TIGHTNESS IN CONTACT METRIC 3–MANIFOLDS

JOHN B. ETNYRE, RAFAL KOMENDARCZYK, AND PATRICK MASSOT

ABSTRACT. This paper begins the study of relations between Riemannian geometry and global properties of contact structures on 3–manifolds. In particular we prove an analog of the sphere theorem from Riemannian geometry in the setting of contact geometry. Specifically, if a given three dimensional contact manifold (M, ξ) admits a complete compatible Riemannian metric of positive $4/9$-pinched curvature then the underlying contact structure ξ is tight; in particular, the contact structure pulled back to the universal cover is the standard contact structure on S^3. We also describe geometric conditions in dimension three for ξ to be universally tight in the nonpositive curvature setting.

1. INTRODUCTION

A contact structure on a 3–manifold is either overtwisted or tight according as it contains, or does not contain, an embedded disk which is tangent to the contact planes along its boundary [9]. The study of the tight/overtwisted dichotomy and its implications for low-dimensional topology is the primary focus of contact topology, one of the newest branches of contact geometry [1, 10]. A driving question in the area concerns the existence of tight contact structures on a given manifold. Though this has been much studied it is still very difficult to distinguish tight from overtwisted contact structures. In this paper we present several results relating tightness of contact structures to curvature constraints on metrics suitably adapted to the contact structure. In particular, we also introduce new ways for a Riemannian metric and a contact structure to be compatible (though our notion of "weakly compatible" discussed below has had precursors in the literature [12, 24], it does not seem to have been formalized until now).

A natural reference point here is the well known sphere theorem in Riemannian geometry [30, 2, 23], which is one of the fundamental results showing how geometry can control the topology of the domain. Recall, the sphere theorem states that every simply connected n-manifold which admits $1/4$-pinched positive sectional curvature is homeomeorphic to the n-sphere. In a similar vein we ask if a suitably chosen class of Riemannian metrics can control the topology of an underlying contact structure. Slightly extending a classical definition [8], we say that a metric is compatible with a contact structure if the corresponding unit contact form α satisfies $*d\alpha = \theta'\alpha$ where $*$ is the Hodge star operator induced by the metric and θ' is some constant. We establish the following theorem (in dimension three), which we refer to as the *contact sphere theorem*.

Theorem 1.1 (Contact sphere theorem)**.** *Let (M, ξ) be a closed contact 3–manifold and g a complete Riemannian metric compatible with ξ. If there is a constant $K_{max} > 0$ such that the sectional curvatures of g satisfy*

$$0 < \frac{4}{9}K_{max} < \sec(g) \leq K_{max},$$

then the universal cover of M is diffeomorphic to the 3–sphere by a diffeomorphism taking the lift of ξ to the standard contact structure on the 3–sphere.

Recall the standard contact structure ξ_{std} on the 3–sphere S^3 is the one induced as the complex tangencies to the unit sphere in $\mathbb{C}^2$. This contact structure can alternately be described as the orthogonal complement, in the round metric on S^3, to the Hopf fibration. Eliashberg proved in [10] that ξ_{std} is the unique (up to contactomorphism) tight contact structure on S^3, and what we actually prove in the above theorem is tightness of the contact structure. So this theorem really gives Riemannian geometric conditions that imply tightness.

Turning to general 3–manifolds, we recall that there are manifolds that do not admit tight contact structures [13] and we know precisely which Seifert manifolds admit them [26]. Since we also know manifolds containing homologically essential surfaces admit tight contact structures [11, Corollary 3.2.11] we are left with the central question: Which hyperbolic manifold (specifically, homology spheres) can support a tight contact structure?

Many hyperbolic manifolds are known to have tight contact structures but their construction and the proof of their tightness has nothing to do with the hyperbolic metric. While we do not directly address this question here, it does provide a strong motivation for better understanding the relation between contact topology and Riemannian geometry. We also state some results that provide a hint as to how to approach this and similar questions. For example, as a corollary of Theorem 1.5 below, we obtain a geometric criterion for a contact structure on a negatively curved manifold to be universally tight, see Theorem 1.9. (A contact structure is universally tight if its pull back to the universal cover of the manifold is tight, this is stronger than tightness.)

From a slightly different perspective, one might also want to go beyond the tight vs overtwisted dichotomy and sort the class of tight contact structures by finding privileged subclasses. In addition to the notion of universal tightness explained above, a classical class is that of symplectically fillable contact structures. In this paper, we seek classes interacting nicely with curvature in Riemannian geometry. Note that such interactions will automatically be inherited by covering spaces, contrasting with fillability properties.

Our approach to the contact sphere theorem and other global questions of tightness is through a quantitative versions of Darboux theorem. Recall that Darboux theorem in contact geometry guaranties that each point in a contact manifold has a neighborhood which is standard, i.e. embeds inside the standard contact structure on $\mathbb{R}^3$. According to Bennequin's theorem this neighborhood is then tight. In a Riemannian setting we can ask for a quantitative version guarantying that balls up to a certain radius are standard and, in particular, tight. Let g be any Riemannian metric and ξ any contact structure on M. We define the *tightness radius of ξ and g at a point p* to be

$$\tau_p(M,\xi) = \sup\{r \mid \text{the contact structure on the geodesic ball } B_p(r) \text{ at } p \text{ of radius } r \text{ is tight}\},$$

and the global *tightness radius*

$$\tau(M,\xi) = \inf_{p\in M} \tau_p(M,\xi).$$

Of course, if we do not assume any compatibility condition between the metric and the contact structure then we cannot estimate the tightness radius. We first concentrate on what happens with the compatibility definition recalled above. The tightness radius is, by definition, always less than the injectivity radius but one could ask if, for compatible metrics, they always coincide. This would explain the following surprising result (which is an important ingredient of the proof of Theorem 1.1).

Theorem 1.2. *Let (M, ξ) be a contact 3–manifold and g a complete Riemannian metric that is compatible with ξ. For a fixed point $p \in M$ let $\tau_p = \tau_p(M, \xi)$ and suppose that $\tau_p < \mathrm{inj}_p(g)$. Then for all radii r with $\tau_p \leq r < \mathrm{inj}_p(g)$, the geodesic sphere $S_p(r)$ contains an overtwisted disk.*

Recall that, a priori, overtwisted disks can have a very complicated geometry and this is what makes it hard to prove tightness of contact structures. We find this theorem somewhat surprising as it says that when a metric is compatible with a contact structure then as soon as a geodesic ball is large enough to be overtwisted one sees the overtwisted disk in a specific place, namely the boundary of the ball. Thus making it easy to determine when such a ball is tight (using Bennequin's theorem).

Despite this surprising result, we have numerical experiments, see Section 7.2, which strongly suggest that the tightness radius can indeed be less than the injectivity radius for compatible metrics, so we search for geometrical quantities controlling the tightness radius. To this end we recall that given a Riemannian metric g on M the *convexity radius* of g is defined to be

$$\begin{aligned} \mathrm{conv}(g) = \sup\{r \mid r < \mathrm{inj}(g) \text{ and the geodesic balls of radius } r \\ \text{are weakly geodesically convex}\}, \end{aligned}$$

where $\mathrm{inj}(g)$ is the injectivity radius of (M, g). For a more complete discussion see Subsection 3.2.

Theorem 1.3. *Let (M, ξ) be a contact 3–manifold and g a complete Riemannian metric that is compatible with ξ. Then,*

$$\tau(M, \xi) \geq \mathrm{conv}(g). \tag{1.1}$$

In particular, if $\sec(g) \leq K$, for $K > 0$, then

$$\tau(M, \xi) \geq \min\{\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\}$$

and $\tau(M, \xi) = \mathrm{inj}(g)$, if g has non-positive curvature.

We note that if M is a compact manifold then one may easily show that a lower bound for $\tau(M, \xi)$ exists. To see this note that M may be covered with Darboux balls (which are tight). Then the Lebesgue number for this open cover provides the desired lower bound. Of course this bound exists for any metric and one has virtually no control over it. Theorem 1.3 shows that if the metric is compatible with the contact structure then one does not need compactness and one can estimate $\tau(M, \xi)$ below in terms of curvature and injectivity radius. In particular, our theorem shows when M is noncompact with bounded curvature and injectivity radius, the tightness radius cannot shrink to zero at infinity.

The above theorem is based on the comparison of Riemannian convexity and almost-complex convexity in symplectizations of contact manifolds. Its proof uses holomorphic curves techniques due to Gromov and Hofer. In the seminal paper [21], Hofer proved that overtwisted disks guaranty the existence of closed Reeb orbits. Here almost-complex convexity controlled by Riemannian convexity allows us to use this argument in balls where we know there are no closed Reeb orbits. (In Section 5, we will see a surprising link with a third type of convexity — Giroux's convexity of surfaces in contact manifolds.)

We notice that our bounds on the tightness radius are especially effective in the case of non-positive curvature.

Corollary 1.4. *Let (M, ξ) be a contact manifold and g a complete Riemannian metric compatible with ξ having non-positive sectional curvature. Then ξ is universally tight and hence tight.*

One should point out that the study of compatible metrics is useful in fluid mechanics, plasma physics and other subjects, see for example [12]. In addition, it has produced a great many questions from the Riemannian geometry perspective, see for example [3]. It appears however, that so far there has been little work connecting properties of compatible metrics with much studied global properties of the contact structure in dimension three, such as tightness (though a exception to this is [24] that provides a lower bound for the volume of overtwisted Seifert fibered manifolds under S^1-symmetry conditions).

The class of compatible metrics is very natural but it is fairly restrictive. In relation to the question of existence of tight contact structure on hyperbolic manifolds, we note right away that a hyperbolic metric cannot be compatible with a contact structure on a closed manifold [3, p. 99]. Furthermore, Blair conjectures that if a metric is compatible with a contact structure on a closed manifold and has non-positive curvature then it is flat. So the above corollary may be of very limited impact[1]. This justifies the introduction of a more general class of metrics that can include the hyperbolic ones. Another motivation for extending the notion of compatibility comes from the theory of curl eigenfields. A curl eigenfield on a Riemannian 3–manifold is a 1–form α satisfying $*d\alpha = \theta'\alpha$ with θ' constant but $\|\alpha\|$ can vary. (We note that this equation is dual to the normal curl eigenfield equation for vector fields.) See [12] for some applications of this concept.

We say that a Riemannian metric and a contact structure ξ are *weakly compatible* if there exist a Reeb vector field for ξ which is perpendicular to ξ. We will show in Section 2 that this condition can be equivalently stated as there exists a contact form α such that

$$* \, d\alpha = \theta'\alpha, \tag{1.2}$$

where θ' is a positive function (which we will see measures the rotation speed of the contact planes). This equivalent definition enables us to see that the class of weakly compatible metrics is an extension of compatible metrics.

This class of metrics includes all the non-singular curl eigenfields and Beltrami fields. In addition, as shown in Section 7 it allows for hyperbolic metrics. It is also stable both under conformal changes and under the modifications used by Krouglov in [25], see Remark 1.7 below.

We will use several measures of how far a weakly compatible metric is from being compatible. First the rotation speed θ' and the norm ρ of the special Reeb vector field R entering

[1] However it does not seem obvious that any contact structure which is compatible with a flat metric is tight, especially since there is no classification of foliations of $\mathbb{R}^3$ by lines, contrasting with the situation in S^3.

in the definition are both constant in the compatible case so their gradient are such measures. We will also use the mean curvature H of the contact plane field. It vanishes for compatible metrics and its definition is recalled in Section 2. Finally we shall also use the normalized Reeb vector field $n = R/\|R\|$, which also happens to be a unit normal vector field to the contact planes. In the compatible case it is a geodesic vector field so we will consider $\nabla_n n$.

We will prove in Section 2 that the following two combinations of these measures give the same vector field:

$$D_g := \nabla_n n + 2Hn = (\nabla \ln \theta')^\perp - \nabla \ln \rho \tag{1.3}$$

where $v^\perp$ is the component of v perpendicular to ξ. We introduce:

$$d_g = \max_M \|D_g\|$$

Note that d_g is finite whenever M is compact and vanishes for compatible metrics (all terms in D_g vanish in this case).

To extend our main theorem to weakly compatible metrics we also introduce the following notation: let $K \geq 0$ and $\sec(g) \leq \pm K$, define

$$\mathrm{ct}_K(r) = \begin{cases} \sqrt{K}\cot(\sqrt{K}r)\,, & \text{for } \sec(g) \leq K,\ r \leq \min\{\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\}, \\ \frac{1}{r}, & \text{for } \sec(g) \leq 0, \\ \sqrt{K}\coth(\sqrt{K}r), & \text{for } \sec(g) \leq -K. \end{cases} \tag{1.4}$$

Here, of course, in the first case we assume $\sec(g)$ is positive somewhere and in the second case that it is 0 somewhere. Also to simplify our notations we will often write ct_K instead of ct_{-K} understanding that we mean the latter in the negative curvature setting. We may now state our result for weakly compatible metrics as follows.

Theorem 1.5. *Let (M, ξ) be a contact 3-manifold (not necessarily closed) that is weakly compatible with a Riemannian metric g. Whenever $d_g < \infty$ the tightness radius admits the following lower bound*

$$\tau(M, \xi) \geq \min\left\{\mathrm{ct}_K^{-1}(d_g)\,, \mathrm{inj}(g)\right\}.$$

Remark 1.6. While the above theorem provides a bound on the tightness radius in a weakly compatible metric, we note that it is not sufficient to prove a version of the the Contact Sphere Theorem 1.1 in this setting; however, it is reasonable to hope that such a theorem holds for weakly compatible metrics, but possibly with weaker bounds. Evidence for this, as well as a discussion of other possible strengthenings of Theorem 1.1, is discussed in Section 6.2.

Remark 1.7. The above theorems can be applied only when we have control over the sectional curvature of all plane fields and not only the sectional curvature of contact planes ξ. This is natural in view of the following very slight sharpening of a result of Krouglov saying that the latter curvature is very flexible. To get this version, start with a compatible metric and observe that Krouglov's modifications do not destroy weak compatibility, although they destroy compatibility (this is another reason to use weakly compatible metrics).

Theorem 1.8 (Krouglov 2008, [25]). *Given a cooriented contact structure ξ on a closed 3–manifold M and any strictly negative function f, there is a weakly compatible metric on M such that the sectional curvatures of ξ are given by f. Moreover, if the Euler class of ξ is zero then any function f may be realized.* □

Observe that since $\mathrm{ct}_K(r) \to \infty$ as $r \to 0$ the bound in Theorem 1.5 is always nonzero. We also notice that if α is actually compatible with g then $d_g = 0$ and thus $\mathrm{ct}_K^{-1}(d_g)$ can be taken to be $+\infty$. A similar situation occurs when $\sec(g) \leq -K$, then $\mathrm{ct}_K^{-1}(r)$ is ill defined for $r \in [-\sqrt{K}, \sqrt{K}]$ and we may assume $\mathrm{ct}_K^{-1}(d_g)$ to be $+\infty$ as well. Recall that for such manifolds, the universal cover is exhausted by geodesic balls. Since an overtwisted disk has to be contained in a compact part of the universal cover, we get the following corollary that we state as a theorem due its potential relevance to the problem of the existence of tight contact structures on hyperbolic manifolds.

Theorem 1.9. *Let (M, ξ) be a contact 3-manifold (not necessarily closed) that is weakly compatible with a complete Riemannian metric g of nonpositive sectional curvature. If*

$$\sec(g) \leq -d_g^2 \tag{1.5}$$

at all points then the contact structure ξ is universally tight.

One remarkable property of compatible metrics is that Reeb orbits are geodesics (see Corollary 2.5) and we use this in our study of compatible metrics. However, this is precisely what rules out closed hyperbolic manifolds in dimension 3: these manifolds cannot have any geodesic vector field [35]. But many hyperbolic manifolds have quasi-geodesic vector fields, see [4] for a recent account. These vector fields also cannot have any contractible Reeb orbits. So if a closed hyperbolic manifold has a quasi-geodesic Reeb field then the corresponding contact structure is universally tight. This observation does not explicitly use any easily defined compatibility between a metric and contact structure. However we can use an easy differential geometric criterion for quasi-geodesicity to get the following theorem which can then be compared to Theorem 1.9.

Theorem 1.10. *Let (M, ξ) be a closed contact manifold. Suppose M admits a metric g such that the sectional curvature of g is bounded above by $-K$ for some constant $K \geq 0$ and there is a Reeb vector field R for ξ such that the normalized Reeb field $N = R/\|R\|$ satisfies*

$$\|\nabla_N N\| \leq \sqrt{K}.$$

Then the universal cover of M is tight.

We note that one can think of the condition $\|\nabla_N N\| \leq \sqrt{K}$ as some type of compatibility between g and ξ. We also note that, while this theorem is stronger than Theorem 1.9 when they both apply, it does require that we are working with a closed manifold. We would lastly like to point out that earlier we used n for the normalized Reeb vector field while here we used N for that purpose. We will always use n to denote a unit normal vector field to the contact planes (which, in a weakly compatible metric, the normalized Reeb vector field always is) and use N if the normalized field does not have to be normal.

Outline: The rest of the paper is structured as follows. Section 2 defines the various notions of compatibility between metrics and contact structures which will be used in this paper and proves some formulas useful for our convexity comparison results. Section 3 contains our results comparing Riemannian convexity and almost complex convexity (Propositions 3.4 and 3.7) and the proofs of Theorems 1.3 and 1.5 as well as their corollaries. Section 4 proves Theorem 1.10. Section 5 centers around characteristic foliations of surfaces in contact 3–manifolds which are used in the proof of Theorem 1.2. Section 6 contains the proof of the contact sphere theorem and further discussion around it. Section 7 describe examples where we can apply our estimates of the tightness radius.

We end the introduction by noting that there are generalizations of some of the theorems presented above to the higher dimensional setting. These appear in [14] along with other methods of proof for some weaker forms of the Darboux theorem with estimates considered here.

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2. Compatible metrics

In this section we discuss various notions of compatibility between a metric and a contact structure. Our notions of compatibility are more general than the usual notions of compatibility, but seem to be more natural from a contact geometry perspective. Several of the results in this section come from, or were inspired by (previously unpublished) parts of [27].

Throughout this section and the rest of the paper we will interchangeably use the notation $g(u,v)$ and $\langle u,v\rangle$ to denote the Riemannian metric evaluated on the vectors u and v.

2.1. **Instantaneous rotation.**

We begin by observing how to detect if a plane field is contact using any Riemannian metric on a 3–manifold. To this end let ξ be an arbitrary oriented plane field on an oriented 3–manifold M. Recall that the Frobenius integrability criterion says that ξ is integrable if and only if the flow of a (local) non-zero vector field tangent to ξ preserves ξ. Given a Riemannian metric g on M we can (locally) choose an oriented orthonormal moving frame u, v, n where u, v is an oriented basis for ξ and n is a unit normal vector to ξ. Denote by ϕ_t the flow of u and define $\theta(t)$ to be the angle between $(\phi_{-t})_* v$ and n. That is

$$\cos\theta(t) = \frac{g((\phi_{-t})_*(v), n)}{\|(\phi_{-t})_*(v)\|}.$$

One may compute that $\theta'(0) = -g([u,v],n)$. Setting $\alpha(\cdot) = g(n,\cdot)$, one can then characterize $\theta'(0)$ by $\alpha\wedge d\alpha = \theta'(0)\mathrm{vol}_g$. In particular, $\theta'(0)$ depends on g and ξ but not on u, v.

We denote the function $\theta'(0)$ by θ' and call it the *rotation* or *instantaneous rotation* of ξ with respect to g. Notice that the Frobenious condition implies that ξ is a (positive) contact structure if and only if $\theta' > 0$.

2.2. **Second fundamental form.** In analogy with foliations we now introduce the second fundamental form of a general plane field. This notion goes back at least as far as [31]. The second fundamental form of ξ is the quadratic form on ξ defined as follows: for vectors u and v in $\xi_p = T_pM \cap \xi$,

$$II(u,v) = \frac{1}{2}\left\langle \nabla_u v + \nabla_v u, n \right\rangle, \tag{2.1}$$

where n is the oriented unit normal to ξ. (We note that u and v will need to be extended to vector fields tangent to ξ in a neighborhood of p to compute $II(u,v)$, but the value of $II(u,v)$ is independent of this extension *i.e.* II is tensorial.)

We note the following geometric interpretation of II inspired by [17] and proved in [27]. For any point $p \in M$ and a unit vector $v \in \xi$ let P_v denote the plane spanned by v and the oriented unit normal to ξ. There is a neighborhood N of the origin in T_pM such that the exponential map pulls ξ back to a plane field $(\exp|_N)^*\xi$ that is transverse to $P_v \cap N$. This plane field induces a foliation on $P_v \cap N$ and $II(v,v)$ is the curvature of the leaf of this foliation through the origin (measured by the flat metric on T_pM given by g_p).

We have the following curvatures derived from II. The *extrinsic curvature*, K_e, of ξ is the determinant of the quadratic form II with respect to g. The *mean curvature*, H, of ξ is half the trace of II (that is, it is the mean of the eigenvalues of II). It is clear that when ξ is an integrable plane field, then II is the standard second fundamental form of the leaves of the foliation associated to ξ and the quantities K_e and H are their classical curvatures.

2.3. **Weak compatibility.** We now want to introduce a notion of weak compatibility between a contact structure and a metric. For the sake of comparisons with other kinds of compatibility and for later computations, we describe several characterizations of weak compatibility.

Proposition 2.1. *Let α be a positive contact form on a 3–manifold M and g be a metric on M. We set $\rho = \|R_\alpha\|$. The following properties are equivalent:*

(1) *The Reeb vector field R_α is orthogonal to ξ.*
(2) *There is some positive function θ' such that*

$$*d\alpha = \theta'\alpha,$$

where $*$ is the Hodge star operator associated to g. (If this is true then θ' is the instantaneous rotation of $\ker\alpha$ *with respect to g.)*

(3) *There is an almost complex structure J on ξ that is compatible with $d\alpha|_\xi$, (that is $d\alpha(Ju, Jv) = d\alpha(u,v)$ for all $u, v \in \xi$ and $d\alpha(v, Jv) > 0$ for all $v \neq 0$ in ξ) such that, using the extension ϕ of J to TM sending R_α to 0, we have:*

$$g(u,v) = \frac{\rho}{\theta'} d\alpha(u, \phi(v)) + \rho^2\alpha(u)\alpha(v).$$

where θ' is the instantaneous rotation of $\ker\alpha$.

Definition 2.2. We say a contact form and a Riemannian metric g on a 3–manifold M are *weakly compatible* if the conditions of the preceding proposition are satisfied. We say that a contact structure ξ and a metric g are weakly compatible if there is a contact form α for ξ that is weakly compatible with g.

The first condition of the proposition is the most concise so it can be taken as *the* definition of weak compatibility. The second condition clearly relates this notion to [8] which asks in addition that $\rho = 1$ and $\theta' = 2$ everywhere. An intermediate notion between this and our weak compatibility is when θ' is any constant but ρ can vary. Then one says that α is a *non-singular curl eigenfield*. One also says that α is a *non-singular Beltrami field* if θ' is arbitrary and $\delta\alpha = 0$, where δ is the formal adjoint of the de Rham differential. (The condition $\delta\alpha = 0$ is equivalent to the metric dual of α being divergence free.) These two notions of compatibility, which can be generalized to arbitrary plane fields, have been made use of in the study of ideal fluid dynamics, see for example [12].

The third condition relates to the alternative presentation of the notion of [8] which can be found in [3]. (Also notice that in much of the literature about contact metric geometry, in particular [3], contact structures are taken to be negative contact structures on 3–manifolds — that is, they induce the opposite orientation to the one given on the manifold. This introduces various signs that differ from ours due to the fact that we take positive contact structures.)

The reason for weakening the definition of compatibility is not only to widen the class of metrics for which meaningful theorems can be proved but also to allows for homothety as well as other alterations of compatible metrics. More specifically, the set of metrics that are weakly compatible with a given ξ is closed under conformal transformations as well as changes of the metric restricted to ξ and $\xi^\perp$.

We will only explain the equivalence of (1) and (2) since we will not crucially use (3).

Proof. We first prove that (1) implies (2). Let v, w be a (local) oriented orthonormal frame for ξ, $\rho = \|R_\alpha\|$ and $n = \frac{1}{\rho} R_\alpha$. So v, w, n is an oriented orthonormal frame for TM. Let v^*, w^*, n^* be the dual basis. Clearly $\rho\,\alpha = n^*$ since the two 1–forms agree on our chosen basis. Since n is parallel to R_α we see that $\iota_n d\alpha = 0$ and hence $d\alpha = a\, v^* \wedge w^*$ for some function a. Notice

$$\begin{aligned} a = d\alpha(v,w) &= -\alpha([v,w]) \\ &= -\frac{1}{\rho}\, g(n,[v,w]) = \frac{\theta'}{\rho}, \end{aligned}$$

where the last equality follows from our discussion of θ' above. Thus $*d\alpha = \frac{\theta'}{\rho}\,(*v^* \wedge w^*) = \frac{\theta'}{\rho}\, n^* = \theta'\alpha$, as claimed.

We now prove that (2) implies (1). Again let v, w be a (local) orthonormal frame for ξ and set n to be the unit normal to ξ for which v, w, n is an oriented basis for TM. Let v^*, w^*, n^* be the dual basis. We again see that $\alpha = m\, n^*$ for some function m. So $*d\alpha = \theta' m\, n^*$, and hence $d\alpha = \theta' m\, v^* \wedge w^*$. From this we clearly have $\iota_n d\alpha = 0$. Thus n is parallel to the Reeb vector field R_α of α, from which we can conclude that R_α is orthogonal to ξ. □

As first observations about weakly compatible metrics we note the following connections between Riemannian geometry (geodesics) and contact geometry (Reeb flow lines) and a formula for the mean curvature H.

Lemma 2.3. *Let g be a metric that is weakly compatible with a contact form α on the 3–manifold M. Set $n = \frac{1}{\rho}R_\alpha$ where $\rho = \|R_\alpha\|$ is the norm of the Reeb vector field of α. Then*

$$\nabla_n n = -(\nabla \ln \rho)^\xi, \tag{2.2}$$

where v^ξ denotes the projection of v to ξ along R_α. In particular, $(\nabla \ln \rho)^\xi$ measures the deviation of Reeb flow lines from tracing out geodesics; and so, if the length of R_α is constant, then the flow lines of R_α are geodesics.

In addition we can express the mean curvature of $\ker \alpha$ *as*

$$H = \frac{1}{2}\operatorname{tr} II = -\frac{1}{2}\mathrm{div}_{dvol_g} n = -\frac{1}{2} n \cdot \left(\ln \frac{\rho}{\theta'}\right). \tag{2.3}$$

Proof. Since n is a unit vector field, we have $\langle \nabla_n n, n\rangle = 0$ and we are left to compute $\langle \nabla_n n, v\rangle$ for unit vector fields v tangent to ξ. Noting that $\langle n, \nabla_v n\rangle = 0$ and $\rho\,\alpha(v) = \langle n, v\rangle$ (since the two 1–forms in the formula agree on ξ and R_α) we compute

$$\begin{aligned}
\langle \nabla_n n, v\rangle &= -\langle n, \nabla_n v\rangle = -\langle n, \nabla_n v - \nabla_v n\rangle = -\langle n, [n,v]\rangle = -\rho\,\alpha([n,v]) \\
&= \rho\left(d\alpha(n,v) - n\cdot\alpha(v) + v\cdot\alpha(n)\right) = \rho\left(v\cdot\alpha(n)\right) = \rho\left(v\cdot\rho^{-1}\right) \\
&= \rho\left(-\rho^{-2}d\rho(v)\right) = -d(\ln\rho)(v) = \langle -\nabla\ln\rho, v\rangle .
\end{aligned}$$

Thus we see $\nabla_n n = -(\nabla \ln \rho)^\xi$.

In order to compute H, we use a (local) moving frame u, v, n, where u and v are tangent to ξ. We denote by $\mathrm{div}_\Omega w$ the divergence a vector field w with respect to a volume form Ω. Recall that $\mathrm{div}_{f\Omega} w = \mathrm{div}_\Omega w + d(\ln f)(w)$ and $\mathrm{div}_\Omega f w = f\,\mathrm{div}_\Omega w + df(v)$. Here we have the Riemannian volume form vol_g and the contact volume form $\nu = \alpha\wedge d\alpha$ which is preserved by the Reeb vector field R_α. These two volume forms are related by $dvol_g = \frac{\rho^2}{\theta'}\nu$. Indeed

$$\alpha\wedge d\alpha(n,u,v) = \alpha(n)d\alpha(u,v) = -\frac{1}{\rho}\alpha([u,v]) = -\frac{1}{\rho^2}g(n,[u,v]) = \frac{\theta'}{\rho^2}.$$

Recalling that on a Riemannian manifold $\mathrm{div}_{dvol_g} n$ can be computed as the trace of the operator $(u,v)\mapsto \langle \nabla_u n, v\rangle$ we have

$$\begin{aligned}
\operatorname{tr} II &= \langle n, \nabla_u u + \nabla_v v\rangle = -\langle \nabla_u n, u\rangle - \langle \nabla_v n, v\rangle = -\mathrm{div}_{dvol_g} n \\
&= -\mathrm{div}_{dvol_g}(\rho^{-1}R_\alpha) = -\rho^{-1}\mathrm{div}_{dvol_g} R_\alpha - R_\alpha\cdot\rho^{-1} \\
&= -\rho^{-1}\left(\mathrm{div}_\nu R_\alpha + \theta'/\rho^2 R_\alpha\cdot(\rho^2/\theta')\right) - R_\alpha\cdot\rho^{-1} \\
&= -\rho^{-2}R_\alpha\cdot\rho + \rho^{-1}/\theta' R_\alpha\cdot\theta' = -\rho^{-1}(R_\alpha\cdot\ln\rho - R_\alpha\cdot\ln\theta')
\end{aligned}$$

So we can conclude that $\operatorname{tr} II = -n\cdot\ln\frac{\rho}{\theta'}$. □

The preceding lemma implies as promised that the two definitions of D_g in Equation (1.3) agree.

2.4. **Compatibility.** We now introduce stronger forms of compatibility.

Definition 2.4. A contact form α and a Riemannian metric g on a 3–manifold M are *compatible* if

$$\|\alpha\| = 1 \quad\text{and}\quad *\,d\alpha = \theta'\alpha$$

for some positive constant θ'.

A contact structure and a Riemannian metric g are compatible if the unit contact form α of ξ and g are compatible. (Notice that is is equivalent to saying that the instantaneous rotation of ξ with respect to g is constant and the Reeb vector field R_α is unit length and orthogonal to ξ.) We may say that g and ξ are *strongly compatible* if in addition $\theta' = 1$ but we will rarely use this notion.

Allowing that θ' be any constant rather than fixing one gives a class of compatible metrics that is stable under homothety. This is slightly different from [8] where $\theta' = 2$. Allowing flexibility in θ' also enables us to absorb various discrepancies stemming from the different conventions in the definition of the exterior product of differential forms and their impact on the definition of the exterior derivative. In the current paper, for any 1–form α and vector fields u and v, we use $d\alpha(u, v) = u \cdot \alpha(v) - v \cdot \alpha(u) - \alpha([u, v])$. This is twice the derivative used in [3].

As a corollary of Lemma 2.3 we have the following important geometric properties of compatible metrics.

Corollary 2.5. *If g is compatible with α then R_α is a totally geodesic and divergence free vector field.* □

We now observe that given any metric g_ξ on a contact structure ξ then there is a canonical way to extend it to all of M so that it is compatible with ξ. This explains in particular the existence of metrics compatible with a given contact structure (which is well known [8]). The metric g_ξ induces a volume form on ξ. Since for any contact form α_0 and positive function f we have $d(f\alpha_0)|_\xi = f(d\alpha_0)|_\xi$, we can find a unique function f such that $d(f\alpha_0)|_\xi$ agrees with the volume form given on ξ by g_ξ. Setting $\alpha = f\alpha_0$ we can extend g_ξ to M by demanding that the Reeb vector field R_α is of unit length and orthogonal to ξ.

Note that if g is a metric strongly compatible with ξ then this extension agrees with the canonical extension of its restriction to ξ. So we have a "projection" from the space of metrics $\mathcal{M}(M)$ to the space of metrics which are strongly compatible with ξ. However this projection can change drastically the geometry of metrics that are compatible with the contact structure but have $\theta' \neq 1$. We now explain how to remedy this when M is compact. We denote by $\mathcal{M}_\xi(M)$ the space of metrics compatible with ξ. We want a projection

$$C : \mathcal{M}(M) \to \mathcal{M}_\xi(M) \subset \mathcal{M}(M).$$

Given a metric g, we denote by I the average instantaneous rotation speed of ξ with respect to g

$$I = \frac{\int_M \theta' \, dvol_g}{\int_M dvol_g}.$$

We restrict g to ξ then choose the unique contact form α for which $\frac{1}{I} d\alpha$ induces the same area form on ξ as g does. Now define

$$C(g) = g|_\xi + \alpha \otimes \alpha.$$

If g is compatible with ξ then $I = \theta'$. One may then use the third condition of Proposition 2.1 to verify that $C(g) = g$. Moreover, it is clear that I depends smoothly on g and, as the construction of $C(g)$ from g can be done locally, it is also easy to see that $C(g)$ depends smoothly on g. Thus C is a continuous projection of $\mathcal{M}(M)$ onto $\mathcal{M}_\xi(M)$.

3. Comparison of geodesic convexity and symplectic pseudoconvexity

In this section we explore various notions of convexity. In Subsection 3.1 we discuss convexity in complex geometry, while in Subsection 3.2 we discuss convexity in Riemannian geometry. In the last subsection we compare these notions and prove Theorems 1.3, 1.5 and 1.9.

3.1. Convexity in almost complex geometry.

Let (W, J) be an almost complex manifold. Assume that $\Omega \subset W$ is an open, relatively compact, domain in W. Let Σ be the hypersurface in W that bounds the region Ω in W and $f : W \to \mathbb{R}$ be any function for which Σ is a regular level set, say $\Sigma = f^{-1}(1)$, and Ω is a sublevel set of f. The complex tangencies $\mathcal{C} = T\Sigma \cap J(T\Sigma)$ to Σ can easily be described as the kernel of the restriction to Σ of the 1–form $df \circ J$. The form

$$L(u, v) = -d(df \circ J)(u, Jv)$$

is called the *Levi form* of Σ. We say Σ is the *pseudoconvex*, or *strictly pseudoconvex*, boundary of Ω if $L(v, v) \geq 0$, respectively $L(v, v) > 0$, for all $v \in \mathcal{C}$.

The weak maximum principle for elliptic operators gives a well-known property of pseudo-convex hypersurfaces: they cannot be "touched from the inside" by holomorphic curves. This can be precisely formulated as follows.

Lemma 3.1. *Suppose $\Sigma = \partial\Omega$ is a strictly pseudo-convex hypersurface in an almost complex manifold (W, J) as above. Then any holomorphic curve C in Ω is transverse to Σ. In particular, the interior of C is disjoint from Σ.* □

The almost complex manifolds we will use are symplectizations of contact manifolds. The symplectization of a contact manifold (M, ξ) is the submanifold W of T^*M of all covectors vanishing on ξ and inducing the given co-orientation. If λ denotes the canonical Liouville 1–form on T^*M then $d\lambda$ is a symplectic form and the fact that W is a symplectic submanifold of T^*M is equivalent to the contact condition. This submanifold is a $\mathbb{R}_+$-principal bundle over M and is trivialized by any choice of section, that is by any contact form α. In such a trivialization, $\mathbb{R}_+ \times M$, the symplectic form induced from T^*M becomes $\omega = d(t\alpha)$, where t is the coordinate on $\mathbb{R}_+$. Let J be a complex structure on ξ that is compatible with $(d\alpha)|_\xi$. Extending J to TW is equivalent to prescribing $J\partial_t$. Gromov's work tells us that, in order to understand compactness issues for spaces of holomorphic curves, we should ask $\omega(\cdot, J\cdot)$ to be a Riemannian metric on W (the so called "tameness condition"). Here this is equivalent to $\iota_{J\partial_t} d\alpha = 0$ and $\alpha(J\partial_t) > 0$. This means $J\partial_t$ should be parallel to the Reeb vector field R_α of α. We will always extend our complex structure from ξ to TW by setting $J\partial_t = n$ where $n = \frac{1}{\rho} R_\alpha$ and $\rho = \|R_\alpha\|$.

3.2. Convexity in Riemannian geometry.

Let S be a hypersurface in an Riemannian manifold (M^n, g) that bounds a region U. We say that U is *geodesically convex at p* in S if any (local) geodesic in a direction $v \in T_pS$ intersects U only at p. The region U is *geodesically convex* if it is geodesically convex at every point $p \in S$.

Lemma 3.2. *Let $f : M^n \to \mathbb{R}$ be a smooth function on a Riemannian manifold (M^n, g) and let U be a sublevel set of f at a regular value. Then U is geodesically convex at $p \in S = \partial U$ if and only* *if the Hessian of f is positive definite:*

$$\nabla^2 f(v,v) > 0,$$

for all non-zero $v \in T_pS$. □

We recall the convexity of the distance function using the functions ct_K in Equation (1.4), see e.g. [29].

Proposition 3.3. *Let (M^n, g) be a Riemannian manifold and K be a positive constant. If g has non-positive curvature then geodesic spheres are geodesically convex as long as they are embedded.*

If $\sec(g) \leq K$ and

$$r < \min\{\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\} \tag{3.1}$$

where $\mathrm{inj}(g)$ is the injectivity radius of (M^n, g), then the Hessian of the distance function

$$\mathsf{r}_p : M^n \to \mathbb{R} : q \mapsto d(p,q)$$

is positive definite on the ball of radius r about p, $B_p(r)$.

More precisely, if $\sec(g) \leq \pm K$, the Hessian of r satisfies

$$\nabla^2 \mathsf{r}_p \geq \mathrm{ct}_K(r) g.$$

3.3. **Comparison of convexities.** We can now compare the two types of convexity that we considered above.

Let S be a regular level set of a smooth function f on M. Let W be the symplectization of (M, ξ), identified with $\mathbb{R}_+ \times M$ using some contact form α. We set $\Sigma = \mathbb{R}_+ \times S \subset W$ and $\mathcal{C} = T\Sigma \cap JT\Sigma$. Recall that the Levi form of Σ is defined on $\mathcal{C}$ by $L(u,v) = -d(df \circ J)(u, Jv)$. We want to compare pseudo-convexity of Σ, measured by its Levi form L, and Riemannian convexity of S, measured by the Hessian $\nabla^2 f$.

Proposition 3.4. *Let g be a metric weakly compatible with the contact form α on the 3–manifold M. Then for any $v \in \mathcal{C}$ we have*

$$L(v,v) = \nabla^2 f(v,v) + \nabla^2 f(Jv, Jv) - \langle \nabla f, D_g \rangle \|v\|^2$$

where $D_g = \nabla_n n + 2Hn$, see also Equation (1.3).

We use the notation established in Subsection 3.1. In particular, J will denote the complex structure on the symplectization W not just on ξ. That is J leaves ξ invariant (and is the $\pi/2$-rotation of $g_{\|\xi}$) and $J\partial_t = n$. We also extend the metric g on M to W by $g + dt \otimes dt$. We note that the extended J and g are still compatible. Using the notation from above, we begin by defining two bundle maps $A : \xi \to TW$ and $B : \xi \to TW$ by

$$A(v) = J[Jv, v] - \nabla_v v - \nabla_{Jv} Jv$$

and

$$B(v) = J[v, n] + \nabla_{Jv} n + \nabla_n Jv.$$

One can easily check that A and B are tensorial, meaning that their value at a point depends on the vector at the point. The main ingredients of the proof of Proposition 3.4 are the following two technical lemmas.

Lemma 3.5. *Under the hypotheses of Proposition 3.4, we have*

$$-L(v,v)+\nabla^2 f(v,v)+\nabla^2 f(Jv,Jv)=df\left(A(v^\xi)+B(aJv^\xi-bv^\xi)-(a^2+b^2)\nabla_n n\right),$$

when the vector $v\in\mathcal{C}$ can be written as $v=v^\xi+a\,n+b\,\partial_t$, with $v^\xi\in\xi$, $\mathbb{R}$-invariant and a and b are constants.

Proof. We first compute

$$\begin{aligned}\nabla^2 f(v,v)+\nabla^2 f(Jv,Jv)&=v\cdot(df(v))-(\nabla_v v)\cdot f+(Jv)\cdot(df(Jv))-(\nabla_{Jv}Jv)\cdot f\\&=v\cdot(df(v))+(Jv)\cdot(df(Jv))-df(\nabla_v v+\nabla_{Jv}Jv).\end{aligned}$$

And, using the formula $d\alpha(u,w)=u\cdot\alpha(w)-w\cdot\alpha(u)-\alpha([u,w])$ we have

$$d(df\circ J)(v,Jv)=-v\cdot(df(v))-(Jv)\cdot(df(Jv))+df(J[Jv,v]).$$

Adding the two preceding equations, we obtain

$$-L(v,v)+\nabla^2 f(v,v)+\nabla^2 f(Jv,Jv)=df(J[Jv,v]-\nabla_v v-\nabla_{Jv}Jv).$$

Decomposing v as $v^\xi+a\,n+b\partial_t$ as in the statement of the lemma and using $\nabla_{\partial_t}v=0$ we compute

$$\begin{aligned}J[Jv,v]&=J[Jv^\xi,v^\xi]+aJ[Jv^\xi,n]+bJ[n,v^\xi],\\\nabla_v v&=\nabla_{v^\xi}v^\xi+a(\nabla_{v^\xi}n+\nabla_n v^\xi)+a^2\nabla_n n,\end{aligned}$$

and

$$\nabla_{Jv}Jv=\nabla_{Jv^\xi}Jv^\xi+b(\nabla_{Jv^\xi}n+\nabla_n Jv^\xi)+b^2\nabla_n n.$$

Thus we see that $-L(v,v)+\nabla^2 f(v,v)+\nabla^2 f(Jv,Jv)$ equals

$$df\left(A(v^\xi)+aB(Jv^\xi)-bB(v^\xi)-(a^2+b^2)\nabla_n n\right),$$

giving the announced formula. □

Lemma 3.6. *Under the hypotheses of Proposition 3.4, for any vector v in ξ we have*

$$A(v)=-\|v\|^2\left((\operatorname{tr}I\!I)\,n+\theta'\partial_t\right)$$

and

$$B(v)=-\left(\theta' v+(\operatorname{tr}I\!I)Jv\right)-\langle Jv,\nabla_n n\rangle\;n-\langle v,\nabla_n n\rangle\;\partial_t.$$

Proof. Because ξ is a plane field and both sides of our formulas are tensorial, we only need to check the announced formulas for a non-zero constant norm vector field v tangent to ξ and it suffices to check scalar products of our formula with v, Jv, n and ∂_t.

We first prove that $A(v)$ is normal to ξ. To this end we compute

$$\begin{aligned}\langle A(v),v\rangle&=\langle J[Jv,v],v\rangle-\langle\nabla_v v,v\rangle-\langle\nabla_{Jv}Jv,v\rangle\\&=-\langle\nabla_{Jv}v,Jv\rangle+\langle\nabla_v Jv,Jv\rangle-\langle\nabla_{Jv}Jv,v\rangle=0.\end{aligned}$$

One may easily compute, or use the fact that A is J–invariant to see, that $\langle A(v), Jv\rangle = 0$, and thus that $A(v)$ is orthogonal to ξ. The normal component of $A(v)$ is

$$\begin{aligned}\langle A(v), n\rangle &= \langle J[Jv, v], n\rangle - \langle \nabla_v v, n\rangle - \langle \nabla_{Jv} Jv, n\rangle\\ &= \langle [Jv, v], \partial_t\rangle - \mathit{II}(v, v) - \mathit{II}(Jv, Jv) = -\operatorname{tr}\mathit{II}\|v\|^2.\end{aligned}$$

Recalling that $\iota_n g = \rho\,\alpha$ and that $\alpha\wedge d\alpha = \frac{\theta'}{\rho^2} dvol_g$ we compute

$$\begin{aligned}\langle A(v), \partial_t\rangle &= \langle J[Jv, v], \partial_t\rangle - \langle \nabla_v v, \partial_t\rangle - \langle \nabla_{Jv} Jv, \partial_t\rangle\\ &= -\langle [Jv, v], n\rangle = \rho\, d\alpha(Jv, v) = \rho\,\alpha\wedge d\alpha(R_\alpha, Jv, v)\\ &= -\rho\frac{\theta'}{\rho^2} dvol_g(\rho\, n, v, Jv) = -\theta'\,\|v\|^2.\end{aligned}$$

The above inner products give the desired formula for $A(v)$.

We now compute $B(v)$ by computing its inner product with the chosen basis. First we have

$$\begin{aligned}\langle B(v), v\rangle &= \langle J[v, n], v\rangle + \langle \nabla_{Jv} n, v\rangle + \langle \nabla_n Jv, v\rangle\\ &= -\langle \nabla_v n, Jv\rangle + \langle \nabla_n v, Jv\rangle + \langle \nabla_{Jv} n, v\rangle - \langle Jv, \nabla_n v,\rangle\\ &= \langle n, [v, Jv]\rangle\,.\end{aligned}$$

As argued above $\langle n, [v, Jv]\rangle = -\theta'\,\|v\|^2$. Next we have that

$$\begin{aligned}\langle B(v), Jv\rangle &= \langle J[v, n], Jv\rangle + \langle \nabla_{Jv} n, Jv\rangle + \langle \nabla_n Jv, Jv\rangle\\ &= \langle \nabla_v n, v\rangle - \langle \nabla_n v, v\rangle - \langle n, \nabla_{Jv} Jv\rangle\\ &= -\langle n, \nabla_v v\rangle - \langle n, \nabla_{Jv} Jv\rangle = -(\operatorname{tr}\mathit{II})\,\|v\|^2.\end{aligned}$$

Continuing we compute that

$$\begin{aligned}\langle B(v), n\rangle &= \langle J[v, n], n\rangle + \langle \nabla_{Jv} n, n\rangle + \langle \nabla_n Jv, n\rangle\\ &= \langle [v, n], \partial_t\rangle - \langle Jv, \nabla_n n\rangle = -\langle Jv, \nabla_n n\rangle\,.\end{aligned}$$

Finally we compute

$$\begin{aligned}\langle B(v), \partial_t\rangle &= \langle J[v, n], \partial_t\rangle + \langle \nabla_{Jv} n, \partial_t\rangle + \langle \nabla_n Jv, \partial_t\rangle\\ &= -\langle [v, n], n\rangle = -\langle \nabla_v n, n\rangle + \langle \nabla_n v, n\rangle = -\langle v, \nabla_n n\rangle\,,\end{aligned}$$

These computations yield the stated formula for $B(v)$. □

Proof of Proposition 3.4. Given $v \in \mathcal{C}$ since we are only concerned with v at a point so we may assume that it is of the form $v = v^\xi + an + b\partial_t$ as in Lemma 3.5. Setting $w = aJv^\xi - bv^\xi$ Lemmas 3.5 and 3.6 yield

$$\begin{aligned}-L(v, v) + \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv) = df\Big(&-\operatorname{tr}\mathit{II}(\|v^\xi\|^2 n + Jw) - \theta'(\|v^\xi\|^2\partial_t + w)\\ &- \langle Jw, \nabla_n n\rangle\, n - \langle w, \nabla_n n\rangle\,\partial_t - (a^2 + b^2)\nabla_n n\Big).\end{aligned}$$

Note that $w = -bv + aJv + (a^2+b^2)\partial_t$ and $Jw = -av - bJv + (a^2+b^2)n$. Thus, keeping in mind that v is in $\mathcal{C}$, we see that w is in $T\Sigma$ and Jw is in $(a^2+b^2)n + T\Sigma$. Also ∂_t is in $T\Sigma$. So continuing the above computation we have

$$\begin{aligned}
&-L(v,v) + \nabla^2 f(v,v) + \nabla^2 f(Jv,Jv)\\
&\qquad = df\Big(-\operatorname{tr} I\!I \|v\|^2 n - \langle Jw, \nabla_n n\rangle\, n - \langle w, \nabla_n n\rangle\, \partial_t - (a^2+b^2)\nabla_n n\Big).
\end{aligned}$$

Additionally we notice that

$$\langle Jw, \nabla_n n\rangle\, n + \langle w, \nabla_n n\rangle\, \partial_t = \Big\langle v^\xi, \nabla_n n\Big\rangle\,(v - v^\xi) + \Big\langle Jv^\xi, \nabla_n n\Big\rangle\,(Jv - Jv^\xi).$$

By the definition of $\mathcal{C}$, v and Jv are in $T\Sigma$, hence in $\ker df$. So the important part of the right hand-side above is $-\left\langle v^\xi, \nabla_n n\right\rangle v^\xi - \left\langle Jv^\xi, \nabla_n n\right\rangle Jv^\xi$ but this is precisely $-\|v^\xi\|^2\nabla_n n$, so we get the announced formula using the definition $H = \frac{1}{2}\operatorname{tr} I\!I$. $\square$

As a direct corollary we obtain the following result.

Proposition 3.7. *Let g be a metric compatible with the contact structure ξ on the 3–manifold M. Then for any $v \in \mathcal{C}$ we have*

$$L(v,v) = \nabla^2 f(v,v) + \nabla^2 f(Jv,Jv). \tag{3.2}$$

$\square$

The later formula was known before when (W, J) is Kähler, that is the Sasakian case, see [19, Lemma page 646].

With these propositions in hand we can prove our main estimates on the tightness radius of a contact structure. We begin by observing the relation between geodesic convexity and pseudo-convexity.

Lemma 3.8. *Let (M,ξ) be a contact 3-manifold (not necessarily closed) that is weakly compatible with a Riemannian metric g and satisfies $d_g < \infty$. The submanifold $\mathbb{R}_+ \times B_p(r)$ of the symplectization of (M,ξ) with the complex structure discussed in Section 3.1 has strictly pseudo-convex boundary if $r < \min\left\{\mathsf{ct}_K^{-1}(d_g), \operatorname{inj}(g)\right\}$.*

Remark 3.9. *Notice that if $d_g = 0$, as is the case for compatible metrics, then we see that $\mathbb{R}_+ \times \partial B_p(r)$ is pseudo-convex if $\partial B_p(r)$ is geodesically convex.*

Proof. We first use Propositions 3.4 and 3.3 to estimate the Levi form. Denote the radial function from p by $\mathsf{r}(q) = d(p,q)$. For any complex tangency $v = v^\xi + a\, n + b\, \partial_t$ of $\mathbb{R}_+ \times \partial B_p(r)$ we have

$$\begin{aligned}
L(v,v) &= \nabla^2 \mathsf{r}(v,v) + \nabla^2 \mathsf{r}(Jv,Jv) - \langle \nabla \mathsf{r}, D_g\rangle \|v\|^2\\
&\geq \mathsf{ct}_K(\mathsf{r})(2\|v\|^2 + a^2 + b^2) - \|D_g\|\|v\|^2\\
&\geq (\mathsf{ct}_K(\mathsf{r}) - d_g)\|v\|^2.
\end{aligned}$$

Thus $L(v,v) > 0$ if $r < \mathsf{ct}_K^{-1}(d_g)$, $\square$

Proof of Theorem 1.3 and Theorem 1.5. Let $B_p(r)$ be the geodesic ball of radius r about p. From the last lemma we know that for any

$$r < \min\left\{\mathrm{ct}_K^{-1}\left(d_g\right), \mathrm{inj}(g)\right\},$$

$\mathbb{R}_+ \times \partial B_p(r)$ is a strictly pseudo-convex submanifold of $\mathbb{R}_+ \times M$. Assume now for contradiction that $B_p(r)$ contains an overtwisted disk. Arguing as in [21], one can start a family of holomorphic disks near an elliptic singularity in the overtwisted disk. Since these disk cannot touch $\mathbb{R}_+ \times \partial B_p(r)$ thanks to pseudo-convexity, the Gromov–Hofer compactness theorem extends to this setting and we get the existence of a closed Reeb orbit γ in $B_p(r)$ and a J-holomorphic cylinder $C_\gamma = \mathbb{R}_+ \times \gamma$ in $\mathbb{R}_+ \times B_p(r)$. But this is a contradiction because C_γ has to touch $\mathbb{R}_+ \times \partial B_p(r_0)$ from the inside for some $r_0 \leq r$. □

Proof of Theorem 1.9. Note that pull-backs of ξ and the metric to any covering space are (weakly) compatible and the sectional curvature is non-positive. It is well known, by Hadamard Theorem [6], that the universal cover of a three manifold with nonpositive curvature is $\mathbb{R}^3$ and the space is exhausted by geodesic balls. The assumption $d_g \leq \sqrt{K}$ implies

$$\mathrm{ct}_K(r) - d_g > 0, \qquad \text{for all } r \geq 0,$$

since $\mathrm{ct}_K(r) > \sqrt{K}$. Theorem 1.5 says a ball of any radius is tight, but since any potential overtwisted disk will have to be contained in a ball of some radius it cannot exist. □

4. Tightness and quasi-geodesics

Here we establish a geometric "universal tightness" criterion for contact structures even when the metric and contact structure are not compatible.

Proof of Theorem 1.10. Suppose by contradiction (M, ξ) is overtwisted. According to [21] there will be a closed contractible orbit in the flow of the Reeb field R, and hence in the flow of $N = R/\|R\|$. This orbit will lift to a closed orbit γ in the universal cover of M. Of course our metric, contact structure and Reeb field also lift to the universal cover and satisfy the same hypotheses (since we will work exclusively in the cover from now on, we will use the same notation for objects in the cover).

Choose any point p in the cover. There will be some r such that the embedded geodesic ball $B_p(r)$ of radius r about p will contain γ and $\partial B_p(r)$ will have a tangency with γ. Let $\mathsf{r}_p(x) = d(p, x)$ be the radial function measuring the distance from p. The estimate from Proposition 3.3 says

$$\sqrt{K} < \nabla^2 \mathsf{r}_p(\dot\gamma, \dot\gamma).$$

Since we are assuming γ has tangency to $\partial B_p(r)$ we can parameterize γ such that the tangency occurs at $\gamma(0)$. As γ lies inside $B_p(r)$ we see that

$$0 \geq \frac{\partial^2}{\partial t^2}(\mathsf{r}_p \circ \gamma)|_{t=0} = \frac{\partial}{\partial t} g(\nabla \mathsf{r}_p, \dot\gamma) = \nabla^2 \mathsf{r}_p(\dot\gamma, \dot\gamma) + d\mathsf{r}_p(\nabla_{\dot\gamma}\dot\gamma).$$

Thus we can compute

$$\sqrt{K} < \nabla^2 \mathsf{r}_p(\dot\gamma, \dot\gamma) \leq -g(\nabla \mathsf{r}_p, \nabla_{\dot\gamma}\dot\gamma) \leq \|\nabla_{\dot\gamma}\dot\gamma\| = \|\nabla_N N\| \leq \sqrt{K},$$

where the third inequality comes from the fact that $\nabla \mathsf{r}_p$ has unit length so $-g(\nabla \mathsf{r}_p, \nabla_{\dot\gamma}\dot\gamma)$ is one component of $\nabla_{\dot\gamma}\dot\gamma$ in some orthonormal basis. Thus we arrive to the absurd consequence that $\sqrt{K} < \sqrt{K}$ and an overtwisted disk could not exist. $\square$

Remark 4.1. The above proof works also in the higher dimensional setting and shows that the condition of Theorem 1.10 implies the contact manifold is plastikstufe free (see [28] for a definition of plastikstufe, and [14] for further discusion.)

5. Characteristic foliations and contact topology

In this section we recall the definition of the characteristic foliation of a surface in a contact 3–manifold and use it to study the tightness radius of a compatible metric.

5.1. **Characteristic foliations.** Recall that an oriented singular (this adjective will be implicit in the following) foliation on an oriented surface S is an equivalence class of 1–forms on S where $\alpha \sim \beta$ if there is a positive function f such that $\alpha = f\beta$. Let α be a representative for a singular foliation $\mathscr{F}$. A singularity of $\mathscr{F}$ is a point where α vanishes. The singularity p is said to have non-zero divergence if $(d\alpha)_p$ is an area form on T_pS. If ω is an area form on S (compatible with the chosen orientation) then to each singular point p we attach the sign of the unique real number μ such that $(d\alpha)_p = \mu\omega_p$. One can easily check that singular points and their signs do not depend on the choice of α in its equivalence class or on ω if we keep the same orientation.

Let S be an oriented surface in a contact manifold (M, ξ) with $\xi = \ker \alpha$, co-oriented by α. The characteristic foliation ξS of S is the equivalence class of the restriction of α to S. The contact condition ensures that all singularities of characteristic foliations have non-zero divergence. Singularities of ξS correspond to points where S is tangent to ξ and they are positive or negative according as the orientation of ξ and S match or do not match. We also notice that α provides a co-orientation, and hence if S is oriented by an area form ω the orientation of the line field ξS is given by the vector field X which satisfies $\iota_X\omega = \alpha|_S$.

One may dually think of the characteristic foliation on S as coming from the singular line field on S given by $T_pS \cap \xi_p$ for each $p \in S$.

5.2. **Convex spheres and characteristic foliations on spheres.** An oriented foliation $\mathscr{F}$ on a sphere S is *simple* if it has exactly one singularity of each sign (the positive one will be called the north pole and the negative one the south pole). It is *almost horizontal* if, in addition, all its closed leaves are oriented as the boundary of the disk containing the north pole (in other words from "west" to "east"). These are (slight variations on) definitions due to Eliashberg [9].

If ξ is a contact structure on M and S is a sphere in M then we will say that ξ is simple or almost horizontal along S if $\mathscr{F} = \xi S$ has this property. The relevance of these definitions to compatible metrics comes from the following lemma.

Lemma 5.1. *Let g be a Riemannian metric compatible with the contact manifold (M, ξ). Let α be the contact form implicated in the definition of compatibility between g and ξ and R_α its Reeb vector field. Let $r < \mathrm{inj}(g)$ and S be the sphere of radius r around some point p_0. The contact structure ξ is simple along S with poles $\exp_{p_0}(\pm rR)$.*

Proof. Let S be a sphere of radius $r < \mathrm{inj}(g)$ around p_0. Let γ be a geodesic starting at p_0 and denote $\gamma(r) \in S$ by p. Suppose p is a singularity of ξS, that is, ξ_p is tangent to S at p. By Gauss' Lemma, ξ_p is orthogonal to γ at p. As R_α is also orthogonal to ξ_p we must have $\gamma'(r) = \pm R_\alpha$. As the flow of R_α give geodesics we see that $\gamma'(t)$ is equal to $\pm R_\alpha$ for all t and hence $\gamma'(0) = \pm R_\alpha$. Thus p is $\exp_{x_0}(\pm r R_\alpha)$. □

We now have the following important definition in contact geometry due to Giroux [15]. We say that an hypersurface S in a contact manifold (M, ξ) is *ξ–convex* if there is a vector field transverse to S whose flow preserves ξ. (Normally such a hypersurface is just called convex, but to distinguish this notion from other types of convexity in this paper we use the term ξ–convex.) In our situation, we can recognize ξ–convex spheres using the following very special case of a criterion by Giroux.

Lemma 5.2 (Giroux 1991, [15])**.** *If a contact structure ξ is simple along a sphere S, then S is ξ–convex if and only if ξS has no degenerate closed leaf.* □

The following proposition explores how a contact structure on a ball can be overtwisted by explaining some relations between simple, almost horizontal and tight. The first point is obvious while the second one was observed by Giroux. The third one will be crucial for the sphere theorem. The second point will not be directly used in this paper but it could prove useful to get tightness radius estimates in later work and its proof is also needed for the third point.

Proposition 5.3. *Let B be a ball in a 3–dimensional contact manifold (M, ξ) which is the disjoint union of a point p and a family of spheres S_t, $t \in (0, 1]$.*

1. *If all foliations ξS_t are simple and $\xi|_B$ is tight then ξ is almost horizontal along all the S_t.*
2. *If all foliations ξS_t are almost horizontal, then $\xi|_B$ is tight.*
3. *If all foliations ξS_t are simple, and $\xi|_B$ is overtwisted then there is some radius t_1 such that all foliations ξS_t for $t \geq t_1$ have closed leaves.*

Proof. To establish the first point we notice that tightness rules out the existence of any closed leaf in any of the characteristic foliations ξS_t, since such a leaf would bound an overtwisted disk. So they are all (trivially) almost horizontal.

We now prove a special case of the second point of the proposition: if there is no closed leaf in any ξS_t then ξ is tight. Indeed, in the absence of closed leaves, Lemma 5.2 ensures that all the spheres S_t are ξ–convex. The dividing curve Γ_t on each S_t can be taken to be any simple closed curve separating the north and south pole and transverse to ξS_t, so we can assume the Γ_t depend smoothly on t. We can then construct a contact vector field that is transverse to all the S_t (this can be done by using the fact that the space of contact vector fields supported in a neighborhood of a convex surface S is contractible and the fact that such a vector field is transverse to all surfaces close to S). We can use this vector field to embed (B, ξ) in the standard contact structure on $\mathbb{R}^3$. Alternatively, once we know that all spheres S_t are ξ–convex, we can directly apply Giroux's reformulation of Bennequin's theorem, [16, Theorem 2.19] to get tightness.

We now assume that closed leaves do appear. Darboux's theorem, coupled with Bennequin's result that the standard contact structure on $\mathbb{R}^3$ is tight, ensures that there is some $\varepsilon > 0$ such that spheres ξS_t for $t \leq \varepsilon$ have no closed leaf. So there is a minimal $t_1 > 0$

such that ξS_{t_1} has a closed leaf. In order to finish the proof of the second point of the proposition, we have to prove that such a leaf has to go to the west (*i.e.* it is oriented as the boundary of the disk containing the south pole), contradicting the almost horizontal assumption. In order to prove the third point, we have to prove that all spheres S_t for $t \geq t_1$ exhibit closed leaves. This will be done using essentially the analysis in Giroux's Birth-Death Lemma, [16, Lemma 2.12]. In order to facilitate this analysis, we observe the following result.

Lemma 5.4. *Let A be an annulus. There is an embedding of $A \times [\varepsilon, 1]$ into B such that*

(i) *each $A_t := A \times \{t\}$ maps into the corresponding S_t,*

(ii) *the complement of A_t in S_t is made of two disks D_t^n and D_t^s (around the poles) whose characteristic foliations is topologically conjugated to a radial foliation and goes transversely out of D_t^n and into D_t^s (recall these foliations can be oriented),*

(iii) *the vector field ∂_t on $A \times [\varepsilon, 1]$ maps to a Legendrian vector field,*

Moreover, by a C^∞ small perturbation of $A \times [\epsilon, 1]$ that is fixed near any finite number of A_t for which the foliation ξA_t is non-degenerate, we may assume in addition that

(iv) *for t outside a finite set $\varepsilon < t_1 < \cdots < t_N < 1$, all closed leaves of ξS_t are non-degenerate and hence stable, and*

(v) *each ξS_{t_i} has exactly one degenerate closed leaf and it indicates the birth or death of a pair of non-degenerate closed leaves.*

We will establish this technical lemma (which belongs purely to the realm of dynamical systems) after we finish the proof of our proposition.

The pulled back contact structure on $A \times [\varepsilon, 1]$ will also be denoted by ξ. The boundary component of each the annuli A_t (or their essential sub-annuli below) next to the disk D_t^n (respectively D_t^s) will be called the northern (respectively southern) border. Similarly, a closed leaf in any ξA_t will be called northernmost if it bounds an open disk containing D_t^n and no closed leaf.

The following lemma is a variation on Giroux Birth-Death Lemma. Here we see the contact condition with specific orientation plays a crucial role by determining the type (birth or death) of bifurcations as t increases.

Lemma 5.5. *Let $A' \subset A$ be a subannulus of A containing the northern border and pick $\bar{t}$ in $(\varepsilon, 1)$. Suppose $\xi A'_{\bar{t}}$ has a closed leaf. If there is some positive δ such that $\xi A'_t$ has no closed leaf when $\bar{t} < t < \bar{t} + \delta$ (respectively $\bar{t} - \delta < t < \bar{t}$) then the northernmost closed leaf of $\xi A'_{\bar{t}}$ is oriented from west to east (respectively east to west).*

Proof. We denote by γ the northernmost closed leaf of $\xi A'_{\bar{t}}$. Suppose it goes to the west (the other case being symmetric) and let A'' be the sub-annulus of A' bounded by γ and the northern border of A.

Because ξ is a positive contact structure and ∂_t is Legendrian, the characteristic foliations on the annuli A_t are rotating strictly clockwise at each point of A. We know the flow is pointing in along the northern border of A''. Moreover it is tangent to γ at $t = \bar{t}$. Thus our assumption on the direction of the flow along γ implies there is some positive δ such that the characteristic foliation of A''_t will point inward along γ for $\bar{t} < t < \bar{t} + \delta$. So, for these t, $\xi A''_t$ points inward along both boundary components. Since $\xi A''_t$ is non singular, the Poincaré-Bendixson theorem then guaranties the existence of an interior closed leaf. $\square$

The second point of the proposition above clearly follows from the above lemma with $A' = A$ and $\bar{t} = t_1$. The lemma also proves that births next to the northern boundary are always births of leaves heading west (here A' is a annulus containing the new closed leaf but no other one at the birth time). The same reasoning with t going backward shows that deaths next to the northern boundary are always deaths of leaves heading east.

Let us now complete the proof of the the third point. Assume for contradiction that closed leaves disappear completely at time some time t'. Considering the region $S \times [\varepsilon, t']$ we may use Lemma 5.4 to find the annulus A and $A \times [\varepsilon, t']$ satisfying the (i)–(iii). Notice that A_ε and $A_{t'}$ have non-degenerate characteristic foliations as there are no singularities or closed leaves. Thus we may C^∞-perturb $A \times [\varepsilon, t']$ relative to $A_\varepsilon \cup A_{t'}$ to satisfy (iv)–(v) too. Let $\varepsilon < t_1 < \ldots, t_k < t'$ be the times from item (iv) in the lemma. We know by the preceding paragraph that the (unique and northernmost) closed leaf at t_1 goes to the west whereas the one at t_k goes to the east. In addition there are finitely many births and deaths of closed leaves, they occur at times t_i. We now prove by induction on these events that the northernmost closed leaf always goes to the west, contradicting what we just mentioned at t_k. We already know this at birth time t_1. Any birth or death not affecting the northernmost orbit is irrelevant. Births next to the northern border spawn leaves going to the west. Finally there cannot be any deaths next to the northern boundary because they would involve a northernmost orbit heading east but they are prohibited by induction hypothesis. □

We now turn to the proof of our technical Lemma 5.4. This proof was kindly provided by Frédéric Le Roux.

Proof of Lemma 5.4. First observe that the poles are both index one singularities. This can be checked directly for very small radii and then follows by continuity since we know that all foliations ξS_t have exactly two isolated singularities. In addition these singularities have non-zero divergence because they come from a contact structure. The first important point concerns the picture near these singularities.

Claim: Let p be an isolated singularity of a vector field X on a surface. If X has non-zero divergence at p and index one then there is a smooth disk D around p such that X is transverse to the boundary of D and defines a foliation which is topologically conjuguated to a radial foliation in D. In addition, if X belongs to a smooth one-parameter family of such vector fields then D can be chosen to depend smoothly on the parameter as well.

Proof of the claim. Let D_0 be a disk around p where the divergence of X (for some fixed auxilliary area form) does not vanish. Shrinking D_0 if necessary, we can assume that p is the only singular point in D_0. Poincaré–Bendixson's theorem then tells us there are only two kinds of orbits of X which are entirely contained inside D_0 (besides p itself): closed orbits and orbits whose α and ω–limits are both p. In both cases there would exist a subdisk D of D_0 such that the flux of X through ∂D vanishes. But Stokes' theorem shows that this cannot happen under our non-zero divergence assumption. So there is no nontrivial orbit contained in D_0.

Now let γ be a smooth simple closed curve going once around p in D_0 and which minimizes the number of tangencies with X among such curves. We will prove that γ cannot have any tangency with X hence the disk D bounded by γ is the disk we sought. A priori, tangencies with X come in two flavors. The orbit of X through the tangency point can be locally inside or outside the interior of D. But the minimizing property of γ forbids interior tangencies. Indeed, if q is such a tangency point then the orbit of X through q has to leave D at some point q' because it cannot be contained in D. Taking a long flow box along the piece of orbit between q and q', we see that we can modify γ to reduce the number of tangency points, a contradiction. Hence γ has only exterior tangencies with X. But the index of X at p is one plus half the number of interior tangencies minus half the number of exterior ones so, using the index assumption, we get that X is transverse to γ. To get the topological picture inside D, it suffices to remark that the Poincaré-Bendixson theorem now guaranties that all trajectories in D go from the boundary to p.

We now want this to work for one-parameter families. Any disk D satisfying the conclusions of the first part with respect to some vector field X_t in our family also satisfies them for $X_{t'}$ when t' is close to t because transversality to the boundary of D is an open condition. So we only need to prove that, for a fixed X, the space of suitable disks is path connected. Let D and D' be two such disks. Any trajectory through the boundary of D hits the boundary of D' exactly once. So we can build an isotopy pushing D to D' following these trajectories. □

We now return to our family of simple spheres S_t. We denote by D_t^n and D_t^s two families of disks given by the above claim around the north and south poles respectively. Notice that each $S_t \setminus (D_t^n \cup D_t^s)$ is an annulus transverse to ξ. Thus if A is an annulus then we can find an embedding of $A \times [\varepsilon, 1]$ such that (i) through (iii) from the lemma hold.

Since the image of this embedding is compact and the characteristic foliation on A_t for all t is non-singular we know that the maximal angle between ξ_p and T_pA_t is bounded below. Thus we can perturb the A_t so as to make the characteristic foliations generic without introducing any singularities. The only thing we want to check is that we can make this perturbation without getting rid of all closed leaves. So we proceed in two steps. First we perturb the family close to any chosen closed leaf to make that leaf non-degenerate. Then we perturb the whole one-parameter family of foliations on A like in [34] using a small enough perturbation so that the stable leaf created during the first perturbation does not disappear. □

Note that the hypotheses of the second point in the proposition are trivially satisfied if no characteristic foliation of a sphere S_t has a closed leaf. This leads to the remarkable fact that if ξ is overtwisted on B then this can be seen clearly on all spheres S_t for sufficiently large t.

Proof of Theorem 1.2. Fixing $p \in M$ consider the geodesic spheres $S_p(r)$ of radius r about p and the geodesic balls $B_p(r)$ that they bound. We can use Lemma 5.1 to conclude that all the spheres $S_p(r), r \leq \mathrm{inj}_p(g)$, have simple characteristic foliation. Recall that we are assuming that $\tau_p < \mathrm{inj}_p(g)$ *i.e.* $B_p(r), r < \tau_p$, is tight and $B_p(r)$, $r > \tau_p$, is overtwisted. Let

$$r' = \inf\{r \mid \text{such that } S_p(r) \text{ has a closed leaf in its characteristic foliation.}\}$$

Notice that $S_p(r')$ does have a closed leaf since simple foliations on spheres without closed leaves form an open set. By Proposition 5.3 the contact structure restricted to $B_p(r), r < r'$, is tight. Thus $r' = \tau_p$ and we see that $S_p(\tau_p)$ has a closed leaf in its characteristic foliation, which of course bounds an overtwisted disk. We then get overtwisted disks on spheres of higher radii using the third point of Proposition 5.3. □

6. THE CONTACT SPHERE THEOREM

In this section we prove the contact sphere theorem and discuss possible generalizations of it.

6.1. **Proof of the contact sphere theorem.** The following proposition is a variation on a similar result used in the proof of the topological sphere theorem [23], it does not involve any contact geometry.

Proposition 6.1. *Let M be complete simply connected Riemannian manifold whose sectional curvature satisfies $\frac{4}{9} < K \leq 1$. If p and q in M are at maximal distance, that is $d(p,q) = \mathrm{diam}(M)$, then there are radii r_p and r_q such that*

- *both closed balls $\overline{B}(p,r_p)$ and $\overline{B}(p,r_p)$ are embedded,* i.e. $r_p, r_q < \mathrm{inj}(g)$,
- *the ball $\overline{B}(q,r_q)$ is convex,* i.e. $r_q < \mathrm{conv}(g)$,
- *$M = \overline{B}(p,r_p) \cup \overline{B}(q,r_q)$, and*
- *the boundary of each ball, $\overline{B}(p,r_p)$ and $\overline{B}(p,r_p)$, is contained in the interior of the other ball.*

We remark that the Bonnet Theorem guaranties that M is compact in the above proposition so points like p and q do exist.

Proof. Because M is compact, there is some δ such that $\frac{4}{9} < \delta < K \leq 1$. Let ε_p and ε_q be positive numbers to be fixed later. We set $r_p = \pi(1-\varepsilon_p)$ and $r_q = \frac{\pi}{2}(1-\varepsilon_q)$. Klingenberg's injectivity radius estimate, see [7, Theorem 5.10], is $\mathrm{inj}(g) \geq \pi$ so that both balls are embedded. The convexity radius is then at least $\pi/2$, see [6, Theorem 7.10], so our ball around q is convex. To prove the remaining two properties, it suffices to prove that, for any x in M, $d(x,q) \geq r_q$ implies that $d(x,p) < r_p$.

Let γ_1 be a geodesic segment between q and x so that $L(\gamma_1) = d(q,x) > r_q$. According to Berger's lemma [7, Lemma 6.2], there is a geodesic segment γ_2 from q to p such that the angle α between γ_1 and γ_2 at q is not greater than $\pi/2$. We want to get an upper bound on $d(p,x)$. According to Toponogov theorem, we only need to get it for the corresponding hinge in the round 2–sphere in Euclidean space with curvature δ, hence radius $R = \frac{1}{\sqrt{\delta}}$.

Rather than considering lengths, it is more convenient to work with the angle under which a segment is seen from the center of the sphere. The actual length is R times this angle. Let a, b and d be angles such that the triangle we are looking at has sides whose length are Rb coming from γ_1, Rd coming from γ_2 and Ra which is not less than $d(p,x)$. All these angle are between 0 and π. The spherical law of cosines tells us

$$\cos(a) = \cos(b)\cos(d) + \sin(b)\sin(d)\cos(\alpha)$$

The second term is non-negative because $\alpha \leq \pi/2$. Since x is not in the ball around q, we have $Rb \geq \pi(1-\varepsilon_q)/2$, so $\cos(b) \leq \cos(\pi\frac{1-\varepsilon_q}{2R})$ and and we get, from the law of cosines above, $a \leq \pi - \pi\frac{1-\varepsilon_q}{2R}$. The corresponding distance is $\pi\left(R - \frac{1-\varepsilon_q}{2}\right)$ which is strictly less than $\pi(1-\varepsilon_p) = r_p$ whenever $2\varepsilon_p + \varepsilon_q < 3 - \frac{2}{\sqrt{\delta}}$. The lower bound hypothesis on δ allows to choose positive ε_p and ε_q satisfying this inequality. □

Proof of Theorem 1.1. We now gather the different ingredients of the contact sphere theorem, highlighting how Riemannian geometry, topological methods in contact geometry and pseudo–holomorphic curves arguments interact in this proof.

Since both the hypotheses and the conclusion of the theorem are scale invariant, we can assume that the curvature is bounded above by 1 so that $\frac{4}{9} < K \leq 1$. Moreover, we can assume that M is simply connected as pulling the contact structure and metric back to the universal cover of M does not affect the curvature pinching.

Deep classical Riemannian geometry gives, through Proposition 6.1, that there are two geometrics balls B_{cvx} and B_{big} whose interior covers our manifold M and such that B_{cvx} is weakly convex.

We assume for contradiction that ξ is an overtwisted contact structure. A priori, it could be that all overtwisted disks intersect both B_{cvx} and B_{big}. However there is no loss of generality in assuming that there is one, which we denote by D, which misses the center q of B_{cvx}.

Our comparison of Riemannian and almost-complex convexity combines with pseudo-holomorphic curves arguments of Gromov and Hofer to tell us, through Theorem 1.3, that B_{cvx} is a tight ball.

We can now use either the classification of tight contact structures on balls by Eliashberg [10] or, more elementarily, our description in Proposition 5.3 and its proof. Either way, we can construct a contact vector field transverse to the concentric spheres that make up $B_{\text{cvx}} \setminus \{q\}$. This contact vector fields generates a contact isotopy that will push any subset of B_{cvx} that misses q into an arbitrarily small neighborhood of ∂B_{cvx}. In particular we can push D into B_{big}.

Based on the special geometry of compatible metrics and a topological argument using Giroux's study of bifurcations for characteristic foliations, Proposition 5.3 then tells us that there is an overtwisted disk on ∂B_{big}. However, as we know $\partial B_{\text{big}} \subset B_{\text{cvx}}$, this contradicts the tightness of B_{cvx}. Hence we see that ξ must be tight.

Although this is does not completely follow from the previous discussion, the ambiant manifold is the 3-sphere as is guarantied by the classical sphere theorem. Now we get that ξ is isomorphic to the standard contact structure because all tight contact structures on the sphere are standard. This later fact is due to Eliashberg [10] and uses purely topological method in contact geometry (see also [16, Remark 2.18] for Giroux's alternative proof). □

Remark 6.2. As an immediate corollary of the contact sphere theorem, we obtain that any contact structure compatible with the round metric on S^3 is isotopic to the standard one ξ_0. While this result is not obvious, it already follows from older results. Indeed, suppose that ξ is compatible with a round metric and denote by R the Reeb vector field involved. Corollary 2.5 guaranties that R is geodesic (its orbits are great circles parametrized by arc length) and divergence free. Although there is an infinite dimensional space of geodesic vector fields on S^3, Gluck and Gu proved in [18] that they become completely rigid if we assume in addition that they are divergence free. So ξ is actually conjugated to ξ_0 by an isometry.

This rigidity is of a completely different nature than the statement of the sphere theorem where the isotopy with the standard structure is unrelated to any rigid structure on the sphere.

6.2. **Extensions of the contact sphere theorem.** We now discuss possible extensions and strengthening of the contact sphere theorem. We first remark that the hypothesis on the metric is open in the space of compatible metrics. This space can be seen as the product of the space of metrics on ξ and the space of positive constants θ'. We now describe one way to alter the statement of Theorem 1.1 so that it applies to an open set in the space of all metrics. It uses the projection C defined at the end of Section 2.

Given any metric g on M we define the *ξ-adapted sectional curvature* for g to be the sectional curvature of $C(g)$, and denote this curvature as $\sec_\xi(g)$. We have the following immediate corollary of Theorem 1.1 and the fact that the projection C depends smoothly on g.

Corollary 6.3. *Let (M, ξ) be a closed contact manifold and g any complete Riemannian metric. If there is a constant $K_{max} > 0$ such that the ξ-adapted sectional curvatures of g satisfy*

$$0 < \frac{4}{9} K_{max} < \sec_\xi(g) \leq K_{max},$$

then the universal cover of M is diffeomorphic to the 3–sphere by a diffeomorphism taking the lift of ξ to the standard contact structure on the 3–sphere.

Moreover, the set of metrics satisfying the the ξ-adapted sectional curvature pinching is open in the set of all metrics.

It would be interesting to see how the sectional curvature and the ξ-adapted sectional curvature of g compare. For instance, there should be a version of the sphere theorem involving weakly compatible metrics, where the pinching constant is weakened by a factor depending on the derivatives of ρ and θ'.

Another natural question is if the pinching constant $4/9$ in Theorem 1.1 is optimal. Recall that the classical pinching constant $1/4$ of the sphere theorem is optimal in even dimensions. Moreover, in all dimensions, the classical proof of the sphere theorem shows that any simply connected complete Riemannian manifold whose sectional curvature is $1/4$-pinched can be covered by two balls. In trying to extend this idea to the contact geometric setting, notice that the techniques of Eliashberg mentioned above allow one to prove that a contact 3–manifold which can be covered by two standard balls is tight, hence we can hope to get a contact sphere theorem using this strategy for the pinching constant $1/4$. Unfortunately, our tightness radius estimate is not strong enough for this approach to work directly and extra topological considerations were needed to prove Theorem 1.1. It may be that there are better tightness radius estimates in pinched manifolds analogous to what happens with the classical injectivity radius (there is no good general injectivity radius estimate in odd dimensions and Klingenberg's estimate used above really needs pinched curvature).

While our proof does not give any clues as to how one might improve the pinching constant from $4/9$ to $1/4$, we notice that one might hope for an even better pinching constant given that in dimension 3, Hamilton's Ricci flow [20] allows one to prove that any pinching constant (or even asking only the Ricci curvature to be positive) will suffice in the topological sphere theorem.

Question 6.4. *What is the optimal curvature condition in the contact sphere theorem?*

One might also ask for a higher dimensional analog of the contact sphere theorem, but at the moment we seem to know too little about contact topology outside of dimension 3 for an approach similar to the one taken here to work (see [14] for some steps in this direction).

7. Examples

In this section we apply our theorems to several examples. In particular in the first subsection we look at situations where Theorem 1.9 can be used to prove tightness of contact structures. In the following subsection we consider overtwisted contact structures and compare the estimate we have on the tightness radius to the actual tightness radius.

7.1. **Examples in nonpositive curvature.**

Our first goal is to provide examples when Theorem 1.9 is applicable.

Flat 3-torus: We begin by investigating the well-known family of contact structures ξ_k on the 3-torus T^3 in standard coordinates, defined as $\xi_k = \ker\alpha_k$, for $k \in \mathbb{N}$, where $\alpha_k = \cos(kz)\,dx - \sin(kz)\,dy$. The Reeb field is given by $R_{\alpha_k} = \cos(kz)\,\partial_x - \sin(kz)\,\partial_y$. In the flat metric $g_{T^3} = dx^2 + dy^2 + dz^2$ on T^3, one may compute

$$* \, d\alpha_k = k\,\alpha_k \quad \text{and} \quad \|\alpha_k\| = \|R_k\| = 1.$$

Therefore by definition the flat metric is compatible with all ξ_k. Thanks to Equation (1) we get $d_g = 0$ and since $\sec(g_{T^3}) \equiv 0$, the Inequality (1.5) of Theorem 1.9 holds and the theorem concludes that ξ_k are universally tight (as is well-known).

Hyperbolic 3-space: Consider the upper half space model of the hyperbolic 3-space i.e. $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ equipped with the metric

$$g_{\mathbb{H}^3} = \frac{1}{z^2}\big(dx^2 + dy^2 + dz^2\big).$$

Consider the same family of contact structures ξ_k and contact forms α_k as above. Since the hyperbolic metric $g_{\mathbb{H}^3}$ is conformal to the flat metric, the Reeb field R_k is orthogonal to ξ_k, thus $g_{\mathbb{H}^3}$ is weakly compatible with each ξ_k. Let us determine the "parameters" of Theorem 1.9. One may check that

$$* \, d\alpha_k = (kz)\,\alpha_k,$$

and thus obtain that $\rho = \frac{1}{z}$ and $\theta' = k\,z$. Hence we see that

$$d\ln(\rho) = -\frac{1}{z}\,dz, \qquad \nabla \ln\rho = -z\partial_z,$$
$$d\ln(\theta') = \frac{1}{z}dz \quad \text{and} \quad (\nabla \ln\theta')^{\perp} = 0.$$

Since $\sec(g_{\mathbb{H}^3}) \equiv -1$ and $d_g = \max_{\mathbb{H}^3} \|\nabla\ln\rho - (\nabla\ln\theta')^{\perp}\| = 1$, Inequality (1.5) of Theorem 1.9 holds and the theorem concludes again that ξ_k are tight.

Curl eigenfields are of special interest in hydrodynamics and other branches of physics [12]. As an example we point out is the the family of 1-forms:

$$\alpha_k' := \cos(k \ln(z))\, dx - \sin(k \ln(z))\, dy,$$

which are curl eigenfields on $\mathbb{H}^3$ with eigenvalues $\theta' = k$.

Product $\mathbb{R}\times\mathbb{H}^2$*:* For the third nonpositive curvature example let $M = \mathbb{R}\times\mathbb{H}^2$ be equipped with the product metric

$$g_{\mathbb{R}\times\mathbb{H}^2} = dt^2 + \frac{1}{y}\big(dx^2 + dy^2\big), \qquad (t,x,y) \in \mathbb{R}\times\mathbb{H}^2,\ y > 0.$$

We consider the following 1-form on M:

$$\beta = y^{\frac{1}{2}}\, dt + y^{-\frac{1}{2}}\, dx,$$

since β is a positive rescaling of $dx + y\, dt$, $\xi = \ker\beta$ is just the standard tight contact structure on $M \cong \mathbb{R}^3$. One may compute $*d\beta = \frac{1}{2}\beta$ and that $\rho = \|\beta\|^{-1} = (2\,y)^{-\frac{1}{2}}$, and $\theta' = \frac{1}{2}$. Therefore, β defines a $\frac{1}{2}$-curl eigenfield on M. Calculation of the parameters of Theorem 1.9 yields

$$d\ln(\rho) = -\frac{dy}{2\,y}, \quad \nabla \ln\rho = -\frac{1}{2} y\partial_y, \quad d\ln(\theta') = 0, \text{ and}$$
$$d_g = \max_M \|\nabla \ln \rho\| = \frac{1}{2}.$$

Since, $\sec(g_{\mathbb{R}\times\mathbb{H}^2}) \leq 0$, the condition of Theorem 1.9 fails, and from Theorem 1.5 we can only get a tightness radius estimate ($\mathrm{conv}(g_{\mathbb{R}\times\mathbb{H}^2}) = +\infty$) of

$$\tau(\mathbb{R}\times\mathbb{H}^2,\xi) \geq \mathrm{ct}_0^{-1}\big(\frac{1}{2}\big) = 2.$$

7.2. **Overtwisted contact structures in flat 3-space.** On $\mathbb{R}^3$ with the Euclidean metric $g = dr^2 + r^2\, d\theta^2 + dz^2$ in cylindrical coordinates (r,θ,z) we consider contact forms which are invariant under vertical translations, rotations around the z-axis and tangent to rays orthogonal to the z axis:

$$\alpha = a(r)\, dz + b(r)\, d\theta.$$

This defines a smooth 1-form if $b(r) = r^2\tilde{b}$ for some smooth function $\tilde{b}$. Let vol be the usual euclidean volume form on $\mathbb{R}^3$ and δ be defined by $\alpha\wedge d\alpha = \delta\, vol_g$. Here we will assume that α is a positive contact form, i.e. $\delta > 0$. Observe that

$$\delta = \frac{ab' - a'b}{r}, \quad \|\alpha\| = \sqrt{a^2 + \left(\frac{b}{r}\right)^2}, \text{ and } \quad R_\alpha = \frac{1}{\delta}\left(b'/r\partial_z - a'/r\partial_\theta\right).$$

The line field normal to $\ker\alpha$ is directed by the vector field $a\partial_z + \frac{b}{r^2}\partial_\theta$ dual to α. So $\xi = \ker\alpha$ is weakly compatible with the Euclidean metric if

$$\begin{cases} b' &= ra \\ a' &= -\frac{b}{r}. \end{cases} \tag{7.1}$$

Note that the equation is sufficient but not necessary, we have considerably more freedom: a priori we could have multiplied both right hand sides by any non vanishing function of r.

A priori Equation (7.1) is not well behaved at $r = 0$. But if a solution exists then we have $b = -ra'$ so $b' = -ra'' - a'$ which when combined with the first equation yields

$$ra'' + a' + ra = 0 \tag{7.2}$$

This is equivalent to $r^2a'' + ra' + r^2a = 0$ and a continuous at zero. We recognize the Bessel equation of order 0. Thus, for any choice of $b(0)$ we can use the solution $a = a(0)J_0$, where J_0 is the zeroth Bessel function of the first kind, and set $b = -ra'$ as required. Both a and b are analytic functions defined for all $r \geq 0$. We may check that a and b cannot both vanish simultaneously so that α is nonsingular, and a contact form. So we have a contact structure $\xi_{\mathrm{ot}} = \ker \alpha$ which is overtwisted at infinity because of the oscillatory behavior of $J_0(r)$, which is well known to vanish for arbitrarily large r. This must be compared to Corollary 1.4 which implies that any contact structure compatible with the flat metric on $\mathbb{R}^3$ is tight. Again we see that weak compatibility is much more flexible.

Numerical experiments performed with the above solution suggest the tightness radius estimate given by Theorem 1.5 is about 0.15 whereas we know the contact structure is tight when restricted to the cylinder $D_r \times \mathbb{R}$ where r is less than the first zero of $J_0' = -J_1$. This first zero is slightly greater than 3.

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SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332
E-mail address: etnyre@math.gatech.edu
URL: http://www.math.gatech.edu/~etnyre

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: rako@tulane.edu
URL: http://www.math.tulane.edu/~rako

UNIVERSITÉ PARIS SUD 11, 91430 ORSAY, FRANCE
E-mail address: patrick.massot@math.u-psud.fr
URL: http://www.math.u-psud.fr/~pmassot/