CONTACT TOPOLOGY AND HYDRODYNAMICS I:
Beltrami fields and the Seifert Conjecture

JOHN ETNYRE
Department of Mathematics
Stanford University
Palo Alto, CA 94305

ROBERT GHRIST
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160

Abstract

We draw connections between the field of contact topology (the study of totally non-integrable plane distributions) and the study of Beltrami fields in hydrodynamics on Riemannian manifolds in dimension three. We demonstrate an equivalence between Reeb fields (vector fields which preserve a transverse nowhere-integrable plane field) up to scaling and rotational Beltrami fields (nonzero fields parallel to their nonzero curl). This immediately yields existence proofs for smooth, steady, fixed-point free solutions to the Euler equations on all 3-manifolds, and all subdomains of $\mathbb{R}^3$ with torus boundaries.

This correspondence yields a hydrodynamical reformulation of the Weinstein Conjecture from symplectic topology, whose recent solution by Hofer (in several cases) implies the existence of closed orbits for all C^∞ rotational Beltrami flows on S^3. This is the key step for a positive solution to a "hydrodynamical" Seifert Conjecture: all C^ω steady flows of a perfect incompressible fluid on S^3 possess closed flowlines. In the case of spatially periodic Euler flows on $\mathbb{R}^3$, we give general conditions for closed flowlines derived from the algebraic topology of the vector field.

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1 Introduction

The goal of this paper is to delineate several applications of classical and recent theorems from contact topology to the field of hydrodynamics on Riemannian manifolds of odd dimension. For convenience and the sake of applications, we restrict to dimension three, although the basic relationships remain true in any odd dimension.

In hydrodynamics, we consider the class of vector fields whose behavior is most fascinating and whose analysis has been most incomplete: these are the *Beltrami fields*, or fields which are parallel to their own curl. All such fields are steady solutions to Euler's equations of motion for a perfect incompressible fluid. Flows generated by such fields have several noteworthy properties, such as extremization of the L^2 energy functional (Equation 22), as well as the potential to display the phenomenon sometimes called *Lagrangian turbulence*, in which a volume-preserving flow has flowlines which fill up regions of space ergodically [7, 62].

We relate the study of such flows with the rapidly developing field of contact topology. On an odd-dimensional manifold, a contact structure is a maximally nonintegrable hyperplane field and is the odd-dimensional analogue of a symplectic structure (see §1.2 for background). Though long a fixture in the literature on geometric classical mechanics (dating back to the work of Lie), genuine applications of the theory of contact geometry are rare, if they exist at all. A few authors have noted the existence of contact structures in hydrodynamical settings [31, 36], but genuine applications of such structures would appear to be limited to (Marsden-Weinstein) reductions [57, 1, 46, 5], which allow one to simplify integrable or near-integrable systems. In the present hydrodynamical context, the integrable case is the least interesting class of flows — the genuine interest (and difficulty) lies in analyzing nonintegrable dynamics. In this paper we initiate the use of contact topology for nonintegrable fluid flows.

The field of contact geometry/topology has of late been quite active and successful in understanding and classifying contact structures. The present state of affairs, brought about via the work of Eliashberg, Gromov, Hofer, and others [15, 16, 17, 35, 38], encompasses several very strong results in this area. In particular, the method of analysing contact structures via *Reeb fields*, or transverse vector fields whose flows preserve the contact form, has recently proven useful [38, 40].

In §1.1 and §1.2, we provide the necessary background information from each field, since we hope to promote interaction between both areas of mathematics. Then, in §2, we derive an equivalence between Beltrami fields and Reeb fields:

Theorem *The class of (nonsingular) vector fields on a three-manifold parallel to their (nonsingular) curl is identical to the class of Reeb fields under rescaling.*

This immediately yields an elegant existence proof:

Theorem *Every closed 3-manifold admits smooth, steady,* nonsingular *solutions to the Euler equations for a perfect incompressible fluid. Furthermore, every compact domain in* $\mathbb{R}^3$ *with toroidal boundary components likewise admits such solutions which leave the boundaries invariant.*

Furthermore, one has a novel reformulation of the Weinstein Conjecture from symplectic topology into a hydrodynamical context: namely, that rotational Beltrami flows on closed Riemannian manifolds of odd dimension must have closed flowlines. We spell out this reformulation in §3. The question of whether vector fields on three-manifolds are forced to exhibit periodic solutions has a rich history (see, *e.g.*, [45, 44]). The following conjecture of Seifert has been a focus of inquiry:

The Seifert Conjecture *Every* C^k *vector field on* S^3 *possesses a closed orbit.*

This conjecture was shown to be *false* for $k = 1$ by Schweitzer [58], for $k = 2$ by Harrison [37], for C^∞ by K. Kuperberg [45], and for $k = \omega$ (*i.e.*, real-analytic) by G. Kuperberg and K. Kuperberg [44]. There is in addition a C^1-counterexample for volume preserving fields due to G. Kuperberg [43]. In contrast, a recent theorem of Hofer [38] positively resolves the Weinstein Conjecture on S^3. Via the correspondence between Beltrami and Reeb fields, we derive the startling corollary that all C^∞ rotational Beltrami flows on any Riemannian three-sphere (and certain other manifolds) possess closed flowlines, independent of the Riemannian metric and the volume form preserved. In other words, the Kuperberg plug cannot be parallel to its curl under *any* metric. Using this corollary, we provide a positive solution to a version of the Seifert Conjecture for perfect incompressible fluids:

Theorem *Every* C^ω *steady solution to the Euler equations for a perfect incompressible fluid on* S^3 *possesses a closed flowline.*

This result is independent of the Riemannian metric as well as the volume form which is conserved: it is a truly robust statement. Although we can relax the smoothness condition to C^∞ for Beltrami flows, it appears very difficult to improve the above theorem to C^∞, since, for the non-Beltrami flows, we employ techniques from singularity theory which are dependent upon real-analyticity.

Finally, in §4, we apply contact-topological methods to a fundamental class of examples in hydrodynamics: spatially periodic flows on $\mathbb{R}^3$ (*i.e.*, flows on the 3-torus T^3). In the case of Euclidean geometry, these include the *ABC flows*. Based upon the classification theorems of Giroux and Kanda [32, 42], we give conditions which force the existence of closed orbits on $\mathbb{R}^3$ in terms of the algebraic topology of the vector field:

Theorem *Any C^∞ steady rotational Beltrami field on T^3 which is homotopically nontrivial must have a contractible closed flowline. More generally, any C^ω steady Euler flow on T^3 which is homotopically nontrivial must have a closed flowline.*

Throughout this paper, we use $\mathcal{L}_u$ to denote the Lie derivative along the vector field u, and we use ι_u to denote contraction by u.

1.1 Topological hydrodynamics

The principal business of hydrodynamics is to understand the dynamical properties of fluid flows, the simplest class of which are incompressible, perfect flows described by the Euler equation. The following treatment of hydrodynamics on Riemannian manifolds is based on the approach of [9, 8], in which the reader will find several excellent references.

In order to apply our results to manifolds other that $\mathbb{R}^3$, we adopt the following convention: the *curl* of a vector field X on a Riemannian 3-manifold M with metric g and (arbitrary) distinguished volume form μ is the unique vector field $\nabla\times X$ given by $\iota_{(\nabla\times X)}\mu = d\iota_X g$. By taking the curl with respect to an arbitrary volume form, one may apply the subsequent machinery to more general classes of fluids (*e.g.*, the *barotropic* flows, for which density is a function of pressure) which are incompressible with respect to a rescaled volume form only. Note that in the case where μ is the volume form for the metric g, the definition of $\nabla \times X$ assumes the more familiar form [2]: $\nabla \times X = (*d\iota_X g)^\#$, where $*$ denotes the Hodge star and $^\#$ denotes the isomorphism from 1-forms to vector fields derived from g. The uniqueness of $\nabla \times X$ in the above definition comes from the fact that for a fixed volume form μ, contraction into μ is an isomorphism from vector fields to 2-forms.

The standard Euler equations on Eulclidean $\mathbb{R}^3$ are easily translated into a global setting for use on arbitrary Riemannian 3-manifolds.

Definition 1.1 Let u denote a (time-dependent) vector field on a Riemannian 3-manifold M with metric g and distinguished volume form μ. Then u satisfies *Euler's equation for a perfect incompressible fluid* if

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_u u &= -\nabla p \\ \mathcal{L}_u \mu &= 0, \end{aligned} \tag{1}$$

for some (time-dependent) function p (pressure). Here, ∇_u denotes the covariant derivative associated to g along u. The quantity $\mathcal{L}_u\mu$ vanishes if and only if the flow associated to u is volume-preserving with respect to μ. We call vector fields (or corresponding flows) satisfying Equation (1) *Euler fields* (or *Euler flows* respectively).

It will be most convenient to transform the first portion of the Euler Equation into a more useful form. Using the identity [2, p. 588]

$$\nabla_u u = \mathcal{L}_u u - \frac{1}{2}\nabla(\|u\|^2),$$

and transforming all terms from vector fields to one-forms by means of the Riemannian metric g, we have the following exterior differential system:

$$\frac{\partial(\iota_u g)}{\partial t} + \mathcal{L}_u(\iota_u g) = -d\left(p - \frac{1}{2}\iota_u\iota_u g\right), \tag{2}$$

which, by the Lie formula $L_u = di_u + i_u d$ and the definition of the curl $\nabla \times u$, yields

$$\frac{\partial(\iota_u g)}{\partial t} + \iota_u\iota_{\nabla\times u}\mu = -dP, \tag{3}$$

where $P := p + \frac{1}{2}\iota_u\iota_u g$ is a "reduced" pressure term. A vector field u is thus an Euler field if and only if it satisfies $\mathcal{L}_u\mu = 0$ and Equation (3) for some function $P : M \to \mathbb{R}$.

The field of hydrodynamics has been most successful in understanding steady Euler flows, or flows without time dependence. Such steady solutions are fixed points of the evolution operator associated to the Euler equations in the configuration space of volume-preserving fields. *For the remainder of this work, all vector fields will be assumed to be steady.* The topology of a steady Euler flow is almost always very simple:

Theorem 1.2 (Arnold [6]) *Let u be a C^ω nonsingular Euler field on a closed Riemannian three-manifold M. Then, if u is not everywhere colinear with its curl, there exists a compact analytic subset $\Sigma \subset M$ of codimension at least one which splits M into a finite collection of cells $T^2 \times \mathbb{R}$. Each $T^2 \times \{x\}$ is an invariant set for u having flow conjugate to linear flow.*

The proof is straightforward: the reduced pressure term P is an integral for the flow since $\iota_u dP = \iota_u\iota_u\iota_{(\nabla\times u)}\mu = 0$ (this is Bernoulli's Theorem from classical hydrodynamics). This integral is nondegenerate ($dP \neq 0$) if and only if the vorticity and velocity fields are independent. In the nondegenerate case, for regular values of P, the preimage is an invariant compact two-manifold possessing a nonsingular flow: copies of T^2. The preimage of the (finite number of) critical values of P forms the singular set Σ. By analyticity, Σ may not contain open sets; hence, the topology of the flowlines are almost everywhere highly constrained *except* in the case where the curl of u is colinear with u, *i.e.*, $\iota_u\iota_{\nabla\times u}\mu \equiv 0$.

The class of fields for which the above integral degenerates is extremely important.

Definition 1.3 A vector field u on a Riemannian manifold M^3 is a *Beltrami field* if it is parallel to its curl: *i.e.*, $\nabla \times u = fu$ for some function f on M. A *rotational* Beltrami field is one for which $f \neq 0$; that is, the curl is nonsingular.

Beltrami fields have generated significant interest in the fluids community [7, 14, 52, 53, 26, 13]. As shown by Arnold [7], Beltrami fields extremize an energy functional within the class of flows conjugate by μ-preserving diffeomorphisms. Furthermore, the flowlines of a Beltrami flow on a three-manifold are not always constrained to lie on a 2-torus, as is the case for non-Beltrami steady Euler flows. Hence, the manifestation of "Lagrangian turbulence" can only appear within the class of Beltrami fields (see [31]). In addition, Beltrami fields play an important role in magnetodynamics (where they are known as "force-free fields") [49], the stability of matter [47], and other contexts. Finally, there is the result of [13] that on Euclidean T^3 the Beltrami fields form an L^2-orthogonal basis for all incompressible fluid flows, including solutions to the Navier-Stokes equations.

The classic examples of Beltrami fields which exhibit Lagrangian turbulence are the *ABC flows*, generated by a certain family of vector fields on T^3 which are eigenfields of the curl operator (see Equation 17 in §4). Although the ABC flows have been repeatedly analysed [7, 14, 52, 26, 62], few results are known, other than for a handful of near-integrable examples. Beltrami fields in general are even less well understood. Typically, one wants to restrict to, say, Beltrami fields on Euclidean $\mathbb{R}^3$ or T^3, under the fixed standard metric and volume form. Under these restrictions, it becomes nearly impossible to do any sort of analysis on the class of Beltrami fields. A small perturbation, even within the class of volume-preserving fields, almost always destroys the Beltrami property. We note in particular the difficulty of answering global questions about Beltrami flows, such as the existence of closed orbits, the presence of hydrodynamic instability, and the minimization of the energy functional. In this work, we provide some geometric and topological tools which may prove robust enough to overcome these difficulties.

1.2 Contact topology

For the sake of concreteness and applicability, we will restrict all definitions and discussions to the case of contact structures on three-manifolds, noting that several features hold on arbitrary odd-dimensional manifolds. For introductory treatments, see [51, 3].

A *contact structure* on a three-manifold M is a maximally nonintegrable plane field. That is, to each point $p \in M$, we assign a plane in T_pM, varying smoothly with p in such a manner that the Frobenius condition fails everywhere. In particular, a contact structure is locally twisted at every point and may be thought of as an "anti-foliation" — no disc may be embedded whose tangent planes agree with the plane field.

Definition 1.4 A *contact form* on M is a one-form α on M such that $\alpha \wedge d\alpha \neq 0$. That is, $\alpha \wedge d\alpha$ defines a volume form on M. A *contact structure* is a plane field which is the kernel of a (locally defined) contact form: $\xi = \ker(\alpha) = \{v \in T_pM : \alpha(v) = 0, p \in M\}$.

It is usually sufficient to consider contact structures which are the kernel of a globally defined contact 1-form: these are called *cooriented* contact structures.

The manifestation of the Darboux Theorem in this context implies that every contact structure locally looks like (is contactomorphic to, or diffeomorphic via a map which carries the contact structure to) the kernel of $dz + x\,dy$ on $\mathbb{R}^3$ (see [51]), illustrated in Figure 1.

Definition 1.5 Given a three-manifold M with contact structure ξ, let $F \subset M$ be an embedded surface. Then the *characteristic foliation* on F, F_ξ, is the (singular) foliation on F generated by

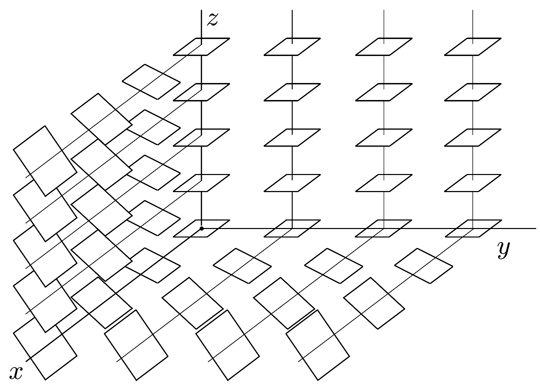

FIGURE 1: Every contact structure is locally equivalent to the kernel of $dz + xdy$.

the (singular) line field

$$\mathcal{F} = \{T_pF \cap \xi_p : p \in F\}.$$

A contact structure ξ is *overtwisted* if there exists an embedded disc $D \subset M$ such that the characteristic foliation D_ξ has a limit cycle (see Figure 2). A contact structure which is not overtwisted is called *tight*.

One thinks of the contact planes as "carving" an embedded surface along the characteristic foliation. This is one of the features that gives contact topology a dynamical flair: there is a strong relationship between the global features of the contact structure and the dynamics of the characteristic foliations on embedded discs.

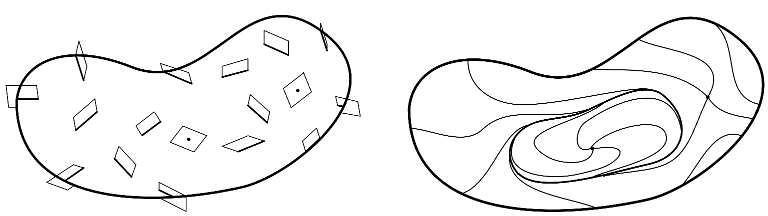

FIGURE 2: An overtwisted disc in a contact structure.

The classification of overtwisted structures up to isotopy coincides with the classification of plane fields up to homotopy [15] and hence reduces to a problem in algebraic topology. The classification of tight structures, on the other hand, is far from complete. Tight contact structures exhibit several "rigid" features which make issues such as classification very difficult. The classification of tight contact structures is completed only on S^3 [17], on certain lens spaces [19], and on the three-torus T^3 [32, 42]. Perhaps the most pressing question in the field is whether every closed orientable 3-manifold supports a tight contact structure.

Given a contact form α generating the contact structure ξ, we may associate to it a vector field which is transverse to ξ and preserves α under the induced flow. Such vector fields were first considered by Reeb [57].

Definition 1.6 Given a contact form α on M, the *Reeb field* associated to α is the unique vector field X such that

$$\iota_X d\alpha = 0 \qquad \text{and} \qquad \iota_X \alpha = 1. \tag{4}$$

The equation $\iota_X \alpha = 1$ is a normalisation. As we are primarily concerned with the topology of the flowlines, which does not depend on the parametrisation, we will also consider the class of *Reeb-like* fields, for which $\iota_X d\alpha = 0$ and $\iota_X \alpha > 0$.

The relationship between the dynamics of a Reeb field and the topology of the transverse contact structure have been analysed most notably by Hofer *et al.* [38, 40]. We will consider these results in detail in §3.

Example 1.7 The standard contact structure on the unit $S^3 \subset \mathbb{R}^4$ is given by the kernel of the 1-form

$$\alpha = \frac{1}{2}\left(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3\right). \tag{5}$$

The Reeb field associated to α is the unit tangent field to the standard Hopf fibration of S^3; hence, the contact structure ξ can be visualised as the plane field orthogonal to the Hopf fibration (orthogonal with respect to the metric on the unit 3-sphere induced by the standard metric on $\mathbb{R}^4$). It is a foundational result of Bennequin [10] that this structure is tight; furthermore, by Eliashberg [17], this is the unique tight contact structure on S^3 up to contactomorphism.

2 Geometric properties of Beltrami flows

2.1 A correspondence theorem

It is not coincidental that the Beltrami condition on a vector field — that the flow must continually twist about itself — is reminiscent of the notion of a nonintegrable, everywhere twisting plane field. We obtain a general equivalence between Beltrami and Reeb fields for arbitrary three-manifolds by working with moving frames and metrics adapted to the flow.

Theorem 2.1 *Let M be a Riemannian three-manifold. Any smooth, nonsingular rotational Beltrami field on M is a Reeb-like field for some contact form on M. Conversely, given a contact form α on M with Reeb field X, any nonzero rescaling of X is a smooth, nonsingular rotational Beltrami field for some Riemannian metric on M.*

Proof: Assume that X is a Beltrami field where $\nabla \times X = fX$ for some $f > 0$. Let g denote the metric and μ the volume form on M. On charts for M, choose an orthonormal frame $\{e_i\}_1^3$ such that $e_1 = X/\|X\|$.[1] Denoting by $\{e^i\}_1^3$ the dual 1-form basis, we have that $\iota_X g = \|X\| e^1$. Let α denote the one-form $\iota_X g = \|X\| e^1$, which is globally defined since X is nonsingular.

[1] Note that we must work on charts only in the case where the Euler class of the vector field ($e(X) \in H^2(M; \mathbb{Z})$) is nonzero.

The condition $\nabla \times X = fX$ translates to $d\iota_X g = f\iota_X\mu$, or, $d\alpha = f\iota_X\mu$. Since μ is a volume form, its representation on each chart is of the form $he^1 \wedge e^2 \wedge e^3$, with h a nonzero function. Hence, α is a contact form since

$$\alpha \wedge d\alpha = \iota_X g \wedge f\iota_X\mu = fh\|X\|^2 e^1 \wedge e^2 \wedge e^3 \neq 0. \tag{6}$$

Finally, X is Reeb-like with respect to α since

$$\iota_X d\alpha = f\iota_X\iota_X\mu = 0. \tag{7}$$

Conversely, assume that α is a contact form for M having Reeb field X with $Y = hX$ a rescaling by some $h > 0$. Then, on charts for M, choose a parallelization $\{e_i\}$ of M such that $e_1 = X$ and e_2 and e_3 are in ξ, the kernel of α, and form a symplectic basis for this plane field (this is always possible since $d\alpha|_\xi$ is a symplectic structure); hence, $d\alpha(e_2, e_3) = 1$. Again let $\{e^i\}$ denote the dual 1-forms to the $\{e_i\}$. On charts, choose the following metric adapted to the flow in the $\{e_i\}$-coordinates:

$$g = \begin{bmatrix} h^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

The transition maps respect this locally defined metric since e_1 is globally defined and the symplectic basis $\{e_2, e_3\}$ differs from chart to chart by an element of $U(1)$, (as the plane field is globally defined). We remark that in this metric, $\|X\| = 1/\sqrt{h}$ and $\|Y\| = \sqrt{h}$.

We claim that, under the metric g, Y is volume preserving and parallel to its curl. First, $\iota_Y g = e^1$ since for $i = 1, 2, 3$,

$$\iota_Y g(e_i) = g(Y, e_i) = g(he_1, e_i) = \delta_{i1}, \tag{9}$$

where δ_{ij} is the Kronecker delta. Next, note that $e^1 = \alpha$, since they act on the basis $\{e_i\}$ identically. Specifically, $\iota_{e_1}\alpha = \iota_X\alpha = 1$ and $\iota_{e_2}\alpha = \iota_{e_3}\alpha = 0$ since e_2 and e_3 were chosen to lie in ξ. We now have that $de^1 = e^2 \wedge e^3$ since for $i < j$,

$$de^1(e_i, e_j) = d\alpha(e_i, e_j) = \delta_{2i}\delta_{3j} - \delta_{2j}\delta_{3i}, \tag{10}$$

since we also chose the pair (e_2, e_3) to form a symplectic basis for ξ.

Let μ denote the volume form $h^{-1}e^1 \wedge e^2 \wedge e^3$. Then,

$$d\iota_Y\mu = d(\iota_{he_1}h^{-1}e^1 \wedge e^2 \wedge e^3) = d(e^2 \wedge e^3) = d^2(e^1) = 0; \tag{11}$$

hence, Y is volume preserving with respect to μ. To show that Y is Beltrami with respect to g and μ, it suffices to note that

$$d\iota_Y g = de^1 = e^2 \wedge e^3, \tag{12}$$

as well as

$$\iota_Y\mu = \iota_{he_1}h^{-1}e^1 \wedge e^2 \wedge e^3 = e^2 \wedge e^3. \tag{13}$$

Hence, $\nabla \times Y = Y$. □

Note that Y is divergence-free under the particular g-induced volume form if and only if the scaling function h is an integral for the flow: *i.e.*, $\mathcal{L}_Y h = 0$. The proof of this theorem generalizes to any odd dimension with little modification. Also, if X is a Beltrami field with singularities, then we may excise the singular points from the manifold and apply Theorem 2.1 to the nonsingular portion of the flow. Then X is still a Reeb flow for a contact form on the punctured manifold.

Corollary 2.2 *Every Reeb-like field generates a nonsingular steady solution to the Euler equations for a perfect incompressible fluid with respect to some Riemannian structure.*

2.2 Existence of nonsingular solutions to the Euler equations

Our first application is to the problem of existence of *nonsingular* solutions to Equation 3: specifically, C^∞ solutions which are free of fixed (stagnation) points. By Theorem 1.2, C^ω nonsingular solutions to the Euler equations are either level sets of integrable Hamiltonian systems or Beltrami fields. Relaxing the condition of this theorem to C^∞ still forces one to have an integrable system on the non-Beltrami portion of the flow — hence, one needs to piece together integrable level sets with boundary and Beltrami fields on manifolds with boundary.

According to Fomenko's theory on integrable systems [24, 23], closed irreducible nonsingular three-dimensional level sets of integrable C^∞ Hamiltonian systems which are nondegenerate (in the Bott-Morse sense [11]) must be graph-manifolds (*i.e.*, they decompose into a collection of Seifert-fibred pieces glued together along incompressible tori). This extends to the real-analytic case without the nondegeneracy restriction [22]. This class of three-manifolds is particularly simple. Since a typical closed irreducible three-manifold is not a graph manifold [59], it follows that "most" closed three-manifolds cannot have nonsingular C^ω solutions to the Euler equations unless there exist nonsingular Beltrami fields. Ostensibly, this would not appear to be easily discernible: functional-analytic arguments about eigenfields of the curl operator have no control over singularities in the vector field. In the C^∞ case, Fomenko's theory also restricts heavily the types of subregions of a manifold which can support integrable dynamics; hence, we are again reduced to the problem of finding nonsingular Beltrami fields in order to have nonsingular solutions to the Euler equations.

Theorem 2.3 *Every closed three-manifold has a C^∞ nonsingular rotational Beltrami flow on it for some Riemannian structure. In fact, there are an infinite number, distinct up to homotopy through nonsingular vector fields.*

Proof: The classical results of Martinet [50] and Lutz [48] imply that every closed three-manifold has a contact structure. Upon choosing a defining 1-form, the associated Reeb field is nonsingular and hence Beltrami by Theorem 2.1.

For the definition of the homotopy class of a vector field, see Definition 4.2. To show that there are a countable collection of homotopically distinct Beltrami fields on any three-manifold, we appeal to the classification of plane fields on three-manifolds, elucidated recently by Gompf [33]. In the case where the rank of the second cohomology $H^2(M;\mathbb{R}) \neq 0$ (*i.e.*, the cohomology is not pure torsional), there are at least a $\mathbb{Z}$'s worth of distinct Euler classes for plane fields on M; hence for (overtwisted) contact structures and the associated Beltrami fields. In the case where $H^2(M;\mathbb{Z})$ has only torsion elements, the Θ-*invariant* of Gompf [33] implies the existence of an infinite number of homotopy classes of plane fields: again, hence, of (overtwisted) contact structures and thus Beltrami fields. □

Note that it is *a priori* unclear how one would construct a (nonzero) Beltrami flow on a nontrivial three-manifold, such as $(S^1 \times S^2)\#(S^1 \times S^2)$. Proving global results about such flows would appear to be even more unlikely.

We may extend Theorem 2.3 to more physically relevant settings: namely, compact domains in $\mathbb{R}^3$ homeomorphic to a solid torus with solid tori removed. It is relevant to note the difficulty of proving the existence of such flows: a recent paper by Alber [4] requires a great deal of difficult analysis to obtain an existence proof of steady C^3 Euler flows on smooth, simply-connected domains in $\mathbb{R}^3$ having nonsingular vorticity (though not necessarily nonsingular velocity) with predetermined boundary conditions in the Euclidean metric.

Theorem 2.4 *Every compact domain $C \subset \mathbb{R}^3$ diffeomorphic to a solid torus with solid tori removed has a C^∞ nonsingular Beltrami field, tangent to the boundary components, for some C^∞ Riemannian structure on $\mathbb{R}^3$. The transverse contact structure associated to such fields may be chosen to be tight.*

Proof: By reembedding C within S^3, assume that the outermost torus boundary of C is unknotted in $\mathbb{R}^3$. Let γ_i denote a finite collection of curves in $S^3 - C$ whose exterior is diffeomorphic to the interior of the domain C. Impose on S^3 the standard tight contact structure $\xi = \ker(\alpha)$ from Equation (5). For each γ_i, a small C^0 perturbation suffices to make γ_i transverse to ξ, without changing the homeomorphism type of the exterior. The techniques of Moser [54] can be used to show that there exists a tubular neighborhood V_i of γ_i and a diffeomorphism $h_i : D^2 \times S^1 \to V_i$ with

$$h^*(\xi|_{V_i}) = \ker(d\phi + r^2 d\theta),$$

where (r, θ, ϕ) are cylindrical coordinates on $D^2 \times S^1$ [3, p. 171]. It must be the case that $h^*(\alpha|_{V_i}) = g(r, \theta, \phi)(d\phi + r^2 d\theta)$ for some positive function g, since this is a contact form with the same kernel as $d\phi + r^2 d\theta$. We thus may rescale α as follows. Choose a C^∞ positive function $f : S^3 \to \mathbb{R}$ (via partitions of unity) such that $h^*(f\alpha|_{V_i}) = d\phi + r^2 d\theta$. Then, since $f > 0$, $f\alpha$ is a contact form on S^3. The Reeb field X for $f\alpha$ is conjugate to the flow $\partial/\partial\phi$ on each V_i. Hence, we may remove an open invariant tubular neighborhood of each γ_i, leaving a domain diffeomorphic to the desired domain C. Pulling back by this diffeomorphism yields a contact form on C whose Reeb field leaves ∂C invariant. By Theorem 2.1, this is a nonsingular Beltrami field for the appropriate Riemannian structure. The original contact structure on S^3 contained no overtwisted discs, and removing tubes from S^3 and modifying with a contactomorphism does not introduce any further overtwisted discs; hence, the transverse contact structure on C is tight. Pulling back this form by the reembedding of C in S^3 does not alter the tightness. □

Most of these solutions to Euler's Equations are nonintegrable (in the Hamiltonian sense), since, by the work of Cassasayas *et al.* [12] (following the work of Fomenko and Nguyen [25]), only certain highly restricted domains in $\mathbb{R}^3$ can support a nonsingular integrable flow: namely, complements of so-called "zero-entropy" links in S^3 (see [30, App. A],[22]).

This proof yields a scholium concerning knotted flowlines in nonsingular flows (*cf.* [53] for a similar result which is only piecewise smooth and relies on the existence of solutions to the Navier-Stokes equation for its proof):

Corollary 2.5 *Any finite link L in S^3 may be realised as a subset of the closed orbits in a C^∞ nonsingular Beltrami flow on S^3 (or $\mathbb{R}^3$).*

In [22] we prove that unknots are *forced* in all sufficiently smooth Euler flows on the 3-sphere. In part III of this paper series, we discuss knotted flowlines in detail, proving the existence of fluid flows possessing all topological knotting and linking simultaneously [21].

3 Hofer's Theorem and the Hydrodynamical Seifert Conjecture

A fundamental conjecture in global symplectic and contact topology is the Weinstein Conjecture [61], which concerns how a symplectic form intersects certain hypersurfaces of a symplectic manifold. We briefly recount this conjecture, following [51]. For excellent treatments, see the books [41, 51].

Let M^{2n-1} be a hypersurface in a symplectic manifold (W^{2n}, ω). Then M is said to be of *contact type* if there exists a transverse expanding vector field X on a neighborhood of M. That is, X is transverse to M and $\mathcal{L}_X\omega = \omega$.

Conjecture 3.1 (Weinstein Conjecture, symplectic version [61]) *Any hypersurface M of contact type has closed characteristics. That is, the foliation generated by the line field*

$$\mathcal{F} = \{v \in T_zM : \omega(v, T_zM) = 0\},$$

has a closed leaf.

Any surface M of contact type has a natural 1-form α associated to it via $\alpha = \iota_X\omega$, where X is the expanding vector field. Since $d\alpha = \omega|_M$, it follows that α is a contact form with Reeb field generating the foliation $\mathcal{F}$ above. Similarly, given a Reeb field on M, one can embed M in a symplectic manifold as a hypersurface of contact type. Hence, the Weinstein Conjecture translates to whether Reeb fields on M have closed orbits. From Theorem 2.1, we thus have:

Conjecture 3.2 (Weinstein Conjecture, hydrodynamics version) *Every smooth[2] rotational Beltrami flow on a Riemannian three-manifold has a closed orbit.*

The simplest example of an Euler flow without closed flowlines is linear irrational flow on the Euclidean three-torus T^3. However, this flow has everywhere vanishing curl. A simple application of Stokes' Theorem implies that this flow (or any flow transverse to a closed surface) cannot be rotational Beltrami under any metric.

Great progress on the Weinstein Conjecture has been made in the past decade (see [41] for a historical account). For example, Viterbo [60] showed that the Weinstein Conjecture is true on any three-manifold of contact type in $\mathbb{R}^4$ equipped with the standard symplectic form. More important for our applications, though, is the following seminal result of Hofer:

Theorem 3.3 (Hofer [38]) *If $\xi = \ker\alpha$ is an overtwisted contact structure on a closed compact orientable three-manifold, then the associated Reeb field has a closed orbit, some multiple of which is contractible.*

The proof of Theorem 3.3 is highly nontrivial, relying on the techniques of pseudo-holomorphic curves pioneered by Gromov [35]. Such techniques yield, among other things, the following partial resolution to Conjecture 3.1.

[2] By "smooth" we mean C^∞. However, the question is certainly interesting in other degrees of regularity.

Theorem 3.4 (Hofer [38]) *The Weinstein Conjecture is true for S^3.*

In the case of S^3, the Reeb fields associated to overtwisted structures are covered by Theorem 3.3. According to Eliashberg's classification of contact structures on S^3 [17], any tight contact structure is contactomorphic to that of Example 1.7. The existence of a closed orbit then follows from an argument which reduces the tight case to that covered by a theorem of Rabinowitz [56].

Theorem 3.4 is the key step in the hydrodynamical Seifert Conjecture:

Theorem 3.5 *Any C^ω steady Euler flow on S^3 has a closed flowline.*

Proof: Given u a C^ω nonsingular Euler field on S^3 under a (likewise C^ω) Riemannian metric g, Theorem 1.2 presents two possible cases.

Case I: If u is not everywhere colinear with $\nabla \times u$, then the reduced pressure function P is an integral for the flow. Hence, S^3 is decomposed by a compact subset Σ into a finite collection of cells diffeomorphic to $T^2 \times \mathbb{R}$, where the each $T^2 \times \{x\}$ is invariant under the flow. The slope of the vector field on each cell may be constant and irrational; hence, we must consider the singular set, Σ, which is the inverse image of the (finite number of) critical values of the function $P : S^3 \to \mathbb{R}$ from Equation 3.

The key tool in this case comes from singularity theory: the inverse image of a critical value of P is a (Whitney) stratified set, since it is a real-analytic variety. That is, although the subset is not necessarily a manifold, it can be decomposed into manifolds of varying dimension (≤ 2 in our case) glued together in a sufficiently regular manner: see [34] for proper definitions and theory.

The structure of Σ is determined by the fact that it is transversally homogeneous in the flow direction. More specifically, let $x \in \Sigma$. Since u is nonsingular, the flow of u defines flowboxes everywhere. Hence, there is a neighborhood $N \cong D^2 \times \mathbb{R}$ of x in S^3 such that $u|_N$ points entirely in the $\mathbb{R}$-component. Denote by D_0 the disc $D^2 \times \{0\}$ containing $\{x\}$. Since $S^3 \setminus \Sigma$ is foliated by invariant tori, Σ is invariant, D_0 is transverse to Σ and $N \cap \Sigma \cong (D_0 \cap \Sigma) \times \mathbb{R}$. Thus, the transverse structure of Σ is invariant along flowlines of u.

By analyticity, Σ is at most two-dimensional. This, along with the Whitney condition for stratified sets [34], implies that the intersection $D_0 \cap \Sigma$ is homeomorphic to a radial k-pronged tree centred at x. If the number k is zero, then Σ is a 1-manifold, which by the compactness of Σ and the flow-invariance of the transverse structure, implies that u has a closed invariant 1-manifold. If $k = 1$, then Σ has an invariant 1-dimensional boundary component, which again by compactness and flow-invariance implies a closed orbit. If $k > 2$, then, since orbits of u are unique, x must lie within an invariant 1-stratum, which as before must continue to a closed invariant 1-manifold via flow-invariance and finiteness of the stratification.

Thus, if u has no closed orbits on Σ, then for every $x \in \Sigma$, the transverse prong number k is precisely two, and Σ is locally homeomorphic to $\mathbb{R}^2$. Hence it is an invariant closed 2-manifold with nonsingular vector field: a collection of 2-tori. Thus, S^3 is decomposed into a finite collection of $T^2 \times \mathbb{R}$ cells glued together (pairwise) along sets diffeomorphic to T^2, and we have expressed the three-sphere as a T^2-bundle over S^1, a contradiction (since S^3 is simply connected).

Case II: If u is everywhere colinear with $\nabla \times u$, then $\nabla \times u = fu$ and u is a Beltrami field. In the case where f is nonzero, u is rotational and the analysis of Case I is completely useless: unlike all other cases, we have a priori no information about the flow. However, by Theorem 2.1, this vector field is, up to a rescaling, a Reeb field. Hence, the associated flows are related by a time reparametrisation, which does not destroy the closed orbit whose existence follows from Theorem 3.4 of Hofer.

If f has zeros, then either f varies, or it is constantly zero. If f is not constant, then f is a nontrivial integral for u as follows. First, $\nabla \times u$ is μ-preserving since

$$d\iota_{\nabla \times u}\mu = d(d\iota_u g) = 0. \tag{14}$$

However, the left hand term simplifies to

$$d\iota_{\nabla \times u}\mu = d(f\iota_u\mu) = df \wedge \iota_u\mu, \tag{15}$$

since u is also μ-preserving. Finally, since we are on a 3-manifold, the 4-form $df \wedge \mu$ vanishes, implying

$$0 = \iota_u(df \wedge \mu) = (\iota_u df)\mu - df \wedge \iota_u\mu = (\iota_u df)\mu. \tag{16}$$

Thus, $\iota_u df = 0$ and f is an integral. The existence of a closed orbit then follows from the singularity-theory arguments of Case I.

Finally, if $\nabla \times u = 0$ everywhere, then the 1-form $\alpha = \iota_u g$ is closed and nondegenerate since $d\iota_u g = f\iota_u\mu = 0$. By the Frobenius condition, $\xi = \ker\alpha$ defines a smooth codimension-one foliation of S^3, to which u is transverse. However, by the Novikov Theorem [55], there is a closed leaf of ξ homeomorphic to T^2. This torus separates S^3 and hence cannot be transverse to the volume-preserving field u, contradicting the assumption that u is nonsingular. □

4 Beltrami flows on T^3

The important feature of Hofer's Theorem is the relationship between the "twistedness" of a contact structure and the dynamics transverse to it. In order to apply contact-topological methods to manifolds which are of greater interest to fluid dynamicists (T^3 or $\mathbb{R}^3$) we consider algebraic obstructions to tightness.

The most important three-manifolds from the point of view of hydrodynamics are the solid torus $D^2 \times S^1$ and the three-torus $T^3 = S^1 \times S^1 \times S^1$. Some authors have suggested using Beltrami "tubes" to model certain domains of turbulent regions in flows, following observations that suggest the vorticity tends to align with velocity in certain domains [29, 53]. In part II of this series, we discuss the applications of contact topology to this case [20]: much less is known of the classification of tight contact structures on $D^2 \times S^1$, and the proof of Hofer's theorem depends on having a manifold without boundary.

Contact structures on the three-torus, however, are more completely understood. In addition, some of the most important examples of interesting Beltrami fields live on T^3. A fundamental observation of Arnold's is the existence of Beltrami fields on the three-torus T^3 which are nonintegrable: these so-called *ABC flows* (and their generalizations) have been the source of a great deal of inquiry:

$$\begin{aligned}\dot{x} &= A\sin z + C\cos y\\ \dot{y} &= B\sin x + A\cos z\\ \dot{z} &= C\sin y + B\cos x\end{aligned}\ , \tag{17}$$

for some $A, B, C \geq 0$. By symmetry in the variables and parameters, we may assume without loss of generality that $1 = A \geq B \geq C \geq 0$. Under this convention, the vector field is nonsingular if and only if [14]

$$B^2 + C^2 < 1. \tag{18}$$

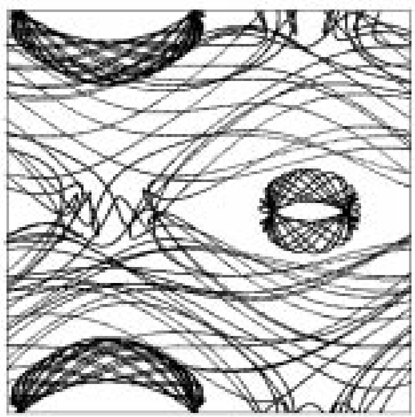

FIGURE 3: The ABC flow on T^3 (as a projected fundamental domain in $\mathbb{R}^3$): three orbits are drawn — two integrable and one nonintegrable.

Though the list of publications concerning ABC flows is extensive, there is very little known about the global features of these flows, apart from cases where one of the constants (say C) is zero or a perturbation thereof. Of the generalizations of ABC flows to other eigenfields of the curl operator, even less is known – there are seemingly no global results. However, thanks to the following classification theorem of Giroux (and, independently, Kanda), we may apply contact-topological methods to the most general case of Beltrami flows on T^3:

Theorem 4.1 (Giroux [32], Kanda [42]) *Any tight contact structure on T^3 is, up to a choice of fundamental class*[3] *in $H_2(T^3)$, isotopic through contact structures to the kernel of α_n for some integer $n > 0$, where*

$$\alpha_n = \sin(nz)dx + \cos(nz)dy. \tag{19}$$

Definition 4.2 Let X be a nonsingular vector field on M. Recall that since all three-manifolds are parallelisable, X gives a map from M to $\mathbb{R}^3$ (well-defined up to choice of framing). Under the normalization $\mathbb{R}^3 \setminus \{0\} \to S^2$, X induces a map $M \to S^2$. The *homotopy class* of X is defined to be the homotopy class of the map $M \to S^2$.

[3]That is, up to a choice of global coordinates x, y, and z.

In other words, a nonsingular vector field is homotopically trivial if it can be deformed through nonsingular vector fields in such a way that all the vectors "point in the same direction" with respect to the chosen framing, which is canonical for T^3.

Theorem 4.3 *Every homotopically nontrivial C^∞ rotational Beltrami field on T^3 has a contractible closed orbit.*

Proof: Let u denote a homotopically nontrivial field on a Riemannian T^3 satisfying $\nabla \times u = fu$ with $f > 0$. Then by Theorem 2.1, there is a natural contact structure ξ transverse to u and uniquely defined up to homotopy. If u is homotopically nontrivial, then so is ξ as a plane field, since the homotopy class of an oriented plane field is defined as the homotopy class of the vector field transverse to it.

By Theorem 4.1, any tight contact structure on T^3 is isotopic to the kernel of the 1-form $\sin(nz)dx + \cos(nz)dy$. The Reeb field X_n associated to this contact form α_n is

$$X_n = \sin(nz)\frac{\partial}{\partial x} + \cos(nz)\frac{\partial}{\partial y}. \tag{20}$$

As the image of the induced map $T^3 \to S^2 \subset \mathbb{R}^3$ lies on the equator $z = 0$, it follows that every tight contact structure on T^3 is homotopically trivial. Hence, u is a Reeb-like field for an overtwisted contact structure, which, by Theorem 3.3, must have a closed orbit which is contractible in T^3. □

Remark 4.4 The existence of closed orbits which are *contractible* is of particular importance. One typically works on T^3 in order to model spatially periodic flows on $\mathbb{R}^3$. The existence of a contractible closed orbit on T^3 implies that when lifted to the universal cover $\mathbb{R}^3$, the orbit remains closed. We note that upon numerically integrating examples of Beltrami fields on T^3 (*e.g.*, the ABC equations), the integrable regions are certainly homotopically nontrivial in T^3, whereas the closed orbits in the nonintegrable regions are completely obscured.

Remark 4.5 The determination of the homotopy type of a nonsingular vector field on a three-manifold reduces to a problem of algebraic topology. In [33], Gompf assigns to a vector field a pair of invariants: a two-dimensional refinement of the Euler class, and a more subtle invariant (Θ), which is derived from what type of four-manifold M bounds. Together, these invariants, which can be computed in many cases, completely classify the homotopy type. Hence, we have a computable criteria for the existence of contractible orbits for spatially periodic Beltrami flows on $\mathbb{R}^3$.

Remark 4.6 It is by no means the case that Theorem 4.3 holds for vector fields in general. As mentioned earlier, the results of Kuperberg [45, 44] allow one to insert "plugs" to break isolated closed orbits. Since the Kuperberg plugs do not change the homotopy type of the vector field, there are smooth nonsingular vector fields on T^3 in *every* homotopy class which have no closed orbits.

We may extend Theorem 4.3 to the analogue of the Seifert Conjecture for Euler flows on T^3.

Theorem 4.7 *Any steady C^ω Euler flow on T^3 which is homotopically nontrivial has a closed orbit.*

Proof: By Theorems 4.3, 1.2, and the proof of Theorem 3.5, the only remaining cases are when one has an integrable Eulerian field u on T^3, or when $\nabla \times u = 0$.

Consider first the integrable case. As in the proof of Theorem 3.5, we know that either there exists a closed invariant 1-manifold, or we have constructed T^3 as a union of cells $T^2 \times \mathbb{R}$ glued together along invariant 2-tori. In the latter case, if there are no closed orbits, then the rotation number for the flow on each T^2 is irrational and thus constant. The vector field is thus homotopically trivial, since the Gauss map $T^2 \to S^2$ for this vector field is nullhomotopic.

In the remaining case where $\nabla \times u = 0$, we again know that the 1-form $\alpha = \iota_u g$ defines a smooth codimension-one foliation. Here, use the fact from foliation theory that any smooth foliation possessing a transverse volume-preserving vector field[4] can be perturbed to a tight contact structure [18]. This perturbation does not change the homotopy type of the plane field, and thus, by Theorem 4.1, the foliation (and the corresponding transverse vector field u) is homotopically trivial. □

We close with the specific case of the ABC flows. Unfortunately, we cannot apply Theorem 4.3 to these equations.

Proposition 4.8 *Every nonsingular ABC field is transverse to a tight contact structure.*

Proof: By normalizing the coefficient A to 1 and using Equation 18, we have that the parameter space $\{(B, C) : 0 \leq B^2 + C^2 < 1\}$ is path-connected; hence, if we show that some ABC field satisfies the proposition, then every other ABC field is homotopic to this through nonsingular Beltrami fields, and so the transverse contact structures are isotopic through contact structures. In the particular case where $B = C = 0$, we have the equations

$$\begin{array}{l} \dot{x} = A \sin z \\ \dot{y} = A \cos z \end{array} , \tag{21}$$

which is the Reeb field for the (tight) contact form $\alpha = A \sin z\, dx + A \cos z\, dy$ (*cf.* Equation 20). □

The ABC fields are the eigenfields of the curl operator on the Euclidean 3-torus with eigenvalue one. It would be interesting to determine whether the higher-order eigenfields are also tight.

Remark 4.9 The most important perspective which we hope to initiate is the distinction between tight and overtwisted fluid flows. The genius of this dichotomy in the field of contact structures lies in the fact that seemingly intractable questions about contact structures are greatly simplified when restricted to the overtwisted class (*e.g.*, existence, classification). We anticipate a similar philosophy to ring true in hydrodynamics. Besides the Seifert-type theorems of this paper, deeper questions may be similarly simplified — we propose the following as tests of this idea:

[4] *i.e.*, a *taut* foliation.

Energy: An important feature of any Beltrami field u is the fact that it extremizes the L^2 energy functional

$$E(\tilde{u}) = \frac{1}{2}\int_M \|\tilde{u}\|^2 d\mu \qquad (22)$$

among the class of all vector fields $\tilde{u}$ obtained from u by μ-preserving diffeomorphisms of M [7, 9]. It is very challenging to prove theorems about which smooth fields *minimize* the energy functional. It follows from remarks in Arnold [7] that the Reeb field associated to the standard tight contact form on S^3, as well as the ABC flows each *minimize* energy. It thus follows from Proposition 4.8 that every known example of a smooth energy-minimizing field is the Reeb field for a *tight* contact structure. This leads to the conjecture that one can always reduce the energy of a Beltrami field associated to an overtwisted contact structure by a volume-preserving diffeomorphism: *i.e.*, the minimal energy representative can only be smooth in the case of a tight fluid.

Stability: Hydrodynamic stability for Euler fields is a particularly difficult problem. Conventional wisdom says that generically Euler flows should be hydrodynamically unstable, but there has been no formalization of this into a rigorous picture. However, the recent theorems of Friedlander and Vishik [28, 27] give a remarkable criterion: the presence of a single periodic orbit of hyperbolic type forces unstable modes. They have used this, for example, to conclude that large families of nonintegrable ABC fields with fixed points are hydrodynamically unstable. The results of this paper do not include any spectral information about the periodic orbits. However, in the case of an overtwisted Beltrami field, the periodic orbit forced by Hofer's theorems is not of elliptic type [39]: it is either hyperbolic or degenerate. Hence, it is a reasonable conjecture that *every* overtwisted Beltrami field on a three-manifold is hydrodynamically unstable.

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Upon dissemination of a preprint version of this paper, we were informed of the paper of Ginzburg and Khesin [31], in which Euler flows are considered from the point of view of symplectic geometry. In particular, they remark on the fact that certain flows on 4-manifolds with level sets of contact type might be forced to have a closed orbit.

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