int_0^1 \frac{\partial^2 f_t}{\partial x_i \partial x_j}(sx)\, ds}_{(B_x)_{ij}} \\
&= \sum_{i,j} (v_t)_i\, x_j\, (B_x)_{ij}.
\end{aligned}
$$

so $(df_t)_x(v_t) = x^T B_x v_t$

note $B_0 = \left(\frac{\partial^2 f_t}{\partial x_i \partial x_j}(0) \right)$ so is invertable $\therefore B_x$ invertable near 0

now $(**)$ becomes

$$x^T B_x v_t = x^T G_x x$$

Set $v_t = B_x^{-1} G_x x$

and note v_t satisfies $(**)$ and hence the flow ϕ_t of v_t

satisfies $(*)$