SYMPLECTIC CONVEXITY IN LOW DIMENSIONAL TOPOLOGY

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ABSTRACT. In this paper we will survey the the various forms of convexity in symplectic geometry, paying particular attention to applications of convexity in low dimensional topology.

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1. INTRODUCTION

For quite some time it has been known that there is a relation between the topology of a manifold and the geometry it supports. In recent years there has been much work to indicate that low dimensional topology is closely related to symplectic and contact geometry. The advent of Seiberg-Witten theory (see [D]) has done much to strengthen these ties. In particular, the work of Taubes [T] has shown that symplectic manifolds are basic building blocks in 4-dimensional topology (In the sense that a closed minimal simply connected symplectic 4-manifold is irreducible). Recently there has been a great deal of work constructing symplectic manifolds. Most of these methods have involved symplectic convexity in some way. In 3-dimensions *tight* contact structures also have something to say about topology. For example Eliashberg [E4] gave a proof of Cerf's theorem using contact geometry. Cerf's theorem says that any diffeomorphism of S^3 extends over the 4-ball. The structure of contact 3-manifolds is influenced by symplectic convexity: it can be used to construct and distinguish contact structures. This paper is a survey of various forms and uses of convexity in symplectic geometry, paying particular attention to what happens in low dimensions.

We begin in Section 2 by discussing the strongest form of convexity, that of ω-convexity. It is ω-convexity that is a necessary component in most cut-and-paste constructions of symplectic manifolds; however, ω-convexity is not sufficient for these constructions. Indeed, this lack of sufficiency can be exploited to understand contact manifolds better (see [LM] or page 19 below). In Section 3 we give several constructions of ω-convex hypersurfaces and then review several constructions of symplectic manifolds which can be interpreted in terms of ω-convexity. Specifically, we show how the symplectic normal connected sum operation of Gromov [Gr2], used by Gompf [G1] and McCarthy-Wolfson [MW1], follows from ω-convexity. We also see how McCarthy and Wolfson's gluing along ω-compatible hypersurfaces [MW2] can (usually) be seen as an ω-convex gluing. In Sections 4 and 5 we consider three weaker forms of convexity and discuss their relation to ω-convexity. (See Figure 1 for these relationships and the appropriate sections for the relevant definitions.) Unfortunately, these weaker forms of convexity can rarely be used to perform a symplectic cut-and-paste; however, they are useful in constructing tight contact structures. This is the content of Section 6, where we show how symplectic convexity has quite a lot to say about contact geometry in dimension three. In Section 7 we discuss some questions and conjectures.

$$\begin{tikzcd}
\omega\text{-convex} \arrow[r, leftrightarrow] & \text{dominate} \arrow[r] \arrow[d] & \text{weak symplectic filling} \\
& \text{pseudo-convex} \arrow[ur] &
\end{tikzcd}$$

(**a**)

$$\begin{tikzcd}
\omega\text{-convex} \arrow[r] & \text{dominate} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{weak symplectic filling} \\
& \text{pseudo-convex} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

(**b**)

FIGURE 1. Relation between the notions of convexity in dimensions above four (**a**) and in dimension four (**b**).

This paper is intended for a topologically minded reader who might not be an expert in symplectic geometry. Thus we have tried to include complete proofs of most results. A notable exception is in Section 3 where many of the proofs are only indicated as their inclusion would have greatly increased the length of this paper. In addition, good proofs of these results appear elsewhere. Though we do assume familiarity with symplectic geometry we have included an appendix to review a few basic facts.

I would like to acknowledge my indebtedness to the paper [EG]: the first paper to treat symplectic convexity as a subject unto itself. I would also like to thank Bob Gompf who has taught me a great deal about symplectic geometry and four dimensional topology. Lastly I would like to thank Rob Ghrist and Margaret Symington for encouragement during the writing of this paper and many excellent comments and suggestions on the first draft of this paper.

2. Strong Convexity

Let (X, ω) be a symplectic manifold. Given a vector field v we can ask how ω changes along v. We will consider the situation when

$$L_v\,\omega = c\,\omega, \tag{1}$$

where $L_v\omega$ is the Lie derivative of ω in the direction of v and $c \in \mathbb{R}$. If $c = 0$, then v is locally Hamiltonian (i.e. about any point we can find a function H such that $dH = \iota_v\,\omega$). If $c \neq 0$, then we can renormalize v to obtain a vector field $v' = \frac{1}{c}\,v$ so that

$$L_{v'}\,\omega = \frac{1}{c} L_v\,\omega = \omega. \tag{2}$$

A vector field v that satisfies Equation (1) with $c \neq 0$ is called a **symplectic dilation**. Given a symplectic dilation v we will always assume that it has been normalized so that $c = 1$. (Many authors take this to be the definition of a symplectic dilation.) If the dimension of X is larger that two then we more generally could have assumed that c in Equation (1) was a function $c\colon X \to \mathbb{R}$. Since in this case we have

$$dc \wedge \omega = d(c\,\omega) = dL_v\,\omega = 0$$

(the first and last two equalities follow since ω is closed). Thus the nondegeneracy of ω implies that $dc = 0$ when the dimension of X is larger than four (for details on this type of argument see the proof of Proposition 11). So c is once again a constant.

A compact hypersurface S in (X, ω) is said to have **contact type** if there exists a symplectic dilation v in a neighborhood of S that is transverse to S. There is an equivalent definition of contact type but before we can state it we need to observe that S has a distinguished line field, LS, in its tangent bundle called the **characteristic line field**. There are several ways to describe this line field, the simplest being as the symplectic complement of TS in TX. (Since S is codimension one it is coisotropic and thus the symplectic complement lies in TS and is one dimensional.) We could also define LS as follows: since S is a hypersurface it can be cut out by a function $H\colon X \to \mathbb{R}$ (i.e. a is a regular value of H and $M = H^{-1}(a)$). The line bundle LS is spanned by the symplectic gradient v_H of H (v_H is the unique vector field satisfying $dH = \iota_{v_H}\,\omega$). It is a simple exercise in the definitions of symplectic gradient and symplectic complement to see that these two definitions are the same.

Proposition 1 (Weinstein: 1971 [W2])**.** *Let S be a compact hypersurface in a symplectic manifold (X, ω) and denote the inclusion map $i : S \to X$. Then S has contact type if and only if there exists a 1-form α on S such that*

(i) $d\alpha = i^*\omega$ *and*

(ii) *the form α is never zero on the characteristic line field.*

Proof. Suppose S is a hypersurface of contact type and v is the symplectic dilation transverse to S. Then $\alpha' = \iota_v\, \omega$ is a 1-form defined in a neighborhood of S. Moreover,

$$\omega = L_v\, \omega = (d\iota_v + \iota_v d)\, \omega = d\iota_v\, \omega = d\alpha'.$$

Thus the 1-form $\alpha = i^*\alpha'$ satisfies (i) To verify (ii) let $H : X \to \mathbb{R}$ be a function that defines S. Then

$$\begin{aligned}\alpha(v_H) &= (\iota_v\, \omega)(v_H) = \omega(v, v_H) \\ &= -\omega(v_H, v) = -dH(v) \neq 0,\end{aligned} \tag{3}$$

since v is transverse to S.

Conversely, suppose S is a hypersurface and α is a 1-form on S satisfying (i) and (ii). We first extend α to a 1-form α' defined in a neighborhood of S. With a little care we can choose α' so that $d\alpha' = \omega$. (One uses that fact that a tubular neighborhood of S deformation retracts onto S.) The nondegeneracy of ω defines a vector field v satisfying

$$\iota_v\, \omega = \alpha'.$$

Clearly v is a symplectic dilation. Equation (3) and property (ii) show us that v is transverse to S. $\square$

To justify the terminology "contact type" notice that if S is a hypersurface of contact type then the 1-form α guaranteed by Proposition 1 is a contact form on S. To see this we first observe that

$$\ker \alpha \cong TS/LS,$$

which is easy to see since the bundle map that sends a vector in $\ker \alpha$ to its equivalence class in TS/LS is clearly well-defined and injective. Thus since the two bundles have the same dimension they are isomorphic. Now $i^*\omega$ is nondegenerate on TS/LS since LS is the symplectic complement of TS in TX. Thus $d\alpha$ is nondegenerate on $\ker \alpha$ which is equivalent to α being a contact form. Notice, the injectivity of the above map is equivalent to property (ii) in Proposition 1. In fact, if you assume property (i) then property (ii) is equivalent to the nondegeneracy of $d\alpha$ on $\ker \alpha$.

We have seen that a hypersurface of contact type in a symplectic manifold inherits a co-oriented contact structure. (Recall that co-oriented means that there is a nonzero vector field transverse to the contact fields. This is equivalent to the existence of a global 1-form defining the contact structure.) We would like to see to what extent every co-oriented contact structure arises in this fashion. To this end let (S, ξ) be a co-oriented contact manifold. We will now build a symplectic manifold, (Y, ω), in which S sits. Choose a contact 1-form α for ξ (note this is where we need co-oriented) and consider the submanifold of T^*S

$$Y = \{v \in T^*_m S : m \in S, v = t\alpha_m \text{ and } t > 0\}.$$

Clearly, for each $m \in S$, $Y \cap T^*_m S$ is the ray in $T^*_m S$ on which α_m lies and so $Y = (0, \infty) \times S$. Any other contact from α' for ξ can be thought of as a section of Y. Thus the manifold Y depends only on (S, ξ) and not on α. (The form α does however provide an embedding of S in Y.) We now claim that Y is a symplectic manifold. To see this let $\omega = \omega_0|_Y$ where $\omega_0 = d\lambda$ is the canonical symplectic structure on T^*S and λ is the Liouville 1-form on T^*S. Viewing α as a map $\alpha\colon S \to T^*S$, the Liouville 1-form λ satisfies

$$\alpha^*\lambda = \alpha.$$

Hence

$$\pi^*\alpha|_S = \lambda|_S,$$

where $\pi : T^*S \to S$ is projection and S is thought of as sitting in Y by using α as an embedding. Thus

$$t\pi^*\alpha = \lambda|_Y$$

and so

$$\omega = d\lambda|_Y = d(t\pi^*\alpha) = dt \wedge \pi^*\alpha + t\pi^*(d\alpha).$$

If the dimension of S is $2n - 1$ then we may finally compute

$$\omega^n = t^{n-1}\left[dt \wedge \pi^*(\alpha \wedge (d\alpha)^{n-1})\right]$$

which is clearly a volume form on Y since $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on S and $Y = (0, \infty) \times S$. Hence ω is a symplectic form on Y. We define the **symplectification** of S, denoted $\mathrm{Symp}(S, \xi)$, to be the manifold Y with symplectic form ω.

Notice that the vector field $t\frac{\partial}{\partial t}$ is transverse to $\alpha(S) \subset Y$ and

$$\begin{aligned} L_{t\frac{\partial}{\partial t}}\omega &= L_{t\frac{\partial}{\partial t}}(dt \wedge \pi^*\alpha + t\pi^*(d\alpha)) \\ &= d\iota_{t\frac{\partial}{\partial t}}(dt \wedge \pi^*\alpha + t\pi^*(d\alpha)) = d(t\pi^*\alpha) \\ &= \omega. \end{aligned}$$

Thus S is a hypersurface of contact type in $\mathrm{Symp}(S, \xi)$. We summarize the above in the following

Proposition 2. *If (S, ξ) is a co-oriented contact manifold, then there is a symplectic manifold $\mathrm{Symp}(S, \xi)$ in which S sits as a hypersurface of contact type. Moreover, any contact form α for ξ gives an embedding of S into $\mathrm{Symp}(S, \xi)$ that realizes S as a hypersurface of contact type.*

We would also like to note that all the hypersurfaces of contact type in (X,ω) look locally, in X, like a contact manifold sitting inside its symplectification.

Proposition 3. *Given a compact hypersurface S of contact type in a symplectic manifold (X,ω) with the symplectic dilation given by v there is a neighborhood of S in X symplectomorphic to a neighborhood of $\alpha(S)$ in $\mathrm{Symp}(S,\xi)$ where $\alpha = \iota_v\,\omega|_S$ and $\xi = \ker\alpha$.*

Proof. Let $\omega' = d(t\alpha)$ be the symplectic form on $\mathrm{Symp}(S,\xi)$. By the tubular neighborhood theorem we can find a neighborhood of S in X that is diffeomorphic to a neighborhood of $\alpha(S)$ in $\mathrm{Symp}(S,\xi)$ and sends the flow lines of v to the flow lines of $\frac{\partial}{\partial t}$. Now ω' on $\alpha(S)$ is just $d\alpha$ and ω on $S\subset X$ is also $d\alpha$. Finally, choosing the above diffeomorphism between tubular neighborhoods correctly, we can arrange that ω' on $T(\mathrm{Symp}(S,\xi))|_S$ agrees with ω on $TX|_S$. Hence using Moser's method (see appendix) our diffeomorphism may be isotoped into a symplectomorphism. $\square$

The contact structure $\xi = \ker\alpha$ induced on a hypersurface S of contact type in (X,ω) is determined up to isotopy by $S\subset X$ and the co-orientation the symplectic dilation v gives to the normal bundle of S, or in other words the direction of the symplectic dilation v. To see this let w be another symplectic dilation that is transverse to S and pointing in the same direction. Then $v_t = (1-t)\,v + t\,w$ is a family of symplectic dilations that are transverse to S. This gives us a family $\alpha_t = \iota_{v_t}\,\omega|_S$ of contact forms on S. Gray's theorem (see Appendix) then yields the desired isotopy from $\xi_0 = \ker\alpha_0$ to $\xi_1 = \ker\alpha_1$.

Let U be a domain in a symplectic manifold (X,ω) bounded by a hypersurface S. We say that U is **ω-convex** (**ω-concave**) if there exists a vector field v defined in a neighborhood of S that is transverse to S, points out of (into) U and is a symplectic dilation. In other words, S is a hypersurface of contact type and the symplectic dilation points out of (into) U. We will sometimes abuse terminology and say that U has ω-convex (ω-concave) boundary. From the discussion above we know that S will inherit a unique (up to isotopy) contact structure as the ω-convex boundary of U. Knowing the contact structure induced on S is not sufficient to reproduce the symplectic structure in a neighborhood of $S\subset X$; it is, however, sufficient (up to scale) for the purposes of cutting-and-pasting.

Theorem 4. *Let U_i be a domain in the symplectic manifold (X_i,ω_i) with ω_i-convex boundary S_i, for $i=0,1$. If S_0 is contactomorphic to S_1, then there exists a symplectic structure on $(X_0\setminus U_0)\cup_{S_0} U_1$.*

Proof. Let $\alpha_i = \iota_{v_i}\,\omega_i$ be the contact structure induced on $S = S_i$ as the convex boundary of U_i (v_i is the symplectic dilation). Form $\mathrm{Symp}(S,\xi)$

where $\xi = \ker \alpha_0$. The form α_0 allows us to write

$$\mathrm{Symp}(S, \xi) = (0, \infty) \times S$$

where $\alpha_0(S) = \{1\} \times S$. By the proof of Proposition 3 we have a neighborhood N_0 of S in X_0 symplectomorphic to a neighborhood N_0' of $\alpha_0(S)$ in $\mathrm{Symp}(S, \xi)$. See Figure 2. Let $\phi : S \to S$ be the postulated contactomorphism between (S, α_0) and (S, α_1). By rescaling ω_1, if necessary, we have $f\alpha_0 = \phi^*\alpha_1$ where $f : S \to \mathbb{R}$ is a positive function and $f(p) < 1$ for all $p \in S$. So we can think of $\alpha_1(S)$ in $\mathrm{Symp}(S, \xi)$ as the graph of f. Thus $\alpha_1(S)$ is disjoint from $\alpha_0(S)$ (in fact we may take $\alpha_1(S)$ to be disjoint from N_0' as well). Again the proof of Proposition 3 allows us to extend ϕ, thought of as a map from $\alpha_1(S) \subset \mathrm{Symp}(S, \xi)$ to $S \subset X_1$, to a symplectomorphism from a neighborhood N_1' of $\alpha_1(S)$ in $\mathrm{Symp}(S, \xi)$ to a neighborhood N_1 of S in X_1. Let $X_i^0 = X_i \setminus (U_i \setminus N_i)$ and T be the subset of $\mathrm{Symp}(S, \xi)$ bounded by (and including) the N_i', for $i = 0, 1$. We may now use the symplectomorphisms constructed above to glue $N_i \subset X_i^0$ to $N_i' \subset T$, for $i = 0, 1$, forming the manifold

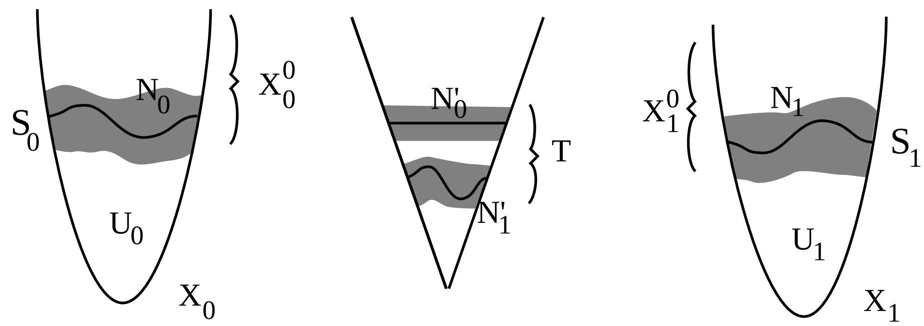

FIGURE 2. The manifolds X_0, $\mathrm{Symp}(S, \xi)$ and X_1

$$Y = X_0^0 \cup_{N_0} T \cup_{N_1} (U_1 \cup N_1).$$

See Figure 3. The manifold Y clearly has a symplectic form on it and is diffeomorphic to $(X_0 \setminus U_0) \cup_{S_0} U_1$ (since T just looks like a collar on $X_0 \setminus U_0$ and U_1 is identified to the other end of T by ϕ). $\square$

Consider a domain U whose boundary, S, is a hypersurface of contact type. Above we said that U has ω-convex boundary if the symplectic dilation is pointing out of U. Notice, we could have equivalently said that U has ω-convex boundary if the orientation induced on S from the contact structure agrees with the orientation induced on S as the boundary of U.

We end this section with a little terminology. If (S, ξ) is a contact manifold then we say that it is **strongly symplectically fillable** if S is the ω-convex

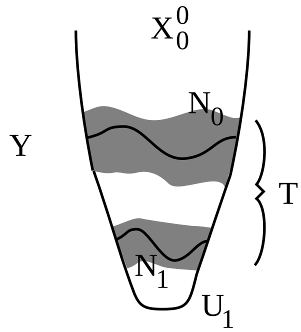

FIGURE 3. The manifold Y.

boundary of some compact symplectic manifold (U, ω) and ξ is the induced contact structure.

3. Examples and Applications

In this section we consider applications of ω-convexity to the constructions of symplectic manifolds. Providing complete proofs for all the theorems stated would add a great deal of length to this paper. So we shall usually just sketch the proofs and provide references to the literature.

Hypersurfaces of contact type exist in abundance. Proposition 2 tells us that given a co-oriented contact manifold (S, ξ) we can always find a symplectic manifold realizing S as a hypersurface of contact type. Finding ω-convex hypersurfaces that bound compact pieces is a little more difficult. The symplectic manifolds guaranteed by Proposition 2 are noncompact. Even worse, S separates the symplectic manifold into two noncompact pieces. Thus Proposition 2 is of no help in finding ω-convex hypersurfaces that bound compact pieces. We do however have one particularly simple example $S^{2n-1} \subset \mathbb{R}^{2n}$. Indeed, if we endow $\mathbb{R}^{2n}$ with the standard symplectic structure then the radial vector field will be a symplectic dilation that is transverse to S^{2n-1} (thought of as the unit sphere) and pointing out of the unit ball in $\mathbb{R}^{2n}$. The next few results give us many more (important) examples.

Proposition 5. *Let E be a rank two symplectic vector bundle over a symplectic manifold (S, ω_S). Denote by ω the symplectic form on the total space of E. Assume $c_1(E) = c\,[\omega_S]$. Then any sufficiently small disk bundle in E has ω-convex or ω-concave boundary according as c is a negative or positive constant.*

By disk bundle we mean the set of all points $(s, v) \in E$ with $|v| \leq \epsilon$, where $s \in S$ and v is in the fiber above s. One may find a proof of this in [M3]. The basic idea is to construct a nice symplectic form ω' on E where it is easy to see the ω'-convexity and then use the symplectic neighborhood theorem to transfer this back to the E with the original form. To construct ω' pick a connection 1-form β on the unit circle bundle in E with $d\beta = -2\pi c\, p^*\omega_s$, where $p\colon E \to S$ is the projection map. Then pull β back to all of E minus the zero section (the pull back will not be well defined along the zero section) and then set $\omega' = d\,(r^2 - \frac{1}{2\pi c})\,\beta$, where r is the radial coordinate in the fiber. We now claim that it is easy to see that this is a well defined symplectic form on all of E and the (rescaled) radial vector field will be a symplectic dilation transverse to the boundary of disk bundles. This proposition has the following very useful consequence.

Corollary 6. *Let (S, ω_S) be a codimension two symplectic submanifold of a symplectic manifold (X, ω). If $c_1(\nu(S)) = c\,[\omega_S]$, where c is a negative (positive) constant, then S has arbitrarily small tubular neighborhoods in X with ω-convex (ω-concave) boundary. In particular, if S is a symplectic surface in a symplectic 4-manifold (X, ω) with negative self-intersection, then inside any tubular neighborhood of S there is another neighborhood of S with ω-convex boundary.*

One may also use Lagrangian submanifolds to find many examples of ω-convex hypersurfaces.

Proposition 7. *Let S be a Lagrangian submanifold in a symplectic manifold (X, ω). Then S has a tubular neighborhood with an ω-convex boundary. Moreover, if S_i are Lagrangian submanifolds of X, for $i = 1, \ldots, n$, with each pair of S_i's intersecting transversely, then $\cup_{i=1}^{n} S_i$ has a neighborhood with ω-convex boundary.*

The proof of this proposition (when $n = 1$) is quite easy once one realizes that any Lagrangian submanifold has a neighborhood symplectomorphic to a neighborhood of the zero section in its cotangent bundle (this result originally appeared in [W1]). When $n > 1$ the proof is more difficult (see [E], cf. [E2]). Proposition 7 brings up the natural question: can Corollary 6 be extended for a family of symplectic submanifolds? In general, this cannot be done. For example, consider two symplectic 2-spheres in a symplectic 4-manifold, both with self-intersection -1 and a single point of transverse intersection between them. A neighborhood N of these two spheres has boundary $S^1 \times S^2$. So this neighborhood would be a strong symplectic filling of $S^1 \times S^2$ if its boundary were ω-convex. Moreover, after blowing down a -1 sphere we would have a strong symplectic filling of $S^1 \times S^2$ by $D^2 \times S^2$. But Eliashberg has shown, [E3], that $S^1 \times S^2$ cannot be symplectically filled by $D^2 \times S^2$. Thus N cannot have ω-convex boundary. There is, however, one very special case when a pair of symplectic surfaces in a symplectic 4-manifold can have an ω-convex neighborhood.

Proposition 8. *Let S_1 and S_2 be two symplectic surfaces in a symplectic 4-manifold (X, ω). Assume that S_2 is a sphere with self-intersection -2, S_1 has negative self-intersection and S_1 and S_2 intersect transversely at one point. Then there exists a tubular neighborhood of $S_1 \cup S_2$ that has ω-convex boundary.*

The idea of the proof is to replace S_2 with a Lagrangian sphere. Then with some care the appropriate neighborhood can be constructed, see [E].

Given two smooth manifolds X_0 and X_1 with embeddings $j_i : \Sigma \to X_i$, of a compact oriented manifold Σ of codimension 2, with the normal disk bundles ν_0 and ν_1 orientation reversing diffeomorphic, one may define the **normal connected sum** of X_0 and X_1 along Σ as follows:

$$X_0 \#_\phi X_1 = (X_0 \setminus \nu_0) \cup_\phi (X_1 \setminus \nu_1),$$

where $\phi : \partial\nu_0 \to \partial\nu_1$ is induced by the aforementioned diffeomorphism from ν_0 to ν_1. In the symplectic case we have the following theorem.

Theorem 9. *Let (X_i, ω_i) be a closed symplectic manifold and $j_i : \Sigma \to X_i$ a symplectic embedding of a closed connected codimension two manifold (Σ, ω), for $i = 0, 1$. Suppose that the normal Euler classes of $j_0(\Sigma)$ and $j_1(\Sigma)$ satisfy $e(\nu_0) = -e(\nu_1)$. Then $X_0 \#_\phi X_1$ admits a symplectic structure for any orientation reversing $\phi : \nu(j_0(\Sigma)) \to \nu(j_1(\Sigma))$.*

This theorem first appeared in [Gr2] and was later exploited by Gompf [G1] and McCarthy and Wolfson [MW1]. For a complete proof of this result and some spectacular applications the reader is referred to [G1]. We sketch a proof in dimension four using ω-convexity. If the normal Euler number $e(\nu_i) \neq 0$ then $j_0(\Sigma)$, say, has a neighborhood with ω_0-convex boundary and $j_1(\Sigma)$ has a neighborhood with ω_1-concave boundary (by Proposition 6). Let S_0 be the ω_0-convex boundary of the neighborhood of $j_0(\Sigma)$ and S_1 be the ω_1-convex boundary of $X_1 \setminus$ (neighborhood of $j_1(\Sigma)$). We now claim that the contact structures induced on S_0 and S_1 are contactomorphic. Let α_0 be the contact 1-form induced on S_0 and α_1 be the pull-back of the contact 1-form on S_1 to S_0 via ϕ. It is not hard to check that $d\alpha_0$ and $d\alpha_1$ are equal to (some positive multiple of) $\pi^*\omega$, where $\pi : S_0 \to \Sigma$ is the bundle projection. Thus if we set $\alpha_t = t\,\alpha_1 + (1-t)\,\alpha_0$ for $0 \leq t \leq 1$, then $d\alpha_t = td\alpha_1 + (1-t)d\alpha_0 = c_t\,\pi^*\omega$ where c_t is some positive constant depending on t. Moreover, $\ker \alpha_t$ is always transverse to the S^1 fibers and hence $d\alpha_t$ is nondegenerate on $\ker \alpha_t$. In other words, α_t are all contact forms on S_0. Thus Gray's theorem tells us that there is a contactomorphism, isotopic to the identity, from (S_0, α_0) to (S_0, α_1). Hence we may conclude that ϕ is isotopic to a contactomorphism. We can now use Proposition 4 to find a symplectic structure on the normal connected sum of X_0 and X_1. When the normal Euler number $e(\nu_i) = 0$ the neighborhoods of $j_i(\Sigma)$ do not have ω_i-convex boundaries. In this case though it is quite easy to glue $X_0 \setminus \nu(j_0(\Sigma))$ to $X_1 \setminus \nu(j_1(\Sigma))$ using the fact that the punctured unit disk in $\mathbb{C}$ can be symplectically turned inside out (one does this in each of fibers in the normal bundle).

In [MW2] McCarthy and Wolfson introduced the notion of an ω-compatible hypersurface. Let (X, ω) be a $2n$-dimensional symplectic manifold and M a $(2n-1)$-dimensional submanifold which supports a fixed point free S^1 action. The manifold M is called ω-**compatible** if the characteristic line field LM is tangent to the orbits of the S^1 action. One can show that if the first Chern class of $M \to (M/S^1)$ is nonzero then M is a hypersurface of contact type (note that one must work with orbifolds since the S^1 action is not necessarily free, see [MW2]). Thus assuming M splits X into two pieces let X^- be the piece for which M is the ω-convex boundary and let X^+ be the other piece. We are now ready to state:

Theorem 10 (McCarthy and Wolfson: 1995 [MW2])**.** *Let (X_i^{2n}, ω_i) be a symplectic manifold, M^{2n-1} a compact manifold with a fixed point free S^1 action and $j_i : M \to X_i$ a map such that $j_i(M)$ is an ω_1-compatible separating hypersurface, for $i = 0, 1$. Further assume that the symplectic forms τ_i induced on M/S^1 by ω_i are symplectomorphic. Then there is a symplectic form on $Y = X_0^- \cup_M X_1^+$.*

Through a similar analysis to the one in the proceeding paragraph it can be shown that the contact structures induced on M as the ω_i-convex boundaries of X_i^- are contactomorphic. Hence in the case of nonzero first Chern class the theorem follows from Proposition 4. In the case of zero first Chern class one must construct a canonical model for a neighborhood of M (as we did in Proposition 3 for the ω-convex case). For details on this see [MW2], where McCarthy and Wolfson prove this theorem using a beautiful theorem of Duistermaat-Heckmann and McDuff.

Weinstein [W3] has given us another nice way to construct ω-convex hypersurfaces. He shows how, given an ω-convex $2n$-manifold, one can add k-handles to it while preserving the ω-convexity, if $k \leq n$. We will indicate how to add a 2-handle to an ω-convex 4-manifold, for the general case see [W3]. First we define a standard 2-handle as a subset of $\mathbb{R}^4$ with symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let $f = x_1^2 + x_2^2 - \frac{1}{2}(y_1^2 + y_2^2)$ and $F = x_1^2 + x_2^2 - \frac{\epsilon}{6}(y_1^2 + y_2^2) - \frac{\epsilon}{2}$, where $\epsilon > 0$. Set $A = \{f = -1\}$ and $B = \{F = 0\}$. We define the standard 2-handle H to be the closure of the component of $\mathbb{R}^4 \setminus (A \cup B)$ that contains the origin, see Figure 4. The attaching region is $A \cap H$ and the core of the handle is the intersection of the y_1y_2-plane with H. Notice that the attaching circle of H (this is the core intersected with the attaching region) is a Legendrian curve in the boundary of H. Now the vector field $v = 2x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2}$ is a symplectic dilation and is transverse to A and B. Given a Legendrian knot L in

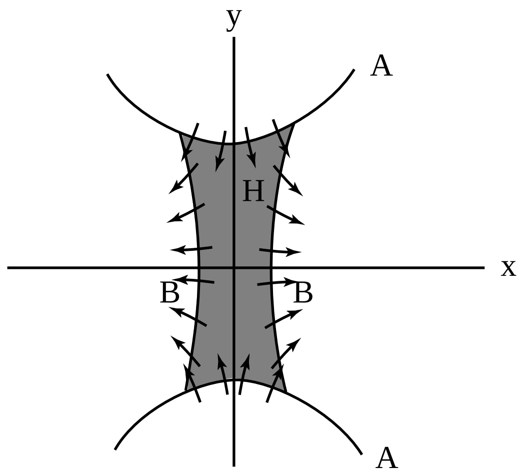

FIGURE 4. The standard 2-handle.

the boundary of a symplectic 4-manifold with ω-convex boundary there is a neighborhood of L contactomorphic to a neighborhood of the attaching circle in A. This contactomorphism is determined by the canonical framing of L (the framing given by the contact structure). Using this contactomorphism and choosing ϵ small enough it is now easy to add H along L to obtain a new symplectic manifold with ω-convex boundary (Figure 5).

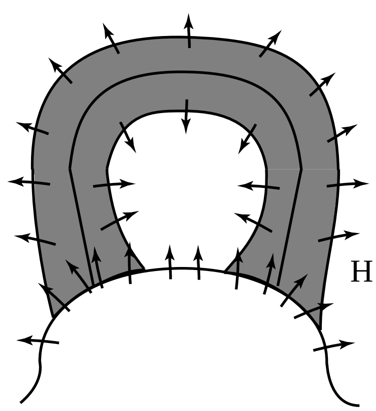

FIGURE 5. The new manifold with handle added.

4. CONVEXITY AND SYMPLECTIC FILLING

In this section we will examine two more notions of convexity. As motivation for the first, recall that a co-orientable contact manifold (M, ξ) is called strongly symplectically fillable if it is the ω-convex boundary of a symplectic "domain" (X, ω), were X is compact and $\partial X = M$. We will say that (M, ξ) is

symplectically fillable (or **weakly symplectically fillable**) when there exists a compact symplectic manifold (X, ω) such that M is the oriented boundary of X and $\omega|_\xi$ is nondegenerate (recall that M is oriented by ξ if the dimension of M is $2n+1$ and n is odd, otherwise the orientation condition should be ignored).

For the second notion of convexity, notice that given a contact manifold (M, ξ) there is a canonical conformal class of symplectic forms on ξ. To see this let α be a contact 1-form for ξ, then $d\alpha|_\xi$ is a symplectic structure on the vector bundle ξ. (This could be taken as the definition of a contact structure.) Given any other contact 1-form α' for ξ there is a positive function f on M so that $\alpha' = f\,\alpha$. Thus $d\alpha'|_\xi = f\,d\alpha|_\xi$. This confirms that $d\alpha$ defines a unique conformal class of symplectic forms on ξ. Finally, given a symplectic manifold (X, ω) with $\partial X = M$ we will say that the symplectic form ω **dominates** the contact structure ξ when $\omega|_\xi$ is in the canonical conformal class of symplectic forms on ξ.

Clearly, if (X, ω) is a symplectic manifold that dominates the contact manifold (M, ξ), then (X, ω) is a (weak) symplectic filling of (M, ξ). In dimension three these two notions actually coincide. To see this let (X, ω) be a symplectic filling of (M, ξ). Thus $\omega|_\xi$ is a symplectic structure on ξ. Let α be any contact 1-form for ξ, then $d\alpha|_\xi$ is also a symplectic structure on ξ. Since ξ is a 2-dimensional bundle, a symplectic structure on it is just an"area form" on each fiber. Thus there is a positive function f on M such that $\omega|_\xi = f\,d\alpha|_\xi$ (to see this one just needs to check that ω and $d\alpha$ give ξ the same orientation), which, of course, implies that (X, ω) dominates (M, ξ). When the dimension is greater that four these two types of convexity are not the same. We will verify this in Section 5.

It is quite clear (in light of the comments after the proof of Proposition 1) that a contact structure ξ on M that is strongly symplectically filled by (X, ω) is also dominated by ω . It is surprising that in dimensions above four these two concepts are equivalent. This was first noticed by McDuff in [M3].

Proposition 11. *Let (X, ω) dominate the contact manifold (M, ξ). If the dimension of M is greater than four, then (X, ω) is also a strong symplectic filling of (M, ω).*

Proof. Let α be a contact 1-form for ξ. Then there exists a positive function $f : M \to \mathbb{R}$ so that $\omega|_\xi = f\,d\alpha|_\xi$. Now replace α with $\frac{1}{f}\,\alpha$, so that we get $\omega|_\xi = d\alpha|_\xi$. Let v be the Reeb vector field of α (i.e. the vector field uniquely determined by $\alpha(v) = 1$ and $\iota_v\,d\alpha = 0$). Then we can find a vector field v' in $TX|_M$ satisfying $\omega(v, v') = 1$ (notice that v' is not in TM). If we now set $\beta = \iota_v\,\omega$, then

$$\omega = d\alpha + \alpha \wedge \beta. \tag{4}$$

Thus

$$d\alpha \wedge \beta - \alpha \wedge d\beta = d\omega = 0. \tag{5}$$

So $\iota_v(\alpha \wedge d\beta) = 0$ since $\iota_v(d\alpha \wedge \beta) = 0$. Expanding this out yields $d\beta - \alpha \wedge \iota_v\, d\beta = 0$. So $\alpha \wedge d\beta = \alpha \wedge \alpha \wedge \iota_v\, d\beta = 0$. Which implies (by Equation (5)) that

$$d\alpha \wedge \beta = 0. \tag{6}$$

This allows us to conclude that $\beta = 0$. Indeed, if $w \in \xi$ then we can find two vectors w_1, w_2 in ξ that are $d\alpha$-orthogonal to w and $d\alpha(w_1, w_2) = 1$. Thus $\beta(w) = d\alpha \wedge \beta(w_1, w_2, w) = 0$. One can similarly check that $\beta = 0$ for vectors not in ξ. We have thus shown that $\omega = d\alpha$ on $TX|_M$. Hence we are done by Proposition 1 and the observations following its proof. □

In dimension three strong symplectic fillability is a stronger notion than domination. We can see this in several ways. First notice that if (X, ω) is a strong filling of (M, ξ) then ω is exact and thus evaluates trivially on two dimensional submanifolds of M. However, if ω just dominates ξ then it might not be exact. For example let X be the unit disk bundle in the cotangent bundle of a surface $S \cong T^2$ with the canonical symplectic form ω. The boundary of X is ω-convex. However, if we perturb ω by adding some small multiple of $\pi^*\omega_S$ to it, where ω_S is any symplectic form on S and π is projection onto S, then ∂X is no longer ω-convex (ω is no longer exact since it evaluates nontrivially on a T^2 in the boundary) but ω still dominates the induced contact structure on the boundary (it is easy to see that (X, ω) is a weak filling of (M, ξ) which, as we have seen, is equivalent (in three dimensions) to dominating ξ).

One might hope that in dimension three if (U, ω) is a weak filling of (M, ξ) and ω is exact then it is also a strong filling. This however is also not true. Once again we delay the proof of this until Section 5.

There is an even stronger sense in which a weak symplectic filling is not a strong filling in dimension three. We have seen that a weak symplectic filling is not necessarily a strong filling, but it still might be possible that a weakly fillable contact manifold (M, ξ) might always be strongly fillable by some other symplectic manifold. This, however, was shown, in 1996, not to be the case by Eliashberg [E5]. To understand this we need to pause a moment and discuss Giroux's classification of contact structures on the 3-torus.

On the 3-torus T^3 for each integer $n > 0$ consider the 1-form

$$\alpha_n = \cos(2n\pi z)\, dx + \sin(2n\pi z)\, dy,$$

where we are thinking of T^3 as the quotient of $\mathbb{R}^3$ by the integer lattice. It is not hard to check that α_n is a contact form on T^3. We claim that each contact structure $\xi_n = \ker \alpha_n$ is weakly symplectically fillable. The contact structure

ξ_1 is actually strongly fillable. To see this first notice that $T^3 = T^2 \times S^1$ is the boundary of the unit disk bundle $T^2 \times D^2$ in the cotangent bundle of T^2. There is a canonical symplectic structure ω on $T^2 \times D^2 \subset T^*T^2$ and we saw in Proposition 7 (since the zero section of the cotangent bundle is Lagrangian) that $\partial(T^2 \times D^2)$ is ω-convex. We leave it as an exercise to check that ξ_1 is the contact structure induced on T^3 as the ω-convex boundary of $T^2 \times D^2$. Now to check the weak fillability of ξ_n for $n > 1$ let $\omega' = \omega + \epsilon\, \pi^* \omega_{T^2}$ be a new symplectic form on $T^2 \times D^2$, where ω_{T^2} is any symplectic form on T^2, $\epsilon > 0$ and π is projection onto T^2. Notice that $T^2 \times \{\text{pt}\}$ is a symplectic submanifold with this form. Thus we now have a weak filling of ξ_1 that is not a strong filling since ω' evaluates nontrivially on $T^2 \subset T^2 \times S^1$. Since $T^2 \times \{0\}$ is symplectic we can take the n-fold cover of $T^2 \times D^2$ branched over $T^2 \times \{0\}$ and obtain a new manifold diffeomorphic to $T^2 \times D^2$ with a symplectic form ω_n. The reader may check that the result of pulling the plane field ξ_1 back using the covering map is the contact structure ξ_n. It is also easy to see that $(T^2 \times D^2, \omega_n)$ is a weak symplectic filling of (T^3, ξ_n). It cannot be a strong filling since, once again, ω_n will evaluate nontrivially on a torus in T^3. Giroux [Gi] and independently Kanda [K] have shown that any weakly fillable , in fact any tight, contact structure on T^3 is contactomorphic to (T^3, ξ_n) for some $n > 0$.

We would now like to give some version of Eliashberg's argument why the ξ_n above cannot be strongly filled by any symplectic manifold. Let T^2 be a Lagrangian torus in $\mathbb{R}^4$. Then T^2 has a neighborhood N in $\mathbb{R}^4$ with ω-convex boundary. Let X be the n-fold cover of $\overline{\mathbb{R}^4 \setminus N}$. Then (X, ω) is a symplectic manifold with n standard ends. By this we mean that there are n ends of X and each one is symplectomorphic to an end of $\mathbb{R}^4$ with its standard symplectic structure. The boundary of X is T^3 which is ω'-concave. Moreover, the contact structure induced on T^3 is ξ_n. Thus if there were a strong symplectic filling, say (U, ω_U), of (T^3, ξ_n) then we could construct a symplectic form ω_Y on the manifold $Y = X \cup U$ that has n standard ends. This contradicts a theorem of Gromov [Gr1] if $n > 1$, thus there is no strong filling of (T^3, ξ_n).

We end this section by discussing a necessary condition for ω-convexity. Let (M, ξ) be the ω-convex boundary of (X, ω). Recall there is a line field LM on M defined as symplectic complement to TM in TX. The ω-convexity of M actually allows us to orient LM. We do this by saying that a vector $w \in LM$ defines a positive orientation if $\omega(v, w) > 0$, where v is the symplectic dilation. Notice that this orientation on LM agrees with the orientation it receives from the contact structure $\xi = \ker \alpha$, where $\alpha = \iota_v\, \omega$. Thus $\alpha(w) > 0$ for any positively oriented vector field in LM. Now we can restate Proposition 1 as follows: M is ω-convex if and only if there exists a 1-form α on M such that $d\alpha = \omega$ (on M) and $\alpha(w) > 0$ for any positively oriented vector field in LM.

5. J-Convexity

Let (X, J) be a $2n$-dimensional almost complex manifold (i.e. $J : TX \to TX$ is a linear isomorphism on each fiber such that $J \circ J = -\mathrm{id}_{TX}$). If M is a codimension one submanifold then there exists a unique hyperplane field of complex tangencies in TM. By this we mean that there is a $(2n-2)$-dimensional subbundle ξ of TM such that $J|_\xi$ is a complex structure on ξ. As a complex bundle ξ is uniquely oriented. Thus there is a 1-form α on M such that $\xi = \ker \alpha$. The **Levi form**, L, is defined to be the restriction of $d\alpha(\cdot, J\cdot)$ to ξ. If L is identically zero we say that M is **Levi flat** (this implies that ξ defines a codimension one foliation of M). If L is positive definite then we say that M is (strictly) ***J*-convex**. (If M is (strictly) J-convex for an unspecified almost complex J then we say that M is (strictly) **pseudo-convex**.) From now on we will not preface convexity with the adjective "strictly," though it is always implied. If M is J-convex, then ξ is an oriented contact structure.

We would now like to consider how pseudo-convexity is related to the notions of symplectic convexity discussed above (see Figure 1). To expect any relation at all, we must of course have some compatibility between our symplectic form and almost complex structure. In particular we say that an almost complex structure J on X is **tamed** by the symplectic form ω if $\omega(v, Jv) > 0$ for all $v \in TX$ that are not equal to zero. Given a symplectic structure one can always find a tame almost complex structure, see [McS]. If J is tamed by ω, then ω is nondegenerate on any J-complex subbundle of TX. Moreover, it is easy to check that the symplectic orientation and complex orientation on the subbundle agree. Thus it is easy to see that if (X, ω) is a symplectic manifold bounded by M and M is J-convex for some J that is tamed by ω, then (X, ω) is a weak symplectic filling of (M, ξ), where ξ is the hyperplane field of complex tangencies to M.

Now let (X, ω) be a symplectic manifold bounded by the contact manifold (M, ξ). Further, suppose that ω dominates ξ. We then claim that M is pseudo-convex. In order to see this we construct an almost complex structure J tamed by ω that has ξ as its field of complex tangencies to M. We begin by noticing that $TX|_M = \xi \oplus \xi^\perp$, where $\xi^\perp$ is the symplectic complement to ξ. On each of ξ and $\xi^\perp$ we can find complex structures tamed by $\omega|_\xi$ and $\omega|_{\xi^\perp}$, respectively, see [McS]. Thus we have J defined on $TX|_M$. It is not hard to extend this J to an ω tame complex structure over the rest of TX (again see [McS]). Hence by construction ξ is the field of complex tangencies to M. Since ω dominates ξ there is some contact 1-form α such that $\omega|_\xi = d\alpha|_\xi$. Thus $d\alpha(v, Jv) = \omega(v, Jv) > 0$ for all $v \in \xi$, verifying that M is J-convex.

Notice that in dimension three we have shown that pseudo-convexity, weak symplectic fillability and domination are equivalent concepts. It is not true though that in dimensions above four pseudo-convexity implies domination.

Let S be a hypersurface in an almost complex manifold (X, J). If S is cut out by a function $f : X \to \mathbb{R}$ (i.e. 0 is a regular value of f and $S = f^{-1}(0)$), then there is a particularly nice way to write down a 1-form that represents the hyperplane field, ξ, of complex tangencies to S. To find this 1-form recall that the kernel of df is TS. Thus a vector $v \in TS$ is in ξ if $J(v)$ is also in $TS = \ker df$. Said another way $v \in TS$ is in ξ if v is in the kernel of J^*df. Thus if we define the 1-form

$$\alpha = -J^* df|_S$$

on S, then the $\xi = \ker \alpha$.

Now consider $\mathbb{C}^n$ with its standard complex structure J. Let γ_t be the circle of radius t in $\mathbb{C} \subset \mathbb{C} \times \mathbb{C}^{(n-1)}$ and $f : \mathbb{C}^n \to \mathbb{R}$ be given by $f(z_1, \dots, z_n) = (|z_1| - 1)^2 + \sum_{i=2}^n |z_i|^2$. Then $f^{-1}([0, \epsilon))$ is a tubular neighborhood of γ_1. The boundary of this neighborhood, $T = f^{-1}(\epsilon)$, is J-convex as can easily be seen since we know the field of complex tangencies is given by $\alpha = -J^*df$. Now if ω is the standard symplectic structure on $\mathbb{C}^n$ then we claim that T is not ω-convex even though J is tamed by ω. To see this notice that $T \cap (\mathbb{C}^1 \times \{0\}) = \gamma_{1-\epsilon} \cup \gamma_{1+\epsilon}$ are two curves in T that are both tangent to the characteristic line field LT. If we let w be a vector field tangent to $\gamma_{1-\epsilon}$ providing the unique orientation to LT that ω-convexity would demand, then $\alpha(w) < 0$. This contradicts the criterion for ω-convexity stated at the end of the last section. This example, along with more subtle versions of it, first appeared in [EG]. Notice we have now shown that J-convexity does not imply ω-convexity in any dimension. Thus in dimensions above four J-convexity does not imply domination either, by Proposition 11. Moreover, we see that weak symplectic fillability does not imply domination in these dimensions. In dimension three we can now see, yet again, that domination does not imply ω-convexity (since J-convexity is equivalent to domination in this dimension).

A **Stein manifold** is a proper nonsingular complex analytic subvariety of $\mathbb{C}^n$. Given a function $\psi : X \to \mathbb{R}$ on a Stein manifold X we define the 2-form $\omega_\psi = -d(J^*(d\psi))$ where $J^* : T^*X \to T^*X$ is the adjoint operator to the complex structure J on X. We call ψ a **plurisubharmonic function** on X if the symmetric form $g_\psi(\cdot, \cdot) = \omega_\psi(\cdot, J\cdot)$ is positive definite. Note that this implies that ω_ψ is a symplectic structure on X; and, moreover, $h_\psi = g_\psi + i\omega_\psi$ is a Hermitian metric on X. Hence we see that X is a Kähler manifold. It is easy to see that any Stein manifold admits a proper exhausting plurisubharmonic function. For example the restriction of the radial distance function on $\mathbb{C}^n$ to X will be such a function. Grauert [Gra1] proved a complex manifold X is a Stein manifold if and only if X admits an exhausting plurisubharmonic function. Thus we know that any Stein manifold admits a symplectic structure. It can in fact be shown that this symplectic structure is essentially unique. In [EG] it was shown that given any two plurisubharmonic functions ψ and ϕ on a Stein manifold X, (X, ω_ψ) is symplectomorphic to (X, ω_ϕ).

Our interest in Stein manifolds is indicated in the next lemma.

Lemma 12. *The gradient vector field ∇_ψ of a plurisubharmonic function ψ on a Stein manifold X is a symplectic dilation for ω_ψ (the gradient is taken with respect to g_ψ).*

Thus the nonsingular level sets of ψ are ω_ψ-convex.

Proof. First by definition we have $\iota_{\nabla_\psi} g_\psi = d\psi$. So

$$\begin{aligned}\iota_{\nabla_\psi}\omega_\psi(\cdot,\cdot) &= \omega_\psi(\nabla_\psi,\cdot) = -g_\psi(\nabla_\psi, J\cdot)\\ &= -J^* g_\psi(\nabla_\psi,\cdot) = -J^* d\psi.\end{aligned}$$

Thus

$$\begin{aligned}L_{\nabla_\psi}\omega_\psi &= d\iota_{\nabla_\psi}\omega_\psi + \iota_{\nabla_\psi} d\omega_\psi\\ &= d\iota_{\nabla_\psi}\omega_\psi = -dJ^* d\psi = \omega_\psi.\end{aligned}$$

Hence ∇_ψ is an expanding vector field for ω_ψ. □

In [E2], Eliashberg demonstrates how to construct Stein manifolds. In particular he proves:

Theorem 13 (Eliashberg: 1990 [E2])**.** *Let J be an almost complex structure on X^{2n}, and $\psi : X^{2n} \to \mathbb{R}$ a proper Morse function all of whose critical points have index less that or equal to n. If $n > 2$, then J is homotopic to a complex structure J' for which ψ is plurisubharmonic. Hence (X^{2n}, J') is a Stein manifold.*

The situation when $n = 2$ was not explicitly discussed in [E2]; however, implicit in this paper was:

Theorem 14. *An oriented 4-manifold is a Stein manifold if and only if it has a handle decomposition with all handles of index less than or equal to 2 and each 2-handle is attached to a Legendrian circle γ with the framing on γ equal to* $\mathrm{tb}(\gamma) - 1$ *(where* $\mathrm{tb}(\gamma)$ *is the Thurston-Bennequin invariant of γ).*

For a complete discussion of Theorem 14 and its may interesting consequences see the paper [G2] of Gompf.

6. Convexity in 4-Dimensions and Contact 3-Manifolds

A contact structure on a 3-manifold falls into one of two classes: tight or overtwisted. The contact 3-manifold (M, ξ) is called **overtwisted** if there is some disk in M whose characteristic foliation contains a limit cycle, otherwise it is called **tight**. It is surprising that these two classes have such different properties. For example, it is quite easy to construct overtwisted structures on any closed 3-manifold [L], where as the existence of tight structures on a given 3-manifold cannot yet be answered in general. The classification of overtwisted contact structures is the same as the classification of homotopy classes of 2-plane fields [E1]. Thus understanding them is reduced to algebraic topology. In contrast, tight structures are much more rigid. For example, on a given 3-manifold there are only finitely many Euler classes that can be realized by a tight contact structure. For more details on what is known about tight contact structures the reader is referred to [E4].

Recall that a contact manifold (M, ξ) is fillable if there is a compact symplectic manifold (X, ω) such that $\partial X = M$, $\omega|_\xi$ is nondegenerate and the orientation on M induced by ξ and X agree. It is a remarkable fact that a fillable contact structure is tight. This is a result of Gromov [Gr1] and Eliashberg [E3]. Thus we have a way of constructing tight contact structures. They will arise as the boundary of any symplectic manifold with convex boundary (notice that any type of convexity discussed above will suffice since they all imply symplectic fillability). For example, Gompf, in [G2], uses Theorem 14 to construct tight contact structures on most Seifert fibered spaces.

One can also use convexity to distinguish contact structures. A simple example of this uses the fact that tight and overtwisted contact structures on a manifold form two distinct classes. Thus we can distinguish two contact structures by showing that one is tight and the other overtwisted. Bennequin [Be] essentially did this to prove the existence of two distinct contact structures on S^3. In general, given a fillable contact structure on M we construct a second contact structure by performing a Lutz twist [L] on M.

A much more subtle example is provided by Lisca and Matić's beautiful use of ω-convexity to distinguish tight contact structures on homology 3-spheres that are homotopic as plane fields. They begin by constructing, using Theorem 14, several contact structures ξ_k, for $1 \leq k \leq n-1$, on the Brieskorn homology sphere $\Sigma(2,3,6n-1)$ that are homotopic as 2-plane fields and strongly symplectically filled by Stein manifolds W_n^k. Then they show that if ξ_k is contactomorphic to $\xi_{k'}$ then $k = k'$ or $k = n - k'$. This is done by constructing a symplectic manifold using the ω-convexity of W_n^k and $W_n^{k'}$ that cannot exist unless the condition on k and k' is satisfied (this nonexistence is due to Seiberg-Witten theory). For more details see [LM]

We end this section by mentioning a result of Rudolph. In the paper [R] he finds an obstruction to smoothly slicing a knot using contact geometry. We can give a proof of his result using, among other things, Theorem 14. Recall a knot $\gamma \subset S^3$ is called **slice** if there is an embedded disk $D \subset B^4$ such that $\gamma = \partial D = D \cap \partial B^4$. The knot γ is called smoothly (topologically) slice if D is a smoothly (topologically) embedded disk. Given any knot γ in S^3 we may isotope γ into a Legendrian knot, where we are using the contact structure on S^3 induced as the ω-convex boundary of B^4 and B^4 is given its standard symplectic structure. In fact, there are many ways to do this. For each Legendrian knot associated to γ there is an associated Thurston–Bennequin invariant (see appendix). Let $\mathrm{TB}(\gamma)$ be the maximum of these invariants. It can be shown that this is always a finite number, thus $\mathrm{TB}(\gamma)$ is clearly an invariant of the isotopy class of γ. We are now ready to state Rudolph's main result from [R].

Theorem 15. *If* $\mathrm{TB}(\gamma) \geq 0$, *then* γ *is not smoothly slice.*

To see why this is true let γ be a knot with $\mathrm{TB}(\gamma) \geq 0$ and assume that it is smoothly slice. Then we find a Legendrian knot isotopic to γ (we will still call it γ) with Thurston–Bennequin invariant equal to 0. Theorem 14 allows us to construct a Stein 4-manifold Y by attaching a 2-handle to γ with framing -1. Now since γ is slice there is an embedded 2-sphere S in Y with self-intersection -1. Corollary 3.3 in [LM] says that we can find a minimal Kähler surface X in which Y embeds. But now S sits in X thus by a result of Taubes [T] we can find a symplectic 2-sphere in X with self-intersection -1, contradicting the minimality of X. Thus γ could not have been smoothly slice. Rudolph uses Theorem 15 to find many examples of topologically slice but not smoothly slice knots (see [R]).

7. Final Remarks

Recall Figure 1. In this paper we have given proofs of all the implications in the figure and shown that any implication in the figure not indicated, except one, is not true in general. The one implication we did not prove or give a counterexample to is

$$\text{weak symplectic filling} \longrightarrow \text{pseudo-convex} \tag{7}$$

in dimensions above four. Thus we have our first

Question 1. *Is the implication in Equation* (7) *true in dimensions above four?*

We have seen in many ways that

$$\text{domination} \longrightarrow \omega\text{-convex} \tag{8}$$

is *not true* in dimension four.

Question 2. *Under what conditions is the implication in Equation* (8) *true in dimension four?*

Or more generally

Question 3. *Given a domain U in a symplectic 4-manifold when does it have an ω-convex boundary?*

These are extremely important and subtle questions. Their importance is clear: to symplectically cut-and-paste we need ω-convexity, but it is usually easier to prove domination (or weakly fillable or pseudo-convex). For example Grauert has shown that a neighborhood of plumbed symplectic spheres has a pseudo-convex boundary if the intersection form of the neighborhood is negative definite [Gra2]. This is precisely the situation one encounters when trying to do a symplectic rational blowdown (see [FS] or [E]). In [E] it was shown that if this neighborhood had an ω-convex boundary then rational blowdowns could be done symplectically. Thus an answer to Question 2 in this case would complete the proof that the important topological operation of rational blowdown can be done in the symplectic category. For partial results along these lines see [E].[1]

There is an answer to Question 3 given by McDuff [M1]. She gives a necessary and sufficient condition for a domain to have ω-convex boundary in terms of structure currents associated to the contact form (see [M1] for the definitions of these terms).

Problem 4. *Understand strongly symplectically fillable contact structures on 3-manifolds.*

Theorem 14 is obviously very useful here. In [G2] Gompf found strongly fillable contact structures on all Seifert fibered spaces. Usually such a structure could be found regardless of the orientation on the Seifert fibered space. A stubborn exception to this led Gompf to ask

Question 5. *Does the Poincaré homology sphere with reversed orientation have a Stein filling?*

Gompf actually conjectured the answer to Question 5 should be no. There are many strong fillings of most Seifert fibered spaces, prompting the following question about which little is known.

Question 6. *When are strong symplectic fillings of contact 3-manifolds unique? When they are not unique, can they be classified?*

[1] Added in proof: Margaret Symington has recently shown that the neighborhoods that arise when performing rational blowdowns along symplectic spheres always have an ω-convex boundary.

Eliashberg's result described above (on page 15) shows that not all tight contact structures are strongly symplectically fillable. But it is still possible that all tight contact structures are weakly symplectically fillable. So we end with

Question 7. *Are all tight contact structures symplectically fillable?*

Appendix

Here we will give a terse overview of a few basic facts we need from symplectic and contact geometry. This is intended to establish notation and terminology. The reader wishing a more thorough introduction should consult [A] or [McS] where in proofs for all the statements below may be found.

A **symplectic manifold** is a pair (X, ω) where X is a manifold and ω is a closed nondegenerate 2-form. We say that ω is a **symplectic form** on X. By closed we mean $d\omega = 0$, and nondegenerate means that for all $x \in X$, ω_x is a nondegenerate form on the vector space T_xX. Since all symplectic vector spaces are even dimensional and ω induces a symplectic structure on each tangent space to X, a manifold must necessarily be even dimensional to admit a symplectic structure. Moreover, ω defines an orientation on X. We will always assume that X is given this orientation. A submanifold Y of a symplectic manifold (X, ω) is called **symplectic** if $\omega|_Y$ is a symplectic form on Y, called **Lagrangian** if $\omega|_Y = 0$ and called **coisotropic** if $TY^\perp \subset TY$. Two symplectic manifolds are called **symplectomorphic** it there is a diffeomorphism between them that sends one symplectic form to the other. Symplectic manifolds have no local structure. For example using Moser's method [Mo] one can show:

Theorem (Moser–Weinstein)**.** *Let X^{2n} be a manifold and C a compact submanifold. If ω_0 and ω_1 are two symplectic forms on X that are equal on each T_xX when $x \in C$, then there exists open neighborhoods U_0 and U_1 of C and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi^*\omega_1 = \omega_0$ and ϕ is the identity on C. More generally, ϕ is the identity wherever ω_0 and ω_1 agree.*

One may use this theorem to prove Darboux's theorem which says that any two points in any two symplectic manifolds have neighborhoods that are symplectomorphic. Another corollary is the symplectic neighborhood theorem.

Theorem. *Let (X_j, ω_j), for $j = 0, 1$, be symplectic manifolds. Assume Y_j is a symplectic submanifold of X_j and $\psi : Y_0 \to Y_1$ is a symplectomorphism. If there is a symplectic bundle map $\Psi : \nu(Y_0) \to \nu(Y_1)$ of the normal bundles that covers ψ, then ψ extends to a symplectomorphism from a neighborhood of Y_0 to a neighborhood of Y_1.*

Contact structures are an odd-dimensional analog of symplectic structures. A **k-dimensional distribution** ξ on an n-manifold M is a subbundle of TM such that $\xi_m \equiv T_mM \cap \xi$ is a k-dimensional subspace of T_mM for every $m \in M$. Note that a codimension one distribution ξ may be defined (at least locally) by a 1-form, say α. By this we mean $\xi = \ker\alpha$. We will say that a $2n$-dimensional distribution ξ on a $(2n+1)$-dimensional manifold M is **maximally nonintegrable** if for any locally defining 1-form α we have $\alpha \wedge d\alpha^n \neq 0$, or equivalently $d\alpha$ in nondegenerate on $\ker\alpha$. A **contact structure** on a $(2n+1)$-dimensional manifold M is a $2n$-dimensional distribution ξ that is maximally nonintegrable. Two contact manifolds are said to be **contactomorphic** if there exists a diffeomorphism that sends one contact distribution to the other. Two contact structures on the same manifold are called isotopic if they are contactomorphic by a contactomorphism that is isotopic to the identity. Contact structures also have no local structure. The analog of Darboux's theorem holds for contact structures and Gray's theorem [G] says that two contact structures that are homotopic (through contact structures) are isotopic.

A submanifold L of a contact manifold (M^{2n+1}, ξ) is called **Legendrian** if $T_mL \subset \xi_m$ for all $m \in L$ and the dimension of L is n. In a 3-dimensional contact manifold (M, ξ) a Legendrian submanifold is a curve. Notice that the contact planes define a canonical framing on a Legendrian curve γ. Framings are in one-to-one correspondence with the integers, but the correspondence is not unique. However, we can specify a unique correspondence if γ is null homologous by choosing a surface that γ bounds. The integer corresponding to the canonical framing is called the **Thurston-Bennequin invariant** of γ and is denoted $\mathrm{tb}(\gamma)$. Finally, note that given a surface Σ in (M^3, ξ) we can get a singular line field $T\Sigma \cap \xi$ on Σ. We can integrate this line field to get a singular foliation Σ_ξ called the **characteristic foliation** of Σ.

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