

KNOTS AND CONTACT GEOMETRY I: TORUS KNOTS AND THE FIGURE EIGHT KNOT

JOHN B. ETNYRE AND KO HONDA

ABSTRACT. We classify Legendrian torus knots and Legendrian figure eight knots in the tight contact structure on S^3 up to Legendrian isotopy. As a corollary to this we also obtain the classification of transversal torus knots and transversal figure eight knots up to transversal isotopy.

1. INTRODUCTION

There have recently been several breakthroughs concerning the classification of tight contact structures on 3-manifolds [Gi3, H1, H2, EH]. The main ingredient in all these advances is the theory of *convex surfaces*, due to Giroux [Gi1]. Convexity enables us to understand neighborhoods of surfaces very well and, more surprisingly, relate the characteristic foliation on one surface to the characteristic foliation on another surface that meets it along a Legendrian curve. Moreover, it reduces the study of characteristic foliations on a surface — often quite delicate — to the study of multi curves on a surface. These *dividing curves* turn out to be more flexible and robust.

In this paper the authors introduce the tools from convex surface theory to the study of Legendrian and transversal knots in the standard tight contact structure on S^3 . Legendrian and transversal knots have played an important role in distinguishing tight contact structures [K1, LM1] and detecting overtwisted contact structures [Be, EH]. Moreover, important topological properties can be detected using Legendrian knots. Specifically, Rudolph [R] has shown how to use invariants of Legendrian knots in a knot type to find obstructions to slicing this knot type. This has been generalized by Lisca and Matić [LM2] and Kronheimer and Mrowka [KM] to find bounds on the slice genus of a knot.

Recently there has been some progress in the classification of transversal knots. Eliashberg [E3] had previously shown that transversal unknots are determined by their self-linking number (the only classical invariant), and, for a few years, this was the only knot type for which a transversal classification existed. Then, in 1998, Etnyre [Et] classified positive transversal torus knots by showing their knot type and self-linking number determine the transversal isotopy class. More recently, Birman and Wrinkle [BW] extended Etnyre's results to transversal iterated torus knots, using a different

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approach — namely, the study of braid foliations and the work of Menasco [Men] on iterated torus knots. Until now, however, the only classification result for Legendrian knots was the classification of Legendrian unknots by Eliashberg and Fraser [EF]. Eliashberg and Fraser had proved that, for Legendrian unknots, the classical invariants (the Thurston-Bennequin invariant and the rotation number) determine the Legendrian isotopy type. In this paper we prove:

Theorem. *Two oriented Legendrian torus knots are Legendrian isotopic if and only if their Thurston-Bennequin invariants, rotation numbers and knot types agree.*

We also find the range of the classical invariants for Legendrian torus knots, thus finishing the classification. In particular we show that, for a negative $(p, -q)$ -torus knot K with $p > q > 0$,

$$\text{tb}(K) \leq -pq.$$

The classical Bennequin inequality only gives

$$\text{tb}(K) \leq pq - p - q,$$

while the bounds discovered by Fuchs and Tabachnikov [FT, Ta] give

$$\text{tb}(K) \leq -pq$$

if q is even but only

$$\text{tb}(K) \leq -pq + p - q$$

when q is odd. We thank Fuchs for informing us of the computations of these bounds in the dissertation [Ep] of one of his students. Thus we give the first class of knots — the $(p, -q)$ -torus knots with q odd — for which all known bounds on the Thurston-Bennequin invariant are not sharp.

As a corollary of the above theorem, we obtain a special case of the Birman-Wrinkle-Menasco result:

Theorem. *Two transversal torus knots are transversally isotopic if and only if their self-linking numbers and knot types agree.*

We also prove:

Theorem. *Two oriented Legendrian figure eight knots are Legendrian isotopic if and only if their Thurston-Bennequin invariants and rotation numbers agree.*

We complete our classification by identifying the range of the Thurston-Bennequin invariant and rotation number for Legendrian figure eight knots. We then obtain the classification of transversal figure eight knots as a corollary.

Theorem. *Two transversal figure eight knots are transversally isotopic if and only if their self-linking numbers agree.*

Continuing our line of inquiry, an open-ended question is:

Question 1. *Which transversal and Legendrian knots are determined by their classical invariants?*

It is known that, at least for Legendrian knots, the answer is not ‘all knot types’. Using the powerful new invariants of contact homology, Hofer and Eliashberg and (independently) Chekanov [C] found Legendrian knots whose classical invariants agree but are not Legendrian isotopic. It is interesting to note that the knot type Chekanov uses is the first knot type in the standard knot tables (e.g. [Ro]) not covered by one of the above theorems! For transverse knots, the authors do not know whether there exist transversal knots which are not distinguished by their classical invariants.

2. BASIC CONTACT GEOMETRY

Recall a *contact structure* on a 3-manifold M is a maximally nonintegrable plane field ξ . Throughout this paper we assume our contact structures are transversely oriented and thus can be globally given as the kernel of a 1-form α where $\alpha \wedge d\alpha \neq 0$. Moreover we always orient M by $\alpha \wedge d\alpha$.

If Σ is a surface in M then $\xi \cap T\Sigma$ is a singular line field on Σ and may be integrated to a singular foliation Σ_ξ called the *characteristic foliation*. The singularities may be assumed to be either elliptic or hyperbolic (depending on the local degree of the foliation) and if Σ is oriented then they also have a sign determined by the compatibility of the orientations of ξ and $T\Sigma$ at the singularities. There are many standard ways to manipulate the characteristic foliation — for details see [A, Gi1, EF]. We also recall that the characteristic foliation determines a contact structure in a neighborhood of the surface.

A contact structure ξ is called *tight* if there are no embedded disks D with a limit cycle in their characteristic foliation. If ξ is not tight, then it is called *overtwisted*. The standard example of a tight contact structure is given by ξ_0 , the complex tangencies to $S^3 \subset \mathbb{C}^2$. This is the unique tight structure on S^3 [E2]. The uniqueness is easily seen using Darboux’s Theorem and the following theorem.

Theorem 2.1 (Eliashberg [E2]). *A tight contact structure on the 3-ball is uniquely determined (up to isotopy) by the characteristic foliation on its boundary.*

In [E2] the group of contactomorphisms was also studied. Fix a point p in S^3 and let $\text{Diff}_0(S^3)$ be the group of orientation-preserving diffeomorphisms of S^3 that fix the plane $\xi_0(p)$, and let Diff_{ξ_0} be the group of diffeomorphisms of S^3 that preserve ξ_0 .

Theorem 2.2 (Eliashberg [E2]). *The natural inclusion of*

$$\text{Diff}_{\xi_0} \hookrightarrow \text{Diff}_0(S^3)$$

is a weak homotopy equivalence.

2.1. Legendrian Knots. A curve γ in M is called a *Legendrian curve* (or *Legendrian knot* if γ is closed) if it is everywhere tangent to the contact plane field ξ . Suppose γ is closed. Our prime interest in this paper is the classification of a Legendrian knot γ up to isotopy through Legendrian knots.

This is equivalent to the classification of Legendrian knots up to global contact isotopies.

Lemma 2.3 (see [E3]). *If $\psi_t: S^1 \rightarrow M$ is a Legendrian isotopy, then there is a contact isotopy $f_t: M \rightarrow M$ such that $f_t \circ \psi_0 = \psi_t$.*

The contact planes ξ provide a canonical framing of the normal bundle of γ . If γ is null-homologous, then we may use the Seifert surface Σ to assign an integer to the canonical framing which we call the *Thurston-Bennequin invariant* of γ and denote by $\text{tb}(\gamma)$. If we orient γ , then the oriented unit tangent vector to γ will provide a section of $\xi|_\gamma$. The Euler class of $\xi|_\Sigma$ relative to this section is called the *rotation number* of γ and is denoted $r(\gamma)$. If γ is a Legendrian knot in \mathbb{R}^3 with the standard contact structure $\xi = \ker(dz + xdy)$, then its projection onto the yz -plane, called the *front projection*, will have no vertical tangencies and at each crossing the arc with the smallest slope will lie over the other arc. See Figure 1. Moreover any

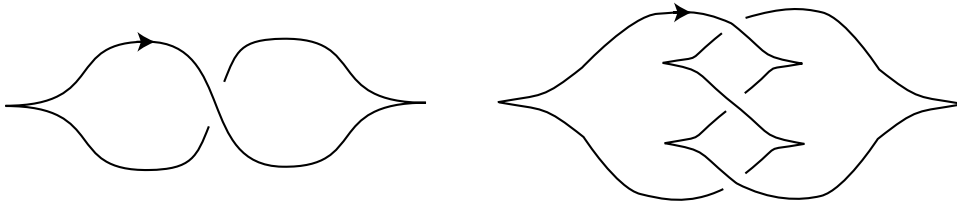


FIGURE 1. Examples of Legendrian knots in the xy -projection.

knot diagram satisfying these conditions will represent a Legendrian knot. In this projection it is easy to compute the Thurston-Bennequin invariant and the rotation number of γ . They are given by

$$(1) \quad \text{tb}(\gamma) = w(\gamma) - r_c,$$

and

$$(2) \quad r(\gamma) = \frac{1}{2}(D_c - U_c),$$

where $w(\gamma)$ is the writhe of γ , r_c is the number of right cusps in the projection, D_c is the number of downward cusps and U_c is the number of upward cusps in the projection. In Figure 1, the unknot has $\text{tb} = -2$ and $r = -1$ and the trefoil has $\text{tb} = -6$ and $r = 1$. It is an interesting exercise to work these formulas out or see [Go].

The only result previously known concerning the classification of Legendrian knot is due to Eliashberg and Fraser [EF]:

Theorem 2.4. *Two oriented Legendrian unknots in a tight contact structure are Legendrian isotopic if and only if they have the same Thurston-Bennequin invariant and the same rotation number. Moreover, a complete list of Legendrian unknots is given in Figure 2.*

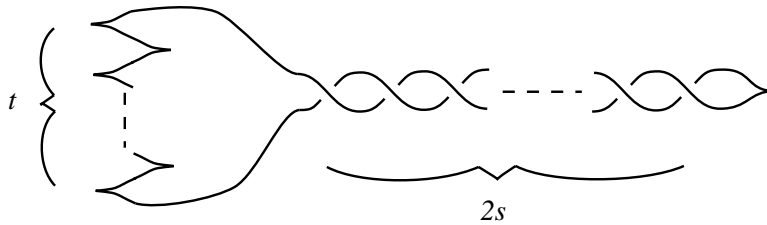


FIGURE 2. Legendrian unknots with $tb = -2s - t$ and $r = \pm(t - 1)$.

We will present a short proof of this theorem in Section 3.5.

Call a knot type \mathcal{K} *Legendrian simple* if (oriented) Legendrian knots in this knot type are determined by their Thurston-Bennequin invariant and rotation number. Thus the unknot is Legendrian simple.

The Bennequin inequality

$$(3) \quad tb(\gamma) + |r(\gamma)| \leq -\chi(\Sigma)$$

provides a (non-optimal, see Theorem 4.1) upper bound on the Thurston-Bennequin invariant of a Legendrian knot. It is easy, however, to decrease the Thurston-Bennequin invariant. Specifically, let γ be an oriented Legendrian knot. We can find a contactomorphism from a neighborhood N of γ to $M = \{(x, y, z) \in \mathbb{R}_y^3 \mid x^2 + z^2 < \epsilon\}$, where \mathbb{R}_y^3 is \mathbb{R}^3 modulo $y \mapsto y + 1$ and $\xi_0 = \{dz + xdy = 0\}$, and γ is sent to the image of the y -axis in M . Now a *positive (negative) stabilization* of γ , $S_+(\gamma)$ ($S_-(\gamma)$) is the curve in N corresponding to the curve in M shown in Figure 3. From Equations 1 and

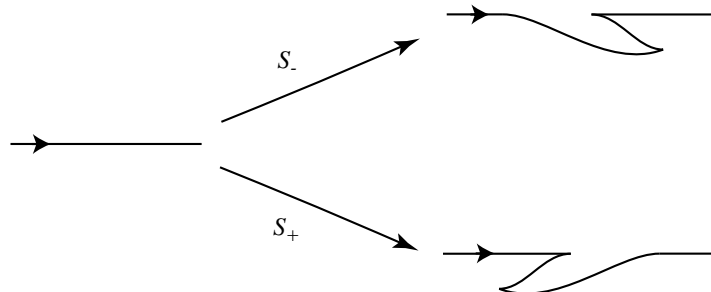


FIGURE 3. Stabilization of Legendrian knots (in yz -projection).

2 it is easy to see that

$$(4) \quad tb(S_{\pm}(\gamma)) = tb(\gamma) - 1$$

and

$$(5) \quad r(S_{\pm}(\gamma)) = r(\gamma) \pm 1.$$

It is important to notice that $S_+(\gamma) \cup \gamma$ cobound a disk D for which $\gamma \setminus \partial D = S_+(\gamma) \setminus \partial D$, $\gamma \cap D$ contains three negative singularities, two elliptic and one

hyperbolic, and $S_+(\gamma) \cap D$ contains the same two elliptic singularities and a positive elliptic singularity. See Figure 4. We also observe the following

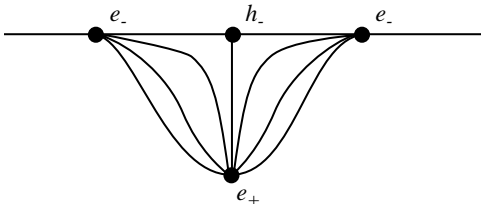


FIGURE 4. A bypass.

simple lemma (cf. [FT]).

Lemma 2.5. *Stabilization is well-defined and $S_+(S_-(K)) = S_-(S_+(K))$.*

In this paper we use a strategy developed in [Et] (for transversal knots) to classify various Legendrian knots. Specifically:

Strategy 2.6. *Given a knot type \mathcal{K} ,*

- (1) *Classify Legendrian knots realizing \mathcal{K} with maximal Thurston-Bennequin invariant.*
- (2) *Show that all Legendrian knots realizing \mathcal{K} without maximal Thurston-Bennequin invariant destabilize (i.e., are stabilizations of other Legendrian knots).*
- (3) *If K and K' are Legendrian knots realizing \mathcal{K} with maximal Thurston-Bennequin invariant, then understand the relationship between their stabilizations.*

From Lemma 2.5, if all three steps can be carried out, then we will have classified Legendrian knots realizing \mathcal{K} . Note that if \mathcal{K} has a unique Legendrian realization with maximal Thurston-Bennequin invariant (as is the case with unknots, positive torus knots, and figure eight knots), then Step 3 is unnecessary.

2.2. Transversal Knots. A knot γ in a contact manifold (M, ξ) is called *transversal* if it is everywhere transverse to the contact planes. One would like to classify transversal knots up to isotopy through transversal knots. The analog to Lemma 2.3 for transversal knots says that this is equivalent to the classification of transversal knots up to global contact isotopy.

In addition to the topological knot type, (nullhomologous) transversal knots have one other classical invariant — the self-linking number. If Σ is the Seifert surface for the transversal knot γ , then we can find a nonzero section σ of $\xi|_\Sigma$. We may use σ to push off a parallel copy γ' of γ and then define the *self-linking number* of γ to be

$$(6) \quad l(\gamma) = \gamma \cdot \Sigma,$$

where \cdot denotes oriented intersection number (*i.e.*, $l(\gamma)$ is just the linking number of γ and γ'). Note if ξ and M are oriented, then γ has a natural induced orientation. If γ is a (generic) transversal knot in \mathbb{R}^3 with the standard contact structure $\xi = \ker(dz + xdy)$, then its projection onto the yz -plane is a diagram in which one does not see the diagram segments in Figure 5. To see this, note that at all but a finite number of points, where the projection has vertical tangents, the projected knot can be locally represented by the graph of a function $(y, f(y))$. Taking the orientation of the knot into account we can parametrically represent the knot by $(t, f(t))$ or $(-t, f(-t))$. Since $(dz + xdy)(\dot{\gamma}) > 0$, we know the x -coordinate of the transversal knot must satisfy $x > -\frac{df}{dt}$ in the first case and $x < -\frac{df}{dt}$ in the second case. This clearly rules out the diagram segment on the left-hand side of Figure 5. The other segments are not allowed since when there is a vertical tangency it must clearly point in the positive z -direction. We also note that any knot

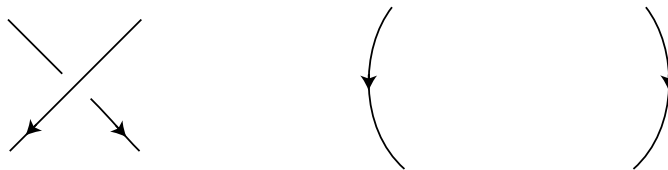


FIGURE 5. Diagram segments not allowed in yz -projection of a transversal knot.

diagram not containing these diagram segments will represent a transversal knot. This can be easily seen since at any crossing not like the one in Figure 5 we can find appropriate x -coordinates for the knot near the crossing. In addition, near the vertical tangencies, we can choose any x -coordinate we like. Now one just has to observe that once the diagram near the crossings and vertical tangencies has been lifted to \mathbb{R}^3 it is easy to lift the rest since one only needs to connect with segments always running in the positive, or negative, y -direction and can thus choose the x -coordinate to be very positive or negative. It is also fairly easy to see that any two transversal knots one constructs from a diagram through this procedure are isotopic through transversal knots.

In the yz -projection it is easy to compute the self-linking number of γ . It is given by

$$(7) \quad l(\gamma) = w(\gamma),$$

where $w(\gamma)$ is the writhe of γ .

Until recently, the only classification results known for transversal knots were the following two results:

Theorem 2.7 (Eliashberg [E3]). *Two transversal unknots are transversally isotopic if and only if their self-linking numbers agree. Moreover, the self-linking numbers of transversal unknots are precisely the negative odd integers.*

Theorem 2.8 (Etnyre [Et]). *Two transversal positive torus knots are transversally isotopic if and only if their self-linking numbers agree and they are of the same topological knot type. Moreover, the self-linking numbers of a transversal positive (p, q) -torus knot ($p, q > 0$) are precisely the odd integers less than or equal to $pq - p - q$.*

Following Birman and Wrinkle [BW], one says a knot type \mathcal{K} is *transversally simple* if transversal knots in this knot type are determined by their self-linking number. Recently, using an interesting connection to braid theory, the above theorem was greatly generalized:

Theorem 2.9 (Birman and Wrinkle [BW], Menasco [Men]). *Iterated torus knots are transversally simple.*

We recall the definition of iterated torus knots. If γ is any knot, let $N = S^1 \times D^2$ be a neighborhood of γ so that $S^1 \times \{p\}$, where $p \in \partial D^2$, bounds a Seifert surface in the complement of N . If the integers p and q are relatively prime, then the homology class $p(\partial D^2) + q(S^1 \times \{p\})$ can be represented by an embedded closed curve $\gamma_{(p,q)} \subset \partial N$. The curve $\gamma_{(p,q)}$ is called the (p, q) -cable of γ . Any iterated cable of the unknot is called an *iterated torus knot*. The above theorem reduces the transversal classification of iterated torus knots to an existence question, i.e., which self-linking numbers are actually realized by transversal knots in these knot types. In private communication with the authors, Birman has finished the classification by showing that the possible self-linking numbers for $(\dots (\gamma_{(p_1, q_1)})_{(p_2, q_2)} \dots)_{(p_n, q_n)}$, where γ is an unknot, are odd integers less than or equal to l_n . Here l_n is computed as follows: Let $b_n = q_1 q_2 \dots q_n$, where we use the convention that $0 < q_i < |p_i|$, and set $a_1 = (q_1 - 1)p_1$ and $a_i = (q_i - 1)p_i - a_{i-1}q_i^2$. Then $l_n = a_n - b_n$.

The Bennequin inequality for transversal knots is

$$(8) \quad l(\gamma) \leq -\chi(\Sigma),$$

where Σ is a Seifert surface for γ . The Bennequin inequality for Legendrian knots, Equation 3, can be obtained from this inequality as follows: If γ' is a Legendrian knot, then we may embed an annulus $A = \gamma' \times [-\epsilon, \epsilon]$ in a neighborhood of γ' so that γ' is the core, $\gamma' \times \{0\}$, of A and the framing A induces on γ' is the same as the one induces by ξ . Now if A is thin enough we can assume the characteristic foliation contains γ' as a closed leaf and the other leaves are transverse to the boundary and spiral to γ' . Now $\gamma' \times \{\epsilon\}$ is a positive transversal unknot which we denote $T_+(\gamma')$ and call the *positive transversal push-off* of γ' . Similarly we have the *negative transversal push-off*, $T_-(\gamma')$, of γ' . One can show ([E2])

$$(9) \quad l(T_{\pm}(\gamma')) = \text{tb}(\gamma') \pm r(\gamma'),$$

and Equation 3 follows from this and Equation 8.

One can define a notion of stabilization for transversal knots, as we did for Legendrian one, and use this to develop a strategy to classify transversal knots. In fact this was used in [Et] to classify transversal positive torus knots. In this paper we use a different approach originally described to the

authors by Eliashberg. We begin with a definition. Two Legendrian knots γ and γ' are called *stably isotopic* if there is some n and n' such that $S_+^n(\gamma)$ and $S_+^{n'}(\gamma')$ are Legendrian isotopic. Since $\text{tb}(\gamma) + r(\gamma) = \text{tb}(S_+(\gamma)) + r(S_+(\gamma))$, a natural invariant of the stable isotopy class of a Legendrian knot is

$$(10) \quad s(\gamma) = \text{tb}(\gamma) + r(\gamma),$$

which we call the *stable Bennequin invariant*. We call a knot type \mathcal{K} *stably simple* if Legendrian knots in this knot type are stably isotopic if and only if their stable Bennequin invariants agree. Note that we allow ourselves to use only *positive* stabilizations in the definition of *stable isotopy*. The reason for this is twofold: first, if we allow negative stabilizations as well, then any two topologically isotopic Legendrian knots would be stably isotopic (cf. [FT]). And, second, we have the following:

Theorem 2.10. *A knot type \mathcal{K} is stably simple if and only if it is transversally simple.*

Proof. Begin by assuming \mathcal{K} is stably simple. Let γ and γ' be two transversal knots in the knot type \mathcal{K} with the same self-linking numbers. Note γ has a neighborhood N contactomorphic to $\{(r, \theta, z) \in \mathbb{R}^2 \times S^1 \mid r < \epsilon\}$ with the contact structure given by $\{\alpha = dz + r^2 d\theta = 0\}$. Now, for large integers n , if T_n are the tori in N with $r = \frac{1}{\sqrt{n}}$, then the characteristic foliation on T_n is by $(-1, n)$ curves. Let L_n be a leaf in this characteristic foliation. Note L_n is a Legendrian knot topologically isotopic to γ and that $S_+(L_n) = L_{n+1}$. Note also that if we have some L_n , then we have L_m for all $m \geq n$. Thus from γ we have a well-defined stable isotopy class of Legendrian knot L_n and from γ' we similarly get L'_n . Now since $s(L'_n) = l(\gamma') = l(\gamma) = s(L_n)$ we know that there is some $m > n$ and $m' > n'$ such that L_m is Legendrian isotopic to $L'_{m'}$. Finally, by observing that $T_+(L_m) = \gamma$ and $T_+(L'_{m'}) = \gamma'$, we see that γ and γ' are transversally isotopic. Thus \mathcal{K} is transversally simple.

The other implication is proved at the end of the next section as it requires the theory of convex surfaces developed there. \square

3. CONVEXITY IN CONTACT GEOMETRY

3.1. Contact vector fields. A vector field v is called a *contact vector field* on a contact manifold (M, ξ) if the flow of v preserves the contact structure ξ . A surface Σ in M is called *convex* if there is a contact vector field transverse to Σ . Generically, $v|_\Sigma$ will be tangent to $\xi|_\Sigma$ along curves Γ that divide Σ in a special way. In general, consider \mathcal{F} a singular foliation on an orientable surface Σ and Γ a disjoint union of simple closed curves on Σ . We say Γ *divides* \mathcal{F} if Γ is transverse to \mathcal{F} , $\Sigma \setminus \Gamma$ is the disjoint union of two (possibly disconnected) surfaces Σ_+ and Σ_- with $\partial\overline{\Sigma_+} = -\partial\overline{\Sigma_-} = \Gamma$, and there is a vector field u and volume form ω on Σ so that u is tangent to \mathcal{F} , $\pm \mathcal{L}_u \omega > 0$ on Σ_\pm and $u|_\Gamma$ points out of Σ_+ . We refer the reader to [Gi1] and [K1] for proofs of the following facts:

- Contact vector fields may be identified with a section of the bundle TM/ξ .
- A closed surface may be isotoped by a C^∞ -small isotopy so that it is convex.
- Let Σ be a compact orientable surface in the contact manifold (M^3, ξ) . Then Σ is a convex surface if and only if there is a tubular neighborhood N of Σ in M that is contactomorphic to $(\Sigma \times (-\epsilon, \epsilon), \beta + u dt)$ taking Σ to $\Sigma \times \{0\}$, where β is a 1-form on Σ and u is a function on Σ and $\epsilon > 0$.
- Let Σ be a closed orientable surface. Then Σ is convex if and only if there is a dividing set Γ for Σ_ξ . (If there is any ambiguity, we will also write Γ_Σ instead of Γ to denote a dividing set of Σ .) Two dividing sets Γ and Γ' for the same \mathcal{F} are isotopic; hence we will slightly abuse language and refer to Γ as ‘the’ dividing set of Σ .

Let v be a contact vector field for (M, ξ) that is transverse to a surface Σ and let Γ be the dividing curves on Σ . An isotopy $F : \Sigma \times [0, 1] \rightarrow M$ of Σ is called *admissible* if $F(\Sigma \times \{t\})$ is transversal to v for all t . The major result concerning convex surfaces says that up to admissible isotopies the dividing set dictates the geometry of ξ near Σ . More specifically, we have the following Flexibility Theorem:

Theorem 3.1 (Giroux [G1], Kanda [K2]). *Let Σ be a closed surface or a surface with Legendrian boundary. Let Γ be the dividing set for Σ_ξ and \mathcal{F} another singular foliation on Σ divided by Γ . Then there is an admissible isotopy F of Σ such that $F(\Sigma \times \{0\}) = \Sigma$, $F(\Sigma \times \{1\})_\xi = \mathcal{F}$ and the isotopy is fixed on Γ .*

Remark 3.2. It is useful to note, and we will frequently implicitly use, that in a tight structure no dividing curve on a convex closed surface or a convex surface with Legendrian boundary can bound a disk unless the surface is a 2-sphere. This observation is due to Giroux. For example, see [H1].

A useful formulation of Giroux’s Flexibility Theorem above is called the Legendrian Realization Principle, which is due to Kanda [K2]:

Theorem 3.3 (Legendrian Realization Principle). *Consider C , a closed curve, on a closed convex surface or a convex surface Σ with Legendrian boundary. Assume $C \pitchfork \Gamma_\Sigma$ and every component of $\Sigma \setminus C$ nontrivially intersects Σ . Then there exists an admissible isotopy $F_t = F(\cdot, t)$, $t \in [0, 1]$ so that*

- (1) $F_0 = id$,
- (2) $F_t(\Sigma)$ are all convex,
- (3) $F_1(\Gamma_\Sigma) = \Gamma_{F_1(\Sigma)}$,
- (4) $F_1(C)$ is Legendrian.

If γ is a closed oriented Legendrian curve in a surface Σ , then we define the *twist of γ relative to Σ* , $t_\Sigma(\gamma)$, to be the twisting of the contact planes ξ along γ measured with respect to the framing induced by Σ . Note that if

Σ is a Seifert surface for γ , then $t_\Sigma(\gamma) = \text{tb}(\gamma)$. We now have the following useful theorem (cf. [K2] as well as [EF] for the first sentence of the theorem):

Theorem 3.4 (Kanda [K2]). *If γ is a Legendrian curve in a surface Σ , then Σ may be isotoped relative to γ so that it is convex if and only if $t_\Sigma(\gamma) \leq 0$. Moreover, if Σ is convex, then*

$$(11) \quad t_\Sigma(\gamma) = -\frac{1}{2}\#(\gamma \cap \Gamma),$$

where Γ is the set of dividing curves for Σ_ξ .

Here, $\#(a \cap b)$ is the cardinality of $a \cap b$. Note that $\gamma \pitchfork \Gamma$ because $\Gamma \pitchfork \Sigma_\xi$.

3.2. Surfaces with boundary and convexity. Let (M, ξ) be a contact manifold. Throughout this section let Σ be a compact surface whose boundary $\partial\Sigma$ is everywhere tangent to ξ . Moreover, whenever we are considering oriented Legendrian curves we assume Σ is oriented and the orientation is consistent with the orientation of $\partial\Sigma$. From Theorem 3.4 we have

Lemma 3.5. *The surface Σ may be made convex if and only if the twist of ξ about each boundary component is less than or equal to zero.*

For Legendrian knots we have

Lemma 3.6 (Kanda [K2]). *Suppose Σ has a single boundary component γ , and γ is Legendrian. Then Σ may be made convex if and only if $\text{tb}(\gamma) \leq 0$. Moreover, if Σ is convex with dividing curves Γ , then*

$$(12) \quad \text{tb}(\gamma) = -\frac{1}{2}\#(\gamma \cap \Gamma)$$

and

$$(13) \quad r(\gamma) = \chi(\Sigma_+) - \chi(\Sigma_-),$$

where Σ_\pm are as in the definition of convexity.

We will also need the following Edge-Rounding Lemma where two convex surfaces meet ‘perpendicularly’ along a common Legendrian boundary. But first we discuss a normal form for two intersecting convex surfaces. Let Σ_1 and Σ_2 be compact convex surfaces with Legendrian boundary in a contact manifold (M, ξ) , Σ_1 and Σ_2 intersect transversally along a common boundary component γ , and $t_{\Sigma_1}(\gamma) = t_{\Sigma_2}(\gamma) < 0$. Then Σ_1, Σ_2 can be C^0 -small perturbed, fixing the boundary, so that the neighborhood $N(\gamma)$ of γ is locally contactomorphic to $\{(x, y, z) \in \mathbb{R}_z^3 \mid x^2 + y^2 < \epsilon\}$ with the contact structure $\alpha_n = \sin(2\pi n z)dx + \cos(2\pi n z)dy$ for some $n \in \mathbb{Z}^+$, where \mathbb{R}_z^3 is \mathbb{R}^3 modulo $z \mapsto z + 1$. $\Sigma_1 \cap N(\gamma) = \{x = 0, 0 \leq y \leq \epsilon\}$ and $\Sigma_2 \cap N(\gamma) = \{y = 0, 0 \leq x \leq \epsilon\}$.

Remark 3.7. This local model for the intersection of two convex surfaces shows how to transfer information from one convex surface to another.

Lemma 3.8 (Edge-Rounding). *In the situation described above, if we join Σ_1 and Σ_2 along $x = y = 0$ and round the common edge, the resulting surface is convex, and the dividing curve $z = \frac{k}{2n}$ on Σ_1 will connect to the*

dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2 , where $k = 0, \dots, 2n - 1$. Here we are assuming that Σ_1 and Σ_2 have been oriented so that the orientations agree after rounding.

3.3. Bypasses and simplifications of the characteristic foliation. We now wish to describe a method for altering the dividing curves of a convex surface. These techniques were first exploited in [H1] and precursors to them appeared in [K1].

Given a Legendrian curve γ a *bypass for γ* (see Figure 6) is a convex disk

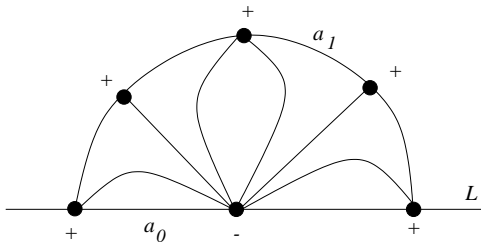


FIGURE 6. A bypass.

D with boundary made up of two smooth Legendrian arcs a_0 and a_1 where:

- $a_0 = (\gamma \cap D) \subset \gamma$,
- along a_0 there are three elliptic singularities in D_ξ — two with the same sign occurring at the endpoints and one with a different sign in the interior of a_0 ,
- along a_1 all the singularities have the same sign, their local degrees alternate, and there are at least three of them,
- there are no interior singularities in D_ξ .

Note all the singularities of D_ξ have the same sign except one — the sign of this singularity is *the sign of the bypass*. Note the relation between the definition of a bypass and a stabilizing disk. In fact, suppose D' is a stabilizing disk for a Legendrian knot γ' and, once stabilized, we obtain the knot γ . Then from γ 's perspective D' is a bypass — *i.e.*, D' shows how to isotop γ so that it twists less. Moreover if D is a bypass for a knot γ and γ' is the knot obtained from pushing γ across D , then D may be isotoped (canceling extra singularities along α_1) so that it is a stabilizing disk for γ' .

Dividing curves are quite helpful in locating bypasses. To see this let Σ be a convex surface with Legendrian boundary. In this situation one may manipulate the the characteristic foliation to make all the singularities along the boundary (half)-elliptic ([EF]). Now if $\text{tb}(\partial\Sigma) = -n \leq 0$ then the dividing curves intersect $\partial\Sigma$, $2n$ times. Suppose one of these dividing curves is boundary-parallel — *i.e.*, cuts off a half-disk which has no other intersections with Γ_Σ . If we think of the characteristic foliation as given by a flow, then we may flow this dividing curve “away” from the boundary and it will limit to a Legendrian curve α . The curve α will separate a disk D

from Σ which will be a bypass for $\partial\Sigma$. (If $t(\partial\Sigma) = -1$, $\Sigma \neq D^2$, and Γ_Σ is a single boundary-parallel curve along $\partial\Sigma$, then we need to exercise a little care. In this case, we first need to perturb Γ and modify Γ_Σ , and then use a more general form of the Legendrian Realization Principle. We remark that this case actually does not appear in this paper.)

Lemma 3.9. *Given a boundary-parallel dividing curve δ on a convex surface Σ with Legendrian boundary, one may find a bypass for the boundary, provided Γ_Σ is not a single arc on $\Sigma = D^2$.*

Remark 3.10. We will frequently abuse terminology and refer to a boundary-parallel dividing curve as a bypass.

If Σ is an annulus, we may find boundary-parallel dividing curves using the following Imbalance Principle (see [H1]):

Proposition 3.11 (The Imbalance Principle). *If $\Sigma = S^1 \times [0, 1]$ is convex and has Legendrian boundary where $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$, then there exists a boundary-parallel dividing curve (and hence a bypass) along $S^1 \times \{0\}$.*

Bypasses may be used to alter the dividing curves of a convex surface as follows:

Proposition 3.12. *Let $A = [0, 1] \times [0, 1]$ be a convex square with three horizontal dividing curves and vertical ruling. Let γ be one of the vertical ruling curves and D a bypass for γ disjoint from A . Then we may isotop A rel boundary by pushing A across D so as to alter the characteristic foliation as shown in Figure 7.*

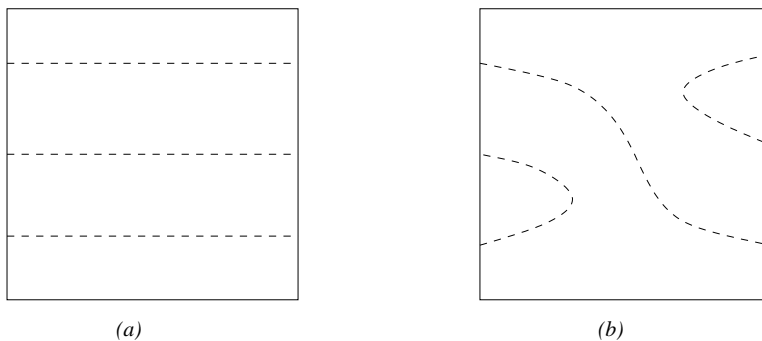


FIGURE 7. Dividing curves on A before the isotopy (a) and after (b).

3.4. The classification of tight contact structures on solid tori. Let T be convex torus. Assuming ξ is tight, we know that no dividing curve bounds a disk, and hence the dividing curves are parallel essential curves. If there are $2n$ parallel curves, then, using Theorem 3.1, we may assume there are $2n$ curves of singularities in T_ξ , one in each region of the complement of the dividing curves. We call these curves the *Legendrian divides*, and their

slope will be called the *boundary slope*. We may also assume the the other leaves in T_ξ form a 1-parameter family of closed curves. We call these curves the *Legendrian ruling curves*. It is important to notice that we can make the slope of these ruling curves whatever we wish, except the boundary slope. If the characteristic foliation on a convex torus has this nongeneric form we say the T_ξ is in *standard form*. If $n > 1$ and we can find a bypass for one of the ruling curves, then we can isotop T so as to reduce n by one. If $n = 1$ we have:

Theorem 3.13 (Honda [H1]). *Let T be a convex torus in standard form with two dividing curves and in some basis for T the slope of the dividing curves is 0. If we find a bypass on a ruling curve of slope between $-\frac{1}{m}$ and $-\frac{1}{m+1}$, $m \in \mathbb{Z}$, then after pushing T across the bypass the new torus has two dividing curves of slope $-\frac{1}{m+1}$.*

This theorem was a key step in proving:

Theorem 3.14 (Honda [H1] (see also [Gi3])). *Let (r_0, \dots, r_k) give the continued fraction decomposition of $-\frac{p}{q}$, where $p > q > 0$. Then there are $|(r_0 + 1) \dots (r_{k-q} + 1)(r_k)|$ tight contact structures on $S^1 \times D^2$ with standard convex boundary having two Legendrian divides of slope $-\frac{p}{q}$. (Here our convention is that the meridian has slope 0.) Moreover, all these structures are distinguished by the number of positive regions on a convex meridional disk with Legendrian boundary.*

It is not hard to see that if we have the ‘standard’ universally tight contact structure on $T^2 \times I$ (with coordinates (x, y, z)) given by $\alpha = \sin(\frac{\pi}{2}z)dx + \cos(\frac{\pi}{2}z)dy$, and we take the one-parameter family of tori $T^2 \times \{z\}$ with linear characteristic foliations, then the slopes of these foliations must decrease as z increases. For convex tori this is a little more complicated but still true.

Lemma 3.15. *If $T^2 \times [0, 1]$ has convex boundary in standard form and the boundary slope on $T^2 \times \{i\}$ is s_i , for $i = 0, 1$, then we can find convex tori parallel to $T^2 \times \{i\}$ with any boundary slope s in $[s_1, s_0]$ (if $s_0 < s_1$ then this means $[s_1, \infty] \cup [-\infty, s_0]$).*

This follows from the classification of tight contact structures on $T^2 \times [0, 1]$ (see [H1, Gi3, EH]). From this Lemma one can easily show:

Lemma 3.16. *If $S = D^2 \times S^1$ has convex boundary with boundary slope $s < 0$, then we can find a convex torus parallel to the boundary of S with any boundary slope in $[s, 0)$.*

We also will make use of the following consequence of the classification of tight contact structures on $T^2 \times I$:

Lemma 3.17. *Consider a tight contact structure on $T^2 \times [0, 1]$ with boundary slopes $s_1 = -\frac{1}{m}$, $s_0 = -\frac{1}{m+1}$ ($m \in \mathbb{Z}^+$) (for $T^2 \times \{1\}$ and $T^2 \times \{0\}$, respectively). If $s_1 < s < s_0$, then there exists a pre-Lagrangian (= linearly foliated) torus T parallel to $T^2 \times \{i\}$, and every convex surface T' in standard*

form with slope s , after contact isotopy, is transverse to T , and $T \cap T'$ is exactly the union of the Legendrian divides of T' .

We shall also have need for the relative Euler class of a contact structure on $U = T^2 \times [0, 1]$. Following [EH], if ξ is a tight contact structure on U for which the boundary is convex and in standard form, then let v be a nonzero vector field transverse to and twisting (with ξ) along the Legendrian ruling curves and tangent to the Legendrian divides. Now let e be the relative Euler class in $H^2(U, \partial U; \mathbb{Z})$ for this section of the bundle $\xi|_{\partial U}$. It is important to note that e is unchanged if we perform a C^0 -small isotopy of ∂U so as to alter the slopes of the ruling curves. Now if c is an oriented curve on $T^2 \times \{0\}$, then we can assume the annulus $A = c \times [0, 1]$ has Legendrian boundary and is convex. A slight generalization of Equation 13 yields

$$(14) \quad e(c) \equiv e(A) = \chi(A_+) - \chi(A_-),$$

where A_{\pm} are the positive and negative regions of A from the definition of convex surface. See [EH, K2] for details.

3.5. Legendrian Unknots. In this section we present a brief proof of Theorem 2.4, using Strategy 2.6. Consider a contact 3-manifold (M, ξ) . The maximal Thurston-Bennequin invariant for a knot type \mathcal{K} is an invariant of the knot type, and will be denoted $\overline{\text{tb}}(\mathcal{K})$. Observe that the tightness of ξ is equivalent to $\overline{\text{tb}}(\mathcal{K}) < 0$, where \mathcal{K} is the knot type for the unknot. Since there exists a Legendrian unknot L with $\text{tb}(L) = -1$ and $r(L) = 0$, $\overline{\text{tb}}(\mathcal{K}) = -1$.

We begin with Step 2 in Strategy 2.6. If $\text{tb}(K') < -1$, then take a spanning convex disk D . There must always exist a bypass by Lemma 3.9 – hence every K' with $\text{tb}(K') < -1$ is a stabilization of a Legendrian unknot with $\text{tb}(K) = -1$.

We now complete the proof by showing that there is a unique Legendrian unknot with $\text{tb}(K) = -1$. In this part of the proof we assume, for clarity, that $M = S^3$. The general case is not much more difficult but obscures the main ideas. Take a Legendrian unknot K with $\text{tb}(K) = -1$ and consider a spanning convex disk D with $\partial D = K$. Since $\text{tb}(K) = -1$, there is only one possible dividing set, and we may uniquely normalize the characteristic foliation using the Flexibility Theorem (Theorem 3.1). If there are two Legendrian unknots K, K' with $\text{tb} = -1$, then there is a diffeomorphism $f : S^3 \rightarrow S^3$ (taking D to D') which is a contactomorphism when restricted to neighborhoods $N(D), N(D')$ of the spanning disks D, D' . The map f may now be isotoped (relative to $N(D)$) into a contactomorphism on all of S^3 using Theorem 2.1, since the tight contact structures on the 3-balls $\overline{S^3 \setminus N(D)}$ and $\overline{S^3 \setminus N(D')}$ induce the same characteristic foliations on their boundaries. Thus, using Theorem 2.2, we find that the $\text{tb} = -1$ Legendrian unknot inside $M = S^3$ is unique up to contact isotopy.

For M arbitrary, we need to use the fact (whose proof we leave to the reader) that two convex 3-balls with the same boundary characteristic foliation are contact isotopic (this essentially follows from Theorem 2.1 as well). We may then reduce to the $M = S^3$ case, thus proving that every Legendrian unknot is determined by tb and r .

3.6. Transversal simplicity implies stable simplicity. In this section we complete the proof of Theorem 2.10 by showing transversal simplicity implies stable simplicity. Begin by assuming that \mathcal{K} is a transversally simple knot type. Let γ and γ' be two Legendrian knots in this knot type with the same stable Bennequin invariants, $s(\gamma) = s(\gamma')$. Let $K = T_+(\gamma)$ and $K' = T_+(\gamma')$ be the positive transversal push-offs of γ and γ' . We can find solid tori neighborhoods N and N' of K and K' with convex boundaries such that γ and γ' lie, respectively, on the boundary of N and N' . Note by thickening the annulus from the definition of $T_+(\gamma)$ we may assume that γ is a Legendrian divide on ∂N that is also a $(-1, n)$ curve. Now, since \mathcal{K} is transversally simple and $l(K) = s(\gamma) = s(\gamma') = l(K')$, we can find a global contact isotopy taking K' to K . Thus we may think of N and N' as solid torus neighborhoods of the same transversal knot K .

In $N \cap N'$ we can find a neighborhood N_ϵ of K contactomorphic to $\{(r, \theta, z) \in \mathbb{R}^2 \times S^1 \mid r < \epsilon\}$ with the contact structure given by $\{(\alpha = dz + r^2 d\theta) = 0\}$. As before for large integers m , if T_m are the tori in N_ϵ with $r = \frac{1}{\sqrt{m}}$, then the characteristic foliation on T_m is by $(-1, m)$ curves. Fix some large m and let L_m be a leaf in the characteristic foliation. We now show that there is some integer k such that $S_+^k(\gamma)$ is Legendrian isotopic to L_m . The same proof will work for γ' , thus proving they are stably equivalent. Isotop T_m to be convex with L_m as a Legendrian divide and the ruling curves all in the class $(-1, n)$. Let $A = \gamma \times [0, 1]$ be an annulus embedded in $N \setminus N_\epsilon$ with $S^1 \times \{0\}$ a ruling curve on ∂T_m and $S^1 \times \{1\} = \gamma$. If we now make A convex, then clearly $t_A(\gamma) = 0$ and $t_A(L_m) < 0$ (if $t_A(L_m) = 0$ then γ is Legendrian isotopic to L_m by Lemma 3.17). Now, by the Imbalance Principle (Proposition 3.11), there exist dividing curves boundary-parallel to $S^1 \times \{0\}$ (and of course non-boundary-parallel to $S^1 \times \{1\}$). This demonstrates that L_m is a stabilization of γ — however, there might be positive and negative stabilizations. To see that there are only positive stabilizations we note that $T_+(L_m)$ is a transversal curve isotopic to the core curve in N_ϵ . Thus $\text{tb}(L_m) + r(L_m) = l(K) = \text{tb}(\gamma) + r(\gamma)$, which of course implies that all the stabilizations were positive.

4. TORUS KNOTS

Let T^2 be a standardly embedded torus in S^3 . By this we mean that T provides a genus one Heegaard splitting of S^3 , so $S^3 = V_0 \cup_T V_1$ where the V_i are solid tori. Let μ be the unique curve on T that bounds a disk in V_0 and λ the unique curve that bounds a disk in V_1 . Orient μ arbitrarily and then orient λ so that μ, λ form a positive basis for $H_1(T)$ where T is

oriented as the boundary of V_0 . Up to homotopy any curve on T can be written as $p\mu + q\lambda$, we shall denote this curve by $K_{(p,q)}$. If p and q are relatively prime then $K_{(p,q)}$ is called a (p, q) -torus knot. We will always assume $|p| > q > 0$ which affords no loss of generality since we may switch the roles of V_0 and V_1 and $K_{(-p,-q)}$ and $K_{(p,q)}$ describe the same knot. One may easily compute that the Seifert surface of minimal genus for $K_{(p,q)}$ has Euler number $|p| + |q| - |pq|$.

4.1. Legendrian Torus Knots. Bennequin's inequality for a Legendrian torus knot says

$$(15) \quad \text{tb}(K_{(p,q)}) + |r(K_{(p,q)})| \leq -|p| - |q| + |pq|.$$

Let $\overline{\text{tb}}_{(p,q)}$ the maximal Thurston-Bennequin invariant of a Legendrian $K_{(p,q)}$ then

Theorem 4.1. *If $p, q > 0$ then*

$$(16) \quad \overline{\text{tb}}_{(p,q)} = pq - p - q,$$

and if $p < 0, q > 0$ then

$$(17) \quad \overline{\text{tb}}_{(p,q)} = pq.$$

Remark 4.2. Note that Bennequin's inequality is sharp for positive torus knots but is not for negative torus knots. As mentioned in the introduction, the upper bounds for the Thurston-Bennequin invariant coming from the work of Fuchs and Tabachnikov [FT, Ta] are also not sharp when $pq < 0$ and q is odd.

We now state our main theorem.

Theorem 4.3. *Legendrian torus knots in the standard tight contact structure on S^3 are determined up to Legendrian isotopy by their knot type, Thurston-Bennequin invariant and rotation number.*

To complete the classification of Legendrian torus knots we can also show:

Theorem 4.4. *Let K be a Legendrian (p, q) -torus knot with maximal Thurston-Bennequin invariant. If $p, q > 0$, then $r(K) = 0$. If $p < 0, q > 0$, then*

$$r(K) \in \left\{ \pm(|p| - |q| - 2qk) : k \in \mathbb{Z}, 0 \leq k < \frac{|p| - |q|}{|q|} \right\}.$$

Remark 4.5. The above results imply that any Legendrian torus knot is some stabilization of a knot in Figure 8. In Figure 8 note the positive torus knot has $\text{tb} = pq - p - q$ and $r = 0$, while the negative (p, q) -torus knot has $\text{tb} = pq$ and $r = q(n_2 - n_1) - e$, where $|p| = nq + e$ and $n = 1 + n_1 + n_2$. To get an idea of the possible values of the invariants for negative torus knots consider the $(-7, 3)$ -torus knot. In Figure 9 we show a graph of the (four largest) Thurston-Bennequin invariants versus the rotation numbers. Each dot corresponds to a pair of invariants that is realized by a Legendrian $(-7, 3)$ -torus knot and each arrow represents a stabilization.

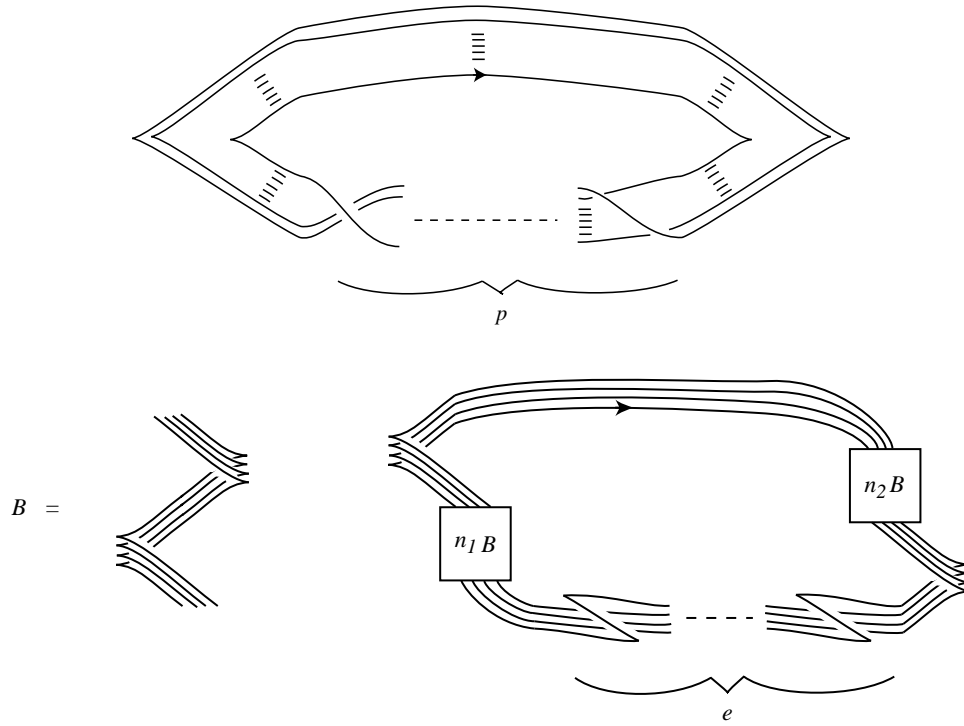


FIGURE 8. Legendrian torus knots.

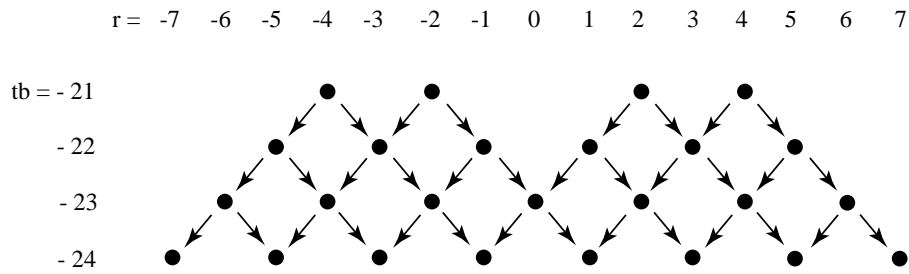


FIGURE 9. Some Thurston-Bennequin invariants and rotation numbers realized by Legendrian $(-7, 3)$ -torus knots.

In this section we classify positive, oriented Legendrian torus knots in (S^3, ξ) up to Legendrian isotopy, leaving the general case for the following subsection. Precisely, in this section we prove:

Theorem 4.6. *If K and K' are two oriented Legendrian positive torus knots then they are Legendrian isotopic if and only if $\text{tb}(K) = \text{tb}(K')$ and $r(K) = r(K')$. Moreover, if K is a (p, q) -torus knot, with $p, q > 0$, then $\text{tb}(K) \leq pq - p - q$ and if $\text{tb}(K) = pq - p - q - n$ for some non-negative integer n then $r(K) \in \{-n, -n + 2, \dots, n\}$.*

We employ Strategy 2.6 described in Section 2.1 to prove this theorem. First note that the maximal Thurston-Bennequin invariant for a Legendrian (p, q) -torus knot, with $p, q > 0$, is $\overline{\text{tb}}_{(p,q)} = pq - p - q$, since there is a Legendrian knot realizing this (see Figure 8) and the Bennequin inequality says that this is as large as it could be. Moreover, note that the Bennequin inequality then implies that a Legendrian knot realizing this Thurston-Bennequin invariant must have rotation number 0. With this in mind we show:

Lemma 4.7. *Let K and K' be two topologically isotopic Legendrian positive torus knots with maximal Thurston-Bennequin invariant. Then K and K' are Legendrian isotopic.*

Proof. Let T and T' be any standardly embedded tori in S^3 on which K and K' respectively sit. We describe everything in terms of T and K but everything also holds for T' and K' . By Theorem 3.4 we may make T convex without moving K , since the twisting of K with respect to T is $-p - q$ (recall p, q are positive). Now that T is convex and $\text{tb}(K)$ is maximal, we may assume T is in standard form. This follows from observing that $\#(K \cap \Gamma) > |K \cap \Gamma|$ implies the existence of a bypass along K , and hence a destabilization. Here $\#(K \cap \Gamma)$ is the unsigned (actual) intersection number and $|K \cap \Gamma|$ is the (minimum) geometric intersection number of the two isotopy classes. Let $-\frac{r}{s}$, ($r, s > 0$) be the slope of the dividing curves Γ and $2n$ be the number of dividing curves. According to Lemma 3.6, $\text{tb}(K) = pq - \frac{1}{2}\#(K \cap \Gamma) = pq - n \det \begin{pmatrix} p & -s \\ q & r \end{pmatrix}$. Thus for $\text{tb}(K)$ to equal $pq - p - q$ we must have $n = 1$ and $r = s = 1$.

Let $V'_0 \cup V'_1$ and $V_0 \cup V_1$ be the Heegaard splittings associated to the tori T' and T . Since the slopes of the dividing curves are the same, we may use the classification of tight contact structures on solid tori in Lemma 3.14 to find a contactomorphism $\phi : V_0 \rightarrow V'_0$ (note that a meridional disk for V_0 of V'_0 has exactly one dividing curve). Applying Lemma 3.14 again, we may extend ϕ to all of S^3 , thus obtaining a contactomorphism of S^3 which takes T to T' . By Eliashberg's result (Theorem 2.2) we may find a contact isotopy of S^3 taking T to T' . So now we may assume that K and K' are two Legendrian knots on the same convex torus T . Note K and K' are both leaves in the ruling foliation of T . The other ruling curves exhibit a Legendrian isotopy from K to K' . \square

The proof of Theorem 4.6 is thus complete by Lemma 2.5 and:

Lemma 4.8. *If K is an oriented Legendrian (p, q) -torus knot, with $p, q > 0$ and $\text{tb}(K) = pq - p - q - n$, then there exist positive integers n_1 and n_2 such that $n = n_1 + n_2$, $r(K) = n_2 - n_1$ and $K = S_-^{n_1}(S_+^{n_2}(K'))$ where K' is the unique Legendrian (p, q) -torus knot with maximal Thurston-Bennequin invariant.*

Proof. Let T be a standardly embedded torus on which K sits. As in the proof of Lemma 4.7 we may assume that T is convex and in standard form, since otherwise K can be destabilized. However, this time the slope $-\frac{r}{s}$ of the dividing curves is not -1 or the number of dividing curves is not 2. We first consider the case when $-\frac{r}{s} \neq -1$. Let $V_0 \cup V_1$ be the Heegaard splitting associated to the torus T . Recall the “slope” of the dividing curves is usually measured thinking of T as the boundary of V_0 . As the boundary of V_0 or V_1 the slope of the dividing curves on T will be less than -1 . Assume V_0 has this property. We know, by Lemma 3.16, that looking at concentric convex tori in V_0 we will see dividing curves with any slope in $[-\frac{r}{s}, 0)$. In particular, there will be a torus $T' \subset V_0$ with two dividing curves having slope -1 . Let $U = T \times [0, 1]$ be the region between T and T' in V_0 and $A = K \times [0, 1]$ be an annulus lying between $T = T \times \{0\}$ and $T' = T \times \{1\}$ in U . The boundary of U is convex and we may assume that the ruling curves on both boundary components have slope $\frac{q}{p}$. Thus we may assume that $\partial A = K \cup K'$ is Legendrian and A is convex. The dividing curves will intersect K , $N = 2n \det \begin{pmatrix} p & -s \\ q & r \end{pmatrix}$ times and K' , $N' = 2(p + q)$ times. As r and s are not both 1, $N' > N$, so we can find a boundary-parallel arc along K among the dividing curves of A . This implies the existence of a bypass for K and hence a destabilization. Specifically, $K = S_{\pm}(K'')$ for some Legendrian K'' . Repeating this argument we will eventually find a sequence of destabilizations which will exhibit K as $S_{-}^{n_1}(S_{+}^{n_2}(K'))$ where K' is the unique Legendrian (p, q) -torus knot with maximal Thurston-Bennequin invariant.

Now for the case when $-\frac{r}{s} = -1$ and $n > 1$. In this case we claim that there is a torus T' in, say V_1 , parallel to T with two dividing curves having slope -1 . To see this just take a copy of T inside V_1 , make the ruling curves meridional, and then look at a convex meridional disk D . We may use the bypasses on D to reduce the number of dividing curves on the copy of T until there are only two. (This follows easily from Proposition 3.12. Also see [H1].) Call the resulting torus T' . Let $U = T \times [0, 1]$ be the region between T and T' . Now if we make the ruling curves on T' have slope $\frac{q}{p}$ then using the annulus $A = K \times [0, 1]$ in U we may repeat the above argument to destabilize K . \square

4.2. Negative Legendrian Torus Knots. We begin by establishing the upper bound on the Thurston-Bennequin invariant.

Lemma 4.9. *For a nontrivial negative (p, q) -torus knot ($-p > q > 0$) we have $\overline{\text{tb}}_{(p,q)} = pq < 0$.*

Proof. First note that the example in Figure 8 shows that $\overline{\text{tb}}_{(p,q)} \geq pq$. We show that $\overline{\text{tb}}_{(p,q)} \leq pq$ by contradiction. If $\overline{\text{tb}}_{(p,q)} > pq$, then we can construct a Stein manifold X (with boundary) by adding a 2-handle to the 4-ball along a (p, q) -torus knot with framing pq . (Given a symplectic

4-manifold with convex boundary, one can add a symplectic 2-handle to a Legendrian knot γ in the boundary with framing $\text{tb}(\gamma) - 1$ and obtain a new symplectic 4-manifold with convex boundary, see [E1, Go].) According to [M], ∂X is the connected sum of two lens spaces (neither of which is S^3 or $q = 1$ and the knot is trivial). Now, a theorem of Eliashberg [E1] says that X must be a boundary sum of two manifolds X_0 and X_1 . Using the Mayer-Vietoris sequence we see that one of the X_i 's, say X_0 , must be an integral homology ball, but this is impossible since ∂X_0 is a nontrivial lens space. \square

This establishes the second part of Theorem 4.1. The easiest part of Strategy 2.6 to execute in this situation is Part 2 concerning destabilization.

Lemma 4.10. *If K is an oriented Legendrian (p, q) -torus knot, with $pq < 0$ and $\text{tb}(K) < pq$, then there is a (p, q) -torus knot K' such that $\text{tb}(K') > \text{tb}(K)$ and K is a stabilization of K' .*

Proof. Since $\text{tb}(K) \leq pq$ we can use Theorem 3.4 to show K lies on a convex standardly embedded torus T . We know the dividing curves Γ on T have slope $-\frac{r}{s} \neq \frac{q}{p}$, since

$$(18) \quad \text{tb}(K) = pq - \frac{1}{2}\#(\Gamma \cap K) < pq.$$

Now as measured on either V_0 or V_1 the dividing curve have slope less than $\frac{q}{p}$. Assume V_0 has such dividing curves. Now since Lemma 3.16 says that we can find tori in V_0 whose dividing curves have any slope in $[-\frac{r}{s}, 0)$ we can find a torus T' in V_0 whose dividing curves have slope $\frac{q}{p}$. Now as in the proof of Lemma 4.8 we can take an annulus A between T and T' with one boundary on K and the other on a Legendrian divide of T' and thus find a bypass for K since the dividing curves on A will intersect the boundary component containing K more than the one containing the Legendrian divide. Hence we may use this bypass to destabilize K . \square

Before proceeding to Parts 1 and 3 of Strategy 2.6 we need to take a detour concerning rotation numbers and relative “Euler classes.” In the following we will assume that K is a Legendrian (p, q) -torus knot with maximal Thurston-Bennequin invariant. Let T be a convex standardly embedded torus on which K sits (Theorem 3.4 assures we can find such a torus). We may put T_ξ in standard form with K being one of the Legendrian divides. We note that all the Legendrian divides of T are Legendrian isotopic. This follows from Lemma 3.17, since all the Legendrian divides sit on a linearly foliated torus. Using an argument like the one at the end of the proof of Lemma 4.8, we may now reduce the number of Legendrian divides on T to two and assume that K is one of these divides.

Now let $S^3 = V_0 \cup_T V_1$, where V_0 (resp. V_1) is the solid torus with meridional curve μ (resp. λ). From Lemma 3.16 we know that inside V_0 there is a solid torus S with two dividing curves of slope $-\frac{1}{m+1}$, where $|p| = mq + e$,

and there is a solid torus S' containing V_0 with two dividing curves of slope $-\frac{1}{m}$. Let $T_m = \partial S'$ and $T_{m+1} = \partial S$. In addition, set $\bar{S} = \overline{S^3 \setminus S'}$ and $\bar{S}' = \overline{S^3 \setminus S}$.

Now we define an invariant of homology classes of curves on T . Let v be any globally non-zero section of ξ and w a section of $\xi|_T$ that is transverse to and twists (with ξ) along the Legendrian ruling curves and is tangent to the Legendrian divides. If γ is a closed oriented curve on T then set $f_T(\gamma)$ equal to the rotation of v relative w along γ . One may check the following properties (cf. [Et]).

- The function f_T is well-defined on homology classes.
- The function f_T is unchanged if we isotop T among convex tori in standard form.
- If γ is a (r, s) -ruling curve or Legendrian divide then $f_T(\gamma) = r(\gamma)$.

We may similarly define f_m and f_{m+1} for curves on T_m and T_{m+1} . The main facts we need concerning these invariants are:

- (1) $f_T(\mu) = 1 - q$ or $q - 1$.
- (2) $f_m(\lambda)$ is in $\{m - 1, m - 3, \dots, 1 - m\}$.
- (3) If $f_T(\mu) = 1 - q$ then $f_T(\lambda) = f_m(\lambda) + (m - |p|)$. So $f_T(\lambda)$ is in $\{2m - |p| - 1, 2m - |p| - 3, \dots, 1 - |p|\}$.
- (4) If $f_T(\mu) = q - 1$ then $f_T(\lambda) = f_m(\lambda) + (|p| - m)$. So $f_T(\lambda)$ is in $\{|p| - 1, |p| - 3, \dots, |p| - 2m + 1\}$.

We will prove these facts at the end of this section. From the above properties we know that

$$(19) \quad r(K) = pf(\mu) + qf(\lambda).$$

Thus the possible values for $r(K)$ lie in

$$\{\pm(|p| - |q| - 2qk) : k \in \mathbb{Z}, 0 \leq k < \frac{|p| - |q|}{|q|}\}.$$

Since all these possible rotation numbers are realized by Legendrian knots in Figure 8, this finishes the proof of Theorem 4.4.

Lemma 4.11. *Let K and K' be two topologically isotopic Legendrian negative torus knots with maximal Thurston-Bennequin invariant. Then K and K' are Legendrian isotopic if and only if $r(K) = r(K')$.*

Proof. Let T and T' be tori on which K and K' respectively sit. We assume that T and T' have been arranged as described above. The proof is finished as we finished the proof of Lemma 4.7. We only need to recall that Theorem 3.14 says that the contactomorphism type of a tight contact structure on a solid torus (with standard convex boundary) is determined by the number of positive bypasses on a meridional disk and the number of positive bypasses on the meridional disks for V_0 and V_1 , where $S^3 = V_0 \cup_T V_1$, are determined by the rotation number of K . \square

We also may show:

Lemma 4.12. *Recall m and e are integers such that $|p| = mq + e$ where $0 < e < q$. Let K and K' be negative Legendrian (p, q) -torus knots with maximal Thurston-Bennequin invariant. If the rotation numbers are r and $r - 2e$, respectively, then $S_-^e(K)$ and $S_+^e(K')$ are Legendrian isotopic. If the rotation numbers are r and $r - 2(q - e)$, respectively, then $S_-^{(q-e)}(K)$ and $S_+^{(q-e)}(K')$ are Legendrian isotopic.*

Proof. We assume $r(K) = r(K') + 2e$ and leave the similar case to the reader. In this situation there must be a k so that $f_T(\lambda) = k + (m - |p|)$ and $f_{T'}(\lambda) = k + (|p| - m)$. Thus $f_T(\mu) = 1 - q$ and $f_{T'}(\mu) = q - 1$. In the notation set up above, let T_m be the convex torus outside V_0 with boundary slope $-\frac{1}{m}$ and T'_m be the corresponding one for T' . From Facts 3 and 4 stated above we know that $f_{T_m}(\lambda) = k$ and $f_{T'_m}(\lambda) = k$. By examining dividing curves on the meridional disks, we also have $f_{T_m}(\mu) = f_{T'_m}(\mu) = 0$ (from Equation 14). Now arrange for the Legendrian ruling curves on T_m to have slope $\frac{q}{p}$ and consider the annulus A between T and T_m with slope $\frac{q}{p}$. We can take one boundary of A to be $K \subset T$ and the other a Legendrian ruling curve on T_m . Once we make A convex, the dividing curves do not intersect the boundary component touching T and intersect the other boundary component $2 \det \begin{pmatrix} p & -m \\ q & 1 \end{pmatrix} = 2(p + qm) = -2e$ times. Since the rotation number (values of f_T and f_{T_m}) of the boundary components of A differ by $-e$, we may use Lemma 3.6 (or a slight generalization of it, see [EH]) to see there are e boundary-parallel dividing curves separating off e negative disks from A . This shows that $K_s = A \cap T_m = S_-^e(K)$. Similarly, we see that $K'_s \subset T'_m$ is $K_+^e(K')$. We may now use the argument in the proof of Lemma 4.11 to find a contact isotopy taking T_m to T'_m . A further contact isotopy takes $K_s = S_-^e(K)$ to $K'_s = K_+^e(K')$. \square

Theorem 4.13. *Negative Legendrian torus knots are determined by their knot type, Thurston-Bennequin invariant and rotation number.*

With this theorem we have completed the proof of Theorem 4.3 and the classification of Legendrian torus knots. Before we begin the proof note that if we graph the Thurston-Bennequin invariants versus the rotation numbers that a fixed negative torus knot realizes, as in Figure 9, we obtain a figure that looks like a mountain range. Above we have shown that the dots at the peak of the mountains (corresponding to maximal Thurston-Bennequin invariant knots) are unique and the dots in the valleys (where stabilizations of maximal Thurston-Bennequin invariant knots first have the same invariants, in Figure 9 there are the points $(-22, -3)$, $(-23, 0)$ and $(-22, 3)$) are unique. The proof of this theorem follows from this and the properties of stabilizations.

Proof. Let K and K' be two (p, q) -torus knots with the same invariants. Let K destabilize to K_d and K' to K'_d . If K_d and K'_d have the same invariants,

then they are Legendrian isotopic and K and K' are the same stabilization of the same knot and hence are Legendrian isotopic. Now suppose the rotation numbers of K_d and K'_d differ by $2e$ as in Lemma 4.12. Then we can realize K as a stabilization of $S_-^e(K_d)$ and K' as a stabilization of $S_+^e(K'_d)$. Thus they both destabilize to $S_-^e(K_d) = S_+^e(K'_d)$ and hence they *both* destabilize to K_d and K'_d , implying they are Legendrian isotopic. Finally, if the rotation numbers of K_d and K'_d differ by something else, then a similar argument will show that K and K' will destabilize to the same Legendrian knot, finishing the proof. \square

Proof of Fact 1: Arrange for all the Legendrian ruling curves on T_m , T_{m+1} , and T to be meridional and let D be a meridional disk for S' that intersects these three tori in Legendrian curves. We may now isotop D (relative to its intersection with the tori) so that it is convex. Let $D = D_V \cup A$ where $D_V = D \cap V_0$ and $A = \overline{D} \setminus D_V$. Orient D so that μ is the oriented boundary of D_V . Now the dividing curves on D_V intersect $\mu = \partial D_V$, $2q$ times (since that is the number of times the dividing curves on T intersect μ). Moreover, they intersect each of ∂D and $D \cap T_m$, 2 times. We claim that all the dividing curves on D_V separate off disks that contain no dividing curves. Note this implies that all the bypasses on D_V are of the same sign. If this were not the case, then we would have bypasses on D_V of both signs and we would be able to glue this to a bypass of the same sign on A , creating an overtwisted disk. To match a bypass on D_V with any bypass on A we might have to “add copies of T to V_D ,” that is, take a copy of T , cut it along μ to obtain an annulus, and glue one of its boundary components to V_D and the other to A . This has the effect of shifting the dividing curves on D_V relative to those on A by $\frac{q}{p}$ (for details see the section on Sliding Maneuvers in [H1]). Since all the bypasses have the same sign, Equation 13 implies that $f_T(\mu) = r(\mu) = -1 + q$ or $1 - q$. \square

Proof of Fact 2: Make the Legendrian ruling curves on $T_m = \partial \overline{S}$ be meridional and let D be the meridional disk for \overline{S} with Legendrian boundary. Note $\partial D = \lambda$ (or at least a translate of it). Now the dividing curves intersect the boundary of D , $2m$ times. And since there are no closed homotopically trivial dividing curves we can conclude that there are exactly m dividing curves. By examining the possible configurations and using Equation 13, one may readily conclude that $f_m(\lambda) \in \{m-1, m-3, \dots, 1-m\}$. Moreover, by recalling how $f_m(\lambda)$ is related to the rotation number of K and noting the values of $r(K)$ realized in Figure 8, we see that all the possible values of $f_m(\lambda)$ are actually realized. \square

Proof of Fact 3: Make the Legendrian ruling curves on T_m and T_{m+1} longitudinal and let A be a longitudinal annulus spanning between T_m and T_{m+1} with Legendrian boundary. After making A convex, the dividing curves will intersect $S_m^1 = T_m \cap A$ in $2m$ points and $S_{m+1}^1 = T_{m+1} \cap A$ in $2m+2$ points. We claim that $2m$ dividing curves run from one boundary component of A to

the other boundary component and there is one boundary-parallel dividing curve with endpoints on S_{m+1}^1 . If this is not the case, then there must be boundary-parallel dividing curves for S_m^1 and hence a bypasses. Carefully applying Lemma 3.13, we can then find a convex torus between T_m and T_{m+1} with slope not lying between $[-\frac{1}{m}, -\frac{1}{m+1}]$. This, by Lemma 3.15, implies that there is a convex torus with boundary slope 0 and the Legendrian divides on this torus will be the boundaries of overtwisted disks. Therefore, the dividing curve configuration on A is as claimed.

Now make the Legendrian ruling curves on T longitudinal and make A intersect T in one of these longitudinal curves, say γ . The curve γ separates A into two annuli — A_m which touches T_m and A_{m+1} which touches T_{m+1} . Note the dividing curves are still as described above. From this we can deduce the structure of the dividing curves on A_m . To this end, note the dividing curves on A_m intersect the boundary component touching T_m , $2m$ times and the boundary component touching T , $2|p|$ times. Due to the structure of the dividing curves on A , we know that the $2m$ dividing curves emanating from T_m run across A_m to the other boundary component. Thus we know there are $|p| - m$ other dividing curves, all of whose boundaries lie on T . As argued in the proof of Fact 1, we can conclude that these all separate off disks that contain no dividing curves and thus give $|p| - m$ bypasses of the same sign.

Let e_U be the relative Euler class for the region $U = T^2 \times I$ between T and T_m . Since $e_U(\lambda) = f_T(\lambda) - f_m(\lambda)$, Equation 14 yields

$$f_T(\lambda) - f_m(\lambda) = (m - |p|) \text{ or } (|p| - m).$$

If $f_T(\lambda) = 1 - q$, then we claim that $f_T(\lambda) - f_m(\lambda) = m - |p|$. Assuming the contrary, we obtain a contradiction to the tightness of ξ as follows: Using Equation 13 one may easily check that $f_m(\mu) = 0$. Thus, if $f_T(\mu) = 1 - q = f_T(\mu) - f_m(\mu) = e_U(\mu)$, then $e_U(p\mu + q\lambda) = 2q(|p| - m) - e > e$. However, looking at the convex annulus A' of slope $\frac{p}{q}$ in U , we see that there are only e non-closed dividing curves, all of which have their boundary on T_m . This means that for $e_U(\mu) > e$ there must be some dividing curves bounding disks, violating the tightness of ξ . Therefore, $f_T(\lambda) - f_m(\lambda) = m - |p|$. \square

Proof of Fact 4: This is quite similar to the proof of Fact 3 and is left to the reader. \square

4.3. Transversal Torus Knots. Since Theorem 4.3 implies that Legendrian torus knots are stably simple, we may use Theorem 2.10 to conclude

Corollary 4.14. *If K and K' are transversal torus knots then there are transversally isotopic if and only if $l(K) = l(K')$. If K is a (p, q) -torus knot, with $p, q > 0$, then $l(K) \leq pq - p - q$ and if $p < 0, q > 0$ then $l(K) = pq + |p| - |q|$. Moreover, all odd integers satisfying these bounds are realized by the appropriate transversal torus knots.*

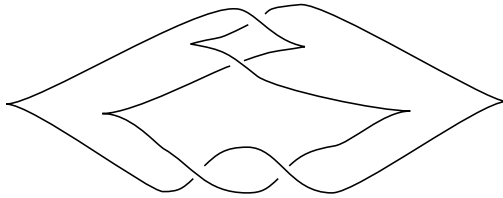


FIGURE 10. Legendrian figure eight knot with $\text{tb} = -3$.

As mentioned earlier, this extends [Et] and reproves a special case of a result of Birman and Wrinkle [BW] and Menasco [Men].

5. FIGURE EIGHT KNOTS

The following is the main theorem of this section:

Theorem 5.1. *The figure eight knot type \mathcal{K} is Legendrian simple.*

Again, Strategy 2.6 will be employed to prove Theorem 5.1.

5.1. Determination of $\overline{\text{tb}}$. The first step in the proof of Theorem 5.1 is to compute the maximal Thurston-Bennequin invariant $\overline{\text{tb}}$ for the figure eight knot.

Lemma 5.2. *The maximal Thurston-Bennequin invariant for the figure eight knot is -3 .*

Proof. To prove the upper bound $\overline{\text{tb}} \leq -3$, use the inequalities due to Fuchs-Tabachnikov [FT, Ta]. They give upper bounds for $\text{tb} + r$ of a Legendrian knot in terms of the HOMFLY and Kauffman polynomials. A simple calculation of these polynomials give the desired upper bound. Figure 10 provides an explicit example of a Legendrian figure eight knot with $\text{tb} = -3$. \square

5.2. Unique maximal tb representative. In this section we prove the following theorem:

Theorem 5.3. *If K and K' are two oriented Legendrian figure eight knots with maximal Thurston-Bennequin invariant $\text{tb} = -3$ (and necessarily $r(K) = r(K') = 0$), then K and K' are Legendrian isotopic.*

Let M be $S^3 \setminus N$ where N is a small standard neighborhood of K so that the Legendrian ruling on ∂M is by longitudes (this is possible since $\text{tb} < 0$) — note that there is a well-defined longitude coming from the Seifert surface. Recall we have a fibration $p: M \rightarrow S^1$ with fiber a punctured torus Σ with Legendrian boundary.

The following is a general procedure for proving that two Legendrian knots K and K' of the same knot type and maximal Thurston-Bennequin invariant are isotopic. We construct a *contactomorphism* $f: S^3 \rightarrow S^3$ taking K to K' as follows: Let f be a map from a standard neighborhood N (as above) of K to a standard neighborhood N' of K' — this is possible because of local uniqueness. Assume N and N' have already been identified via f .

Then we have two tight contact structures on M with identical boundary characteristic foliations. Assume we can prove the following:

Claim. *The two tight contact structures on M are contactomorphic relative the boundary.*

Then the map f may be extended to a contactomorphism of S^3 onto S^3 , taking K to K' . With the map f in hand we may use Theorem 2.2 to show we have a contact isotopy from the identity map $id : S^3 \rightarrow S^3$ to f , exhibiting the desired Legendrian isotopy from K to K' .

We have therefore reduced the proof of Theorem 5.3 to the analysis (and normalization) of the tight contact structure on $M = S^3 \setminus N$. Since M fibers over the circle with punctured torus fiber $\Sigma = T^2 \setminus D^2$, we write $M = (\Sigma \times I)/\sim$, where $(x, 0) \sim (\Psi(x), 1)$, and

$$\Psi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is the monodromy map. Ψ is a diffeomorphism of Σ which fixes ∂D^2 . Note that the matrix by itself does not completely define Ψ , since it is possible to compose with Dehn twists along curves parallel to ∂D^2 . Since

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

we define Ψ as a Dehn twist along a closed curve parallel to $(0, 1)$, followed by a Dehn twist along a closed curve parallel to $(1, 0)$. In particular, we compose with no Dehn twists along ∂D^2 . We will use this particular matrix representative of the conjugacy class of Ψ .

Consider the characteristic foliation on a convex Σ with Legendrian boundary. The following several pages will be devoted to normalizing Σ . Note that the dividing set Γ_Σ will consist of closed curves or arcs with endpoints on $\partial\Sigma$.

First observe the following simple lemma:

Lemma 5.4. *If $\{\gamma_1, \dots, \gamma_n\}$ is a collection of non-boundary-parallel, mutually non-parallel arcs in a punctured torus Σ (with endpoints on $\partial\Sigma$), then $n \leq 3$. Moreover, if there exists a closed curve on Σ disjoint from the γ_i 's, then $n \leq 1$.*

The possible configurations of dividing curves are given below:

Proposition 5.5. *Suppose the dividing curves on a convex $\Sigma \subset M$ with Legendrian boundary consist of arcs $\gamma_1, \dots, \gamma_n$ with endpoints on $\partial\Sigma$, and closed curves c_1, \dots, c_m .*

- I. *If the γ_i 's realize one distinct non-boundary-parallel isotopy class on Σ , then Σ may be isotoped so that (a) $n \geq 1$ (n odd) and $m = 1$, (b) $n = 0$ and $m = 2$, or (c) Σ has a bypass along $\partial\Sigma$.*

- II. If the γ_i 's realize two distinct non-boundary-parallel isotopy classes on Σ , then $m = 0$ and either (a) each isotopy class has two γ_i , or (b) Σ may be isotoped so that it has a bypass along $\partial\Sigma$.
- III. If the γ_i 's realize three distinct non-boundary-parallel isotopy classes on Σ , then $m = 0$ and either (a) one of the isotopy classes has one γ_i and the other two have one or more, or (b) Σ may be isotoped so that it has a bypass along $\partial\Sigma$.

The proposition is preliminary. It will be improved later, after considerably more work.

Proof. Note that any claim we will be making concerning Σ should be interpreted up to a C^0 -small isotopy — in particular, we will make extensive use of the Legendrian Realization Principle (Theorem 3.3) without explicitly mentioning each time that a C^0 -small perturbation is taking place first. Let Σ be a convex fiber in M with Legendrian boundary, and N an open convex neighborhood of Σ . Then in $(M \setminus N) = \Sigma \times [0, 1]$, we look for an annulus A with $\partial A = A_0 - A_1$, where the A_i are Legendrian curves in (a C^0 -isotoped copy of) $\Sigma_i = \Sigma \times \{i\}$, and A contains a bypass. Denote the dividing set of Σ_i by Γ_i .

Let us consider Case I. Observe that $m + n$ is always even. Here we may always find a closed Legendrian curve c on Σ_0 which does not intersect Γ_0 . We take A to be the annulus $c \times [0, 1]$. Note that $\Gamma_1 = \Psi(\Gamma_0)$ and the geometric intersection number $|(c \times \{1\}) \cap \Gamma_1| \geq m + n$, because an Anosov map cannot fix a curve isotopy class. Realize $c \times \{1\}$ as a Legendrian curve which has minimal intersection number with Γ_1 in its isotopy class. Then A will have at least $n + m$ dividing curves emanating from A_1 and none from A_0 , from which we can conclude using the Imbalance Principle (Proposition 3.11) that there is a bypass along A_1 .

If $m + n \geq 4$, then we may always find an embedded bypass, and we apply Proposition 3.12. Let δ be the attaching Legendrian arc $\subset \Sigma_1$ for the bypass, and p, q, r be the intersections with Γ_1 in consecutive order along δ . Since $m + n \geq 4$, p, q, r lie on distinct dividing curves. If p, q both lie on the γ_i 's (or the same for q, r), then there will exist a boundary-parallel dividing curve on the new Σ_1 , after bypass attachment (and thus a bypass for the new $\partial\Sigma_1$). If p, q both lie on the c_i 's (or the same for q, r), then we will be able to reduce m by two. Hence, if $m \geq 2$ and $n \geq 2$, every bypass will be as above, and we may either reduce m by 2 or obtain a bypass for Σ_1 along $\partial\Sigma_1$. Now there are two other cases: $n \geq 3$, $m = 1$, and p, r lie on γ_i 's and q lies on a c_i ; or $n = 1$, $m \geq 3$, and p, r lie on c_i 's and q lies on a γ_i . In the latter case, we reduce m by 2. (Note that, in the former case, the bypass attachment yields a dividing set which puts us in Case III, where two of the isotopy classes of arcs are represented by a single γ_i .)

We eventually have (a) $n \geq 1$ and $m = 1$, (b) $n = 0$ and $m = 2$, (b') $n = 2$ and $m = 0$, or (c) a bypass along $\partial\Sigma_1$. We can actually do better, and remove (b') $n = 2$, $m = 0$. If A_1 intersects Γ_1 exactly twice, then we have a

degenerate bypass along A_1 (one with same endpoints), and attaching this degenerate bypass gives rise to a dividing set which has a boundary-parallel dividing curve. If the intersection number is greater than 2, then keep A but repeatedly attach bypasses until we arrive at boundary-parallel components of (the new) Γ_1 . Observe that, once we have reduced to (a) or (b), there will still exist bypasses that can be attached onto Σ_1 . However, they do not necessarily yield boundary-parallel dividing curves after attachment and cannot be used to destabilize $\partial\Sigma_1$ — at least not immediately.

In Cases II and III, $m = 0$ by Lemma 5.4. Consider Case II. Let γ_1 and γ_2 be the two isotopy classes of curves of Γ_0 , and n_1, n_2 be the number of arcs in each. Note that both n_1 and n_2 are even. For convenience, we will identify $\Sigma = (\mathbb{R}^2/\mathbb{Z}^2) \setminus D^2$, and γ_1, γ_2 with minimal (shortest-length) integral vectors $v_1, v_2 \in \mathbb{Z}^2$. It is easy to find a curve c on Σ so that c intersects Γ_0 and Γ_1 in a different number of points. Indeed consider the subset \mathcal{C}_{min} of the set \mathcal{C} of (isotopy classes of) closed (connected) curves c on Σ , consisting of c which have minimal geometric intersection $|c \cap \Gamma_0|$ among curves in \mathcal{C} . Since the Anosov map Ψ cannot preserve a finite set of closed curves, $\mathcal{C}_{min} \neq \Psi(\mathcal{C}_{min})$. Noting that $\Psi(\mathcal{C}_{min})$ consists of curves c with minimal intersection number $|c \cap \Gamma_1|$ among curves in \mathcal{C} , we pick $c \in \mathcal{C}_{min} \setminus \Psi(\mathcal{C}_{min})$. Using this c we may argue as in Case I to find a bypass along Σ_1 .

As long as $n_1, n_2 > 2$, p, q, r (as above) lie on three distinct dividing curves, and p, q (or q, r) lie on ‘consecutive’ dividing curves of the same type v_1 (or v_2). Thus we can produce a boundary-parallel component of the new Γ_1 after attaching the bypass to Σ_1 . Therefore, at least one of the n_i must equal 2. We will eliminate $n_1 = 2, n_2 > 2$ (or $n_1 > 2, n_2 = 2$). In this case, \mathcal{C}_{min} consists of one curve d_2 parallel to v_2 . As above, $d_2 \notin \Psi(\mathcal{C}_{min})$, so $|\Gamma_1 \cap d_2| \geq n_2 \geq 4$, since d_2 intersects both $\Psi(v_1)$ and $\Psi(v_2)$. We then let $A = d_2 \times [0, 1]$, and find a bypass along $d_2 \times \{1\}$. The only nontrivial case is when our bypass along Σ_1 involves only the two dividing curves isotopic to $\Psi(v_1)$, p, r lie on the same curve, and the bypass attachment changes the dividing curves $\Psi(v_1), \Psi(v_2)$ to $\Psi(v_1 \pm v_2), \Psi(v_2)$. In this case we will have a sequence of ‘nested bypasses’. More precisely, attach the (outermost) bypass onto Σ_1 and take a new $A = d_2 \times [0, 1]$ such that d_2 has fewer intersections with the new Γ_1 . Note that $d_2 \times \{0\}$ intersects Γ_0 twice, but $d_2 \times \{1\}$ still intersects Γ_1 at least four times. In any event, if we reduce this intersection number to four, there can no longer exist any bypasses of the nontrivial type, and we produce a destabilization. The only case left is when $n_1 = n_2 = 2$.

In Case III, if we crush the boundary Σ to a point, then homologically one of the isotopy classes of Γ is the sum of the other two isotopy classes. Therefore we write the three classes as $v_1, v_2, v_1 + v_2$, and the number of dividing curves as n_1, n_2, n_{12} (these must have the same parity). Using the same argument as in Case II (with \mathcal{C}_{min}), we produce an annulus A , together with a bypass along Σ_1 .

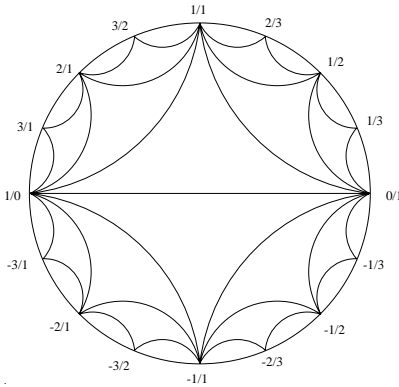


FIGURE 11. The standard tessellation of the hyperbolic unit disk

As long as three of the isotopy classes have more than one dividing curve, any bypass along A_1 gives rise to a boundary-parallel component of Γ_1 on Σ_1 . Thus, one of the isotopy classes will have one dividing curve. \square

Returning now to our normalization, we know that since $\text{tb}(K)$ is maximal, there can be no bypasses on Σ . Thus Proposition 5.5 and Lemma 5.2 imply that there are precisely three dividing curves — we have either I(a) with $n = 3$, $m = 1$ or III(a). In what follows, we need to deal with configurations of both types, but, for the time being, we may assume that all of the dividing curves are arcs and they are non-parallel (we can switch from I(a) to III(a) according to the proof of the previous proposition). Let us still denote the three arcs by the corresponding minimal integral basis vectors v_1 , v_2 , together with their sum $v_1 + v_2$.

We will now normalize these vectors, showing that we can always assume we are in one of two possible situations. To do this we need some way of listing all possible triples of such vectors (or, equivalently, all possible two-triangle “triangulations” of T^2). We do this using the standard (Farey) tessellation of the hyperbolic unit disk $D^2 = \{(x, y) | x^2 + y^2 \leq 1\}$, which we now review. Recall we start by labeling $(1, 0)$ as $0 = \frac{0}{1}$, and $(-1, 0)$ as $\infty = \frac{1}{0}$. We inductively label points on $S^1 = \partial D^2$ as follows (for $y > 0$): Suppose we have already labeled $\infty \geq \frac{p}{q} \geq 0$ (p, q relatively prime) and $\infty \geq \frac{p'}{q'} \geq 0$ (p', q' relatively prime) such that (p, q) , (p', q') form a \mathbb{Z} -basis of \mathbb{Z}^2 . Then, halfway between $\frac{p}{q}$ and $\frac{p'}{q'}$ along S^1 on the shorter arc (the one for which $y > 0$), we label $\frac{p+p'}{q+q'}$. We then connect two points $\frac{p}{q}$ and $\frac{p'}{q'}$ on the boundary by a hyperbolic geodesic, if the corresponding shortest integral vectors form an integral basis of \mathbb{Z}^2 . See Figure 11. An important observation for us is that the tessellation gives a triangulation of D^2 into a union of ideal triangles, and that each geodesic is on the boundary of exactly two ideal triangles — any two integral basis vectors v_1 , v_2 can be

completed into a triple $\{v_1, v_2, v_3\}$ which mutually form an integral basis, in exactly two ways. Observe that the three vectors $v_1, v_2, v_1 + v_2$ in the previous paragraph form the vertices of an ideal triangle in the standard tessellation and any ideal triangle gives a possible triple. Note also that the vertices of the triangles in the tessellation are labeled with the slopes of the vectors in the corresponding triangulation of T^2 . More generally, there is a correspondence between the points on the “circle at infinity” S_∞^1 of the hyperbolic disk and slopes on T^2 .

Proposition 5.6. *There exists an isotopic copy of Σ with dividing set $\Gamma = \{(1, 1), (0, 1), (1, 2)\}$ or $\{(1, 0), (0, 1), (1, 1)\}$.*

To prove proposition 5.6 we start with Γ_0 consisting of three dividing curves corresponding to $v_1, v_2, v_1 + v_2$, where $\{v_1, v_2\}$ form an integral basis of \mathbb{Z}^2 and thus give an ideal triangle in the above tessellation. We will then show how to add bypasses to Σ in such a way that the new dividing curves correspond to a different triangle. After a sequence of such attachments we will eventually arrive at the triangle corresponding to either $\{(1, 1), (0, 1), (1, 2)\}$ or $\{(1, 0), (0, 1), (1, 1)\}$.

Before we begin the proof of Proposition 5.6, we need a preliminary lemma.

Lemma 5.7. *The complete list of bypasses that can be attached to the initial configuration $\{v_1, v_2, v_1 + v_2\}$ are as given in Figures 12 and 13 below. The bypasses are subject to two conditions: (1) they do not give rise to destabilizations, and (2) if $\delta \subset \Sigma_0$ is the attaching Legendrian arc for the bypass, and p, q, r are intersections with Γ_0 in consecutive order along δ , then p, q do not lie on the same dividing curve and q, r do not lie on the same dividing curve.*

The proof of the lemma is left to the reader — one simply needs to check all the possibilities. To do this note that the bypass can only involve two different dividing curves on Σ (or else one gets a boundary-parallel dividing curve on Σ). This observation will yield 12 possible dividing curve attachments, three of which will also result in a boundary-parallel dividing curve on Σ . The nine remaining possibilities are listed in Figures 12 and 13.

Proof of Proposition 5.6. If Σ_0 has dividing set Γ_0 , then act via Ψ to get $\Gamma_1 = \Psi(\Gamma_0)$ on $\Sigma_1 = \Psi(\Sigma_0)$, which is a convex surface which is isotopic to Σ_0 . Hence, we may freely modify $\Gamma \mapsto \Psi^k(\Gamma)$ for any $k \in \mathbb{Z}$, via an isotopy.

Let $p, -\frac{1}{p} \in \mathbb{R} \setminus \mathbb{Q}$ be the slopes corresponding to the two eigendirections of Ψ , with $0 < p < \infty$. On the circle at infinity S_∞^1 of the standard tessellation of D^2 , p is an attracting fixed point and $-\frac{1}{p}$ is a repelling fixed point under the action of Ψ . Suppose that the initial configuration $\{v_1, v_2, v_1 + v_2\}$ corresponds to slopes a, b, c respectively. Then we may act repeatedly via Ψ so that $\infty \geq b > c > a \geq 0$, and we have three possibilities:

- (1) $a, b, c > p$.

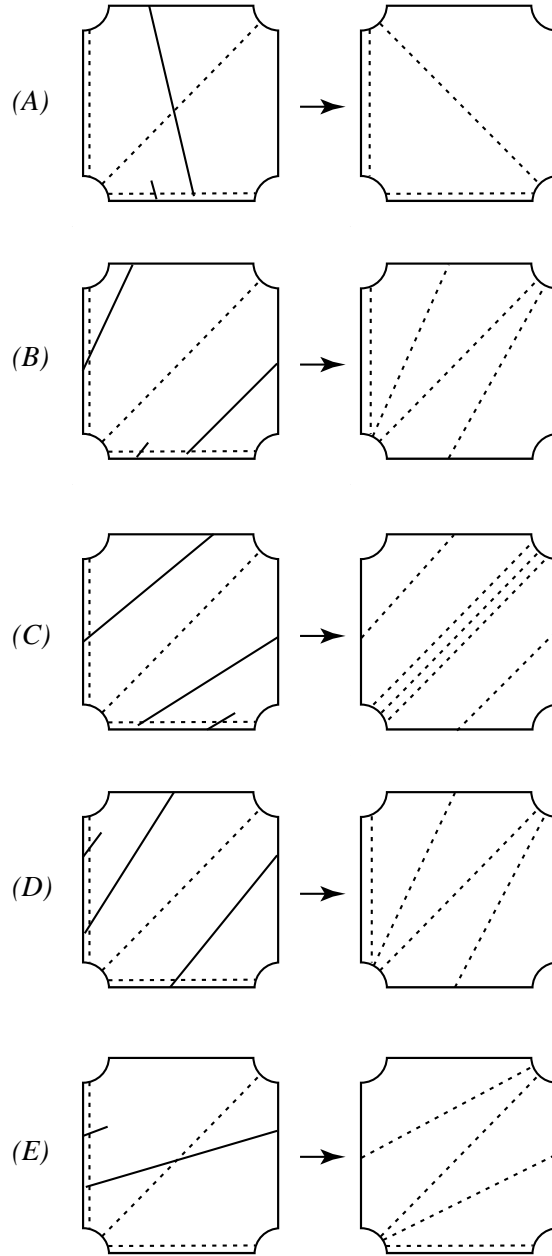


FIGURE 12. Allowable list of dividing curves. The dotted lines are dividing curves, and the Legendrian arcs of attachment for the bypasses are shown (solid lines). The bypasses are attached from the front. The left side and right side of each tile are identified and the top and bottom are also identified.

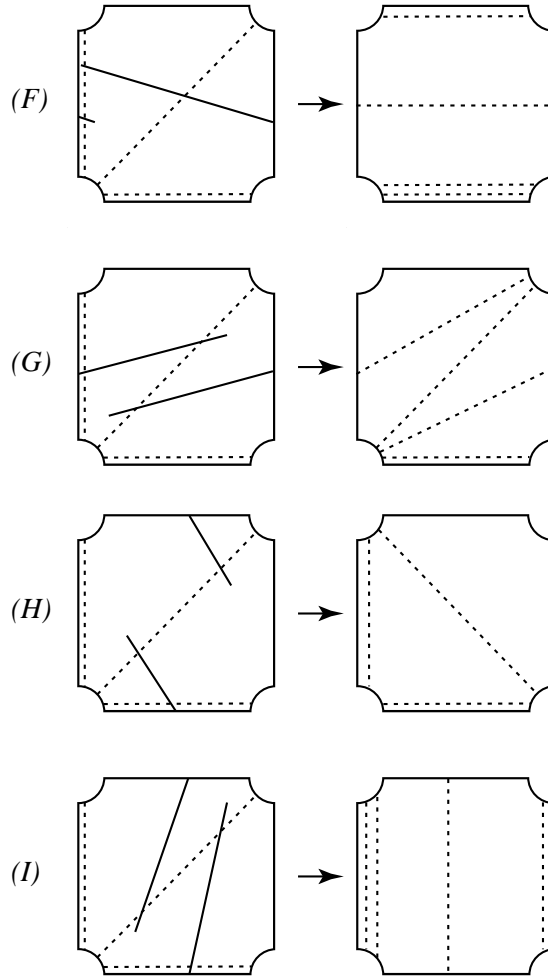


FIGURE 13. Continuation of allowable list of dividing curves.

- (2) $a, b, c < p$.
- (3) $b, c > p, a < p$, or $b > p, a, c < p$.

Assume that in all the cases, bypass attachments do not give rise to boundary-parallel dividing curves (and hence a destabilization), since that would contradict $\overline{tb} = -3$. This means that, for dividing curve configurations of type III(a), we only use moves of type A through I in Figures 12 and 13. For configurations of type I(a), we only use the moves in Figure 15. To simplify notation, write v_s for the shortest integral vector with nonnegative entries, corresponding to a slope $0 \leq s \in \mathbb{Q}$.

Case 1. We may assume in addition that $1 \leq a < c < b \leq \infty$. Indeed, if $a \geq 1$ then we are done, and if $a < 1$ then $p < a, c, b < 1$, since any

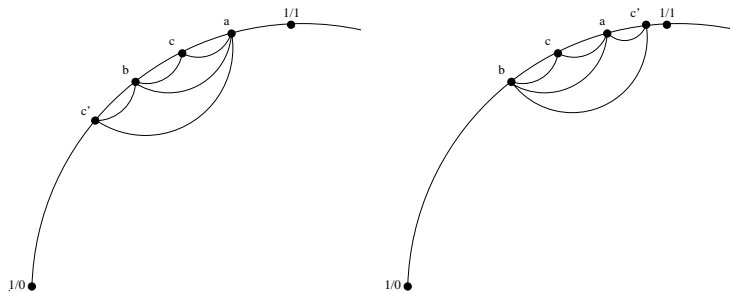


FIGURE 14. Type A and H moves.

triangle in the tessellation whose vertices are between 0 and ∞ and whose clockwise-most vertex < 1 has all its vertices < 1 . Now since $\Psi(\infty) = 1$ we may use Ψ^{-1} to achieve the desired slopes. We will show that using bypasses we can perform a sequence consisting of the following moves to our v_1, v_2 , and $v_3 = v_1 + v_2$ (hence affecting a, b, c):

- (1) replace v_1, v_2, v_3 with v'_1, v'_2, v'_3 , where $v'_1 = v_1 - v_2, v'_2 = v_2$ and $v'_3 = v_1 = v'_1 + v'_2$, (corresponds to right drawing in Figure 14)
- (2) replace v_1, v_2, v_3 with v'_1, v'_2, v'_3 , where $v'_1 = v_1, v'_2 = v_2 - v_1$ and $v'_3 = v_2 = v'_1 + v'_2$, (corresponds to left drawing in Figure 14) or
- (3) replace v_1, v_2, v_3 with v'_1, v'_2, v'_3 , where $v'_3 = v_1$, and v'_1 and v'_2 are determined by $v'_2 = v'_1 + v'_3$, and the triangle corresponding to v'_1, v'_2, v'_3 is the innermost triangle in the tessellation with $v'_3 = v_1$ and all vertices counterclockwise of 1. Here, ‘innermost’ means closest to the center of the disk.

Note that, if we start with a triangle T corresponding to v_1, v_2 and $v_3 = v_1 + v_2$, then there is a finite number N_T of triangles that can be obtained from T by the sequence of moves listed above. Moreover, the triangle with vertices at $(1, 2, \infty)$ is one of the triangles that can be so obtained. Now, if T' is obtained from T by one of the moves above, then $N_{T'} < N_T$. Thus any (sufficiently long) finite sequence of these moves will take us from T to the triangle with vertices at $(1, 2, \infty)$.

We now show how to find the bypasses that allow us to perform the above moves. Since $\Psi(a) < 1 \leq a, b, c$, $|\Psi(v_a) \cap (v_a \cup v_b \cup v_c)| > 2$, and, by using an annulus $d \times I$ with d parallel to $\Psi(v_a)$, we find a bypass of type A, F, or H in Figures 12 and 13. The bypasses are of the stated type, since the slope of the bypass is smaller than a, b , or c . Suppose we do not already have $\{a, b, c\} = \{1, 2, \infty\}$. Moves of type A and H replace the ideal triangle with vertices a, b, c by another triangle with vertices a, b, c' (see Figure 14). Here c' corresponds to the slope of $\pm(v_2 - v_1)$. Thus, v_1, v_2 and v_3 are affected by applying one of the first two moves mentioned above.

A move of type F will give rise to a configuration of type I(a) with $n = 3$ and $m = 1$. Just as we denoted a configuration of type III(a) by $\{a, b, c\}$, we will denote a configuration of type I(a) by $\{a\}$. We deal with the case $a = 1$

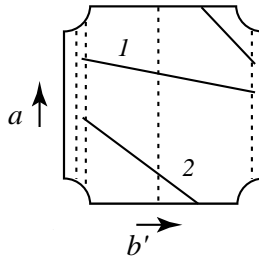


FIGURE 15. Two possibilities for bypass attachment.

at the end of the paragraph. Otherwise, in order to modify $\Gamma_0 = \{a\}$, take an annulus $d \times I$, where d has slope $\Psi(a)$. Note that $p < \Psi(a) < 1$. Since bypass attachments cannot give rise to boundary-parallel dividing curves on Σ , there are two possibilities, given in Figure 15. Here b' is the smallest slope for which there exists an edge of the tessellation from a to b' . (Note: in general there are other possibilities for the slope of the bypass but with this choice of basis for the torus and our choice of d only those shown in Figure 15 are possible.) After the bypass attachment, Σ has dividing curves of type III(a) and two of the dividing curves have slopes a, b' respectively. Let v_1 and v_2 be the minimal integral vectors corresponding to these slopes. Then the third dividing curve on Σ is spanned by $v_3 = v_1 + v_2$ or $v_2 - v_1$. In the first case we have performed a move of type 3 listed above and in the second case we have performed a move of type 3 followed by one of type 2. Thus we may perform a sequence of the above moves eventually obtain $\{1, 2, \infty\}$ or at some point we obtain a configuration of type I(a) with curves of slope 1. In this case use Ψ^{-1} to modify $\{1\} \mapsto \{\infty\}$. Let this be Γ_0 . Then, using the annulus $d \times I$ where d has slope $\Psi(\infty) = 1$, we obtain a bypass which modifies $\{\infty\} \mapsto \{1, 2, \infty\}$.

Case 2. Assume $a < c < b < p$. Without loss of generality, assume $0 \leq a < c < b \leq \frac{1}{2}$ (this argument is the same as the one at the beginning of Case 1). Use the annulus $d \times I$, where d has slope $\Psi(b)$, to obtain a bypass of type A, F, or H. The same argument as in Case 1 allows us to isotop Σ so that the dividing curves are of type I(a) with slope 0 or of type III(a) with slopes $0, \frac{1}{2}, \frac{1}{3}$. In the first case, attaching the only possible bypass along $\Psi(0) = \frac{1}{2}$ which does not yield a boundary-parallel component on Σ_0 , we obtain $\{0\} \mapsto \{0, 1, \infty\}$. Next suppose we have $\{0, \frac{1}{2}, \frac{1}{3}\}$. If we use the annulus $d \times I$ where d has slope $\frac{1}{2}$, then $\Psi(\frac{1}{2}) = \frac{3}{5}$, and F cannot happen (for F to happen, we need $\Psi(\frac{1}{2}) > 1$). Therefore, $\{0, \frac{1}{2}, \frac{1}{3}\} \mapsto \{0, \frac{1}{2}, 1\}$, which is dealt with in Case 3.

Case 3. Here we start with a triangle T in the Farey tessellation whose vertices straddle p . Below we show that attaching bypasses will either put

us back in Case 1, which we have already dealt with, or we perform one of the following moves on v_1, v_2 and $v_3 = v_1 + v_2$:

- (1) replace v_1, v_2, v_3 with $v'_1 = v_1, v'_2 = v_1 + v_2$ and $v'_3 = v_2 + 2v_1$ (if $a < p$ and $b > c > p$) or
- (2) replace v_1, v_2, v_3 with $v'_1 = v_1 + v_2, v'_2 = v_2$ and $v'_3 = v_1 + 2v_2$ (if $a < c < p$ and $b > p$).

Assuming this is true for the moment. There is some k such that the vertices $\{\Psi^k(0), \Psi^k(1), \Psi^k(\infty)\}$ span a triangle G in the Farey tessellation that is disjoint from T and separates T from p . Each time we perform a move of type 1 or 2 to T , we obtain a new triangle T' lying between T and G . Since there are only a finite number of triangles in the tessellation lying between T and G , this process must stop. It can only stop if $T = G$ (or a bypass attachment puts us in Case 1), in which case we are done.

Suppose that $b > c > p$ and $a < p$. Use the annulus $d \times I$, where d has slope $\Psi(c)$, to obtain a bypass of type C, E, or G. C gives $\{a, b, c\} \mapsto \{c\}$, with $c > p$, which was already treated under Case 1. Bypasses of type E or G modify $v_1, v_2, v_1 + v_2$ to $v_1, v_1 + v_2, 2v_1 + v_2$ — and we are still in Case 3 (though we might have $b > p$ and $a < c < p$). Now suppose that $b > p$ and $a < c < p$. Similarly, we have bypasses of type B, D, or I. I puts us in Case 1, and B, D modify $v_1, v_2, v_1 + v_2$ to $v_1 + v_2, v_2, v_1 + 2v_2$ (still in Case 3). \square

Proposition 5.8. (1) *There exists no tight contact structure on $\Sigma \times I$ for which Γ_0 is of type III(a) with slopes $\{0, 1, \infty\}$.* (2) *There exists exactly one tight contact structure on $\Sigma \times I$ for which Γ_0 is of type III(a) with slopes $\{1, 2, \infty\}$, and $(\Sigma_0)_\pm$ — the signs of each region divided by Γ_0 — are fixed.*

Proof. Note that $H = \Sigma \times I$ is a genus two handlebody. In general, when analyzing handlebodies with convex boundary, we use compressing disks D_1 and D_2 so that the cut-open manifold is a 3-ball. To obtain a smooth convex boundary of $H = \Sigma \times I$, we need to round the edges $\partial\Sigma \times \{0, 1\}$. This is done using the Edge-Rounding Lemma (Lemma 3.8). When using Edge-Rounding, we must be careful to remember that, since $\partial\Sigma \times [0, 1]$ was the convex boundary of a neighborhood of a Legendrian curve, there is holonomy (as we go from Σ_0 to Σ_1). We then arrange for D_1, D_2 to have Legendrian boundary — for this we use the Legendrian Realization Principle. Then, D_1, D_2 with Legendrian boundary are perturbed, fixing the boundary, so they become convex. What remains is to study the configuration of dividing curves on each compressing disk.

We have $\partial(\Sigma \times [0, 1]) = \Sigma_0 \cup \Sigma_1 \cup A$ where A is the annulus $\partial\Sigma \times [0, 1]$. Below we discuss the dividing curves on Σ_0 and Σ_1 but first we consider the dividing curves on then annulus A and how they are related to the dividing curves on the Σ_i 's. To this end consider Figure 16. In this figure we identify the right and left edges, then the center rectangle forms the annulus A . The top and bottom parts of the picture form neighborhoods of $\partial\Sigma_i$ in Σ_i and the shaded region is obtained using the Edge-Rounding Lemma. The dividing

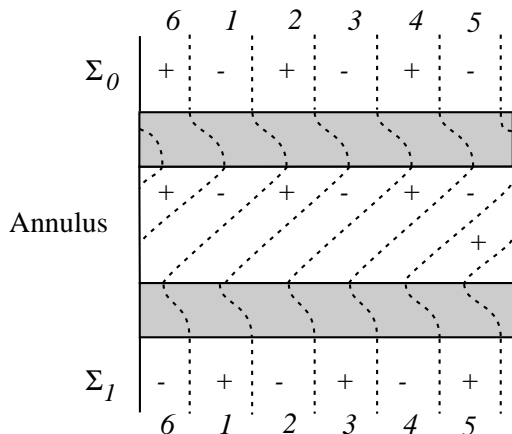


FIGURE 16. Dividing curves (dashed lines) on, and near, the annulus A .

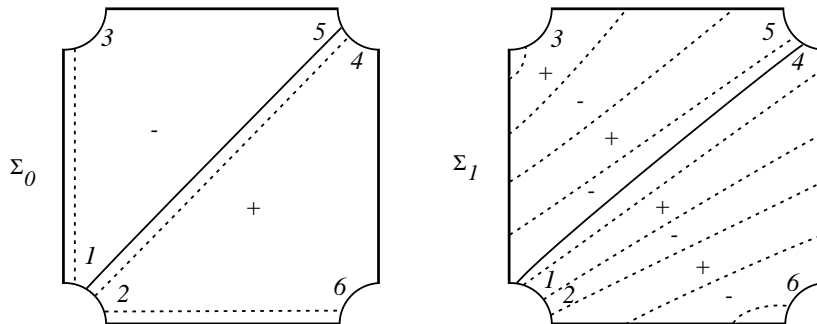


FIGURE 17. Dividing curves on Σ_0 and Σ_1 (dashed lines). The solid line is our choice of δ .

curves on Σ_i divide the $\partial\Sigma_i$ into 6 intervals. We label the intervals 1-6 so we can identify them with the corresponding intervals on the annulus A .

(1) $\Gamma_0 = \{0, 1, \infty\}$ and $\Gamma_1 = \{\frac{1}{2}, \frac{2}{3}, 1\}$. See Figure 17. There exists a disk $D = \delta \times [0, 1]$, where δ is an arc with slope 1, with geometric intersection $|\partial D \cap \Gamma_{\partial H}| = 0$. Note that $\delta \times \{0, 1\} \subset \partial D$ are represented by solid lines in Figure 17, and $(\partial\delta) \times I \subset \partial D$ lies on $A = \partial\Sigma \times [0, 1]$. Though $(\partial\delta) \times I$ is not drawn in Figure 16, one may easily determine that $\partial D \cap \Gamma_{\partial H} = \emptyset$. By the Legendrian Realization Principle, we may realize ∂D as a zero-twisting (rel ∂H) Legendrian curve. This implies that there is an overtwisted disk and that there cannot be a tight contact structure on H with this boundary condition.

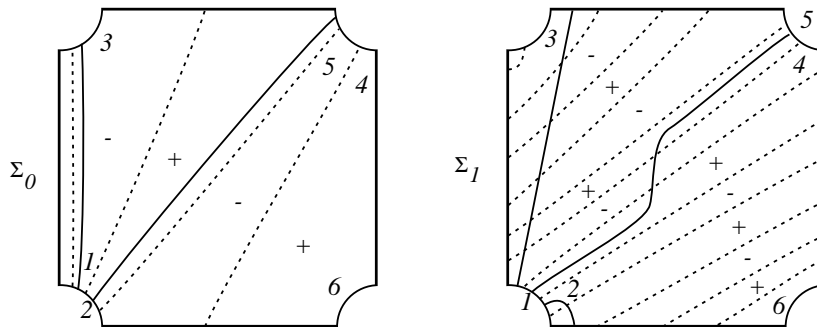


FIGURE 18. Dividing curves on Σ_0 and Σ_1 (dashed lines). The solid line is our choice of δ_i .

(2) $\Gamma_0 = \{1, 2, \infty\}$ and $\Gamma_1 = \{\frac{2}{3}, \frac{3}{4}, 1\}$. Let δ_1 be an arc on Σ with slope 1 and let δ_2 be an arc with infinite slope. Choose the compressing disks D_1 and D_2 for H to be isotopic to $\delta_1 \times I$ and $\delta_2 \times I$, drawn as in Figure 18. After using the Legendrian Realization Principle, we take the boundaries to be Legendrian and ∂D_1 and ∂D_2 to intersect the dividing curves on ∂H (minimal geometric intersection number) 2 and 6 times, respectively.

There is only one possibility for the dividing curves on D_1 . However, there are several possible dividing curve configurations for D_2 . Consider the intersections between $\delta_2 \times \{1\}$ and Γ_1 — count the intersections from bottom to top along $\delta_2 \times \{1\} \subset \Sigma_1$ shown in Figure 18. One may easily check that there cannot exist a bypass along ∂D_2 which straddles the second intersection or the fourth intersection, without immediately yielding a bypass for K . Thus, the only two possibilities for D_2 are shown in Figure 19. Now, if D_2 had the right-hand configuration shown in Figure 19, then a bypass straddles the third intersection between $\delta_2 \times \{1\}$ and Γ_1 . Note this bypass is nested inside another bypass on D_2 . If we added both these bypasses in succession, the resulting copy of Σ would have dividing curves $\{0, 1, \infty\}$. Hence we could find an overtwisted disk as above. Thus, the dividing curves on D_2 are shown on the left-hand side of Figure 19.

Now, since we have normalized each of D_1, D_2 to have a unique dividing curve configuration, we have a unique tight contact structure on H with given configurations Γ_0, Γ_1 , up to isotopy. This is because any two tight contact structures with these boundary conditions can first be matched up along D_1, D_2 , and then matched up inside $H \setminus (D_1 \cup D_2)$ using Theorem 2.1. \square

We have normalized Γ on Σ to be of type III(a) with slopes $\{1, 2, \infty\}$. To finish the proof of Theorem 5.3 we just note that we can also normalize the signs of the two regions of $\Sigma \setminus \Gamma$ by acting via $-id$ on Σ . Since $-id \circ$

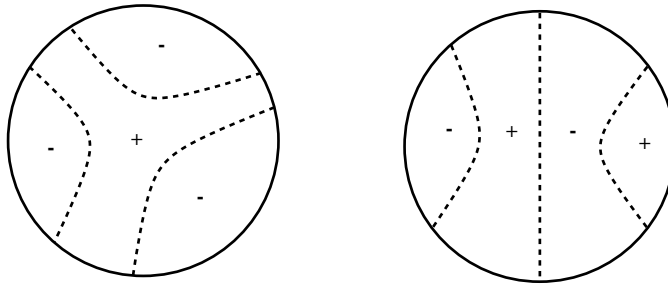


FIGURE 19. Possible dividing curves on Δ_2 .

$\Psi \circ (-id)^{-1} = \Psi$, we may use this *isomorphism* to switch Σ_+ and Σ_- . Therefore, we obtain a contactomorphism $f : S^3 \rightarrow S^3$ taking K to K' as discussed after the statement of Theorem 5.3.

5.3. Destabilization.

Proposition 5.9. *If K is an oriented Legendrian figure eight knot and $tb(K) = -3 - k < -3$, then there exist positive integers k_1 and k_2 such that $k = k_1 + k_2$, $r(K) = k_2 - k_1$ and $K = S_-^{k_1}(S_+^{k_2}(K'))$, where K' is the Legendrian figure eight knot with maximal Thurston-Bennequin invariant.*

Proof. Suppose K is a Legendrian figure eight knot with $tb(K) < -3$. From Lemma 3.6 we know that there must be more than 3 dividing curves on Σ . Thus Proposition 5.5 implies that the dividing curves on the Seifert surface Σ can be normalized so that we have I(a) (with $n \geq 5$), II(a), or III(a). In each of the cases, we prove that either we can normalize Γ into a standard form, or there exists a destabilization — this is done in the same way as in Proposition 5.6. Then, in the same way as in Proposition 5.8, we prove that there always exists a destabilization, when Γ is in the standard form.

Case II(a). Refer again to the tessellation picture, and denote Γ_0 by $\{a, b\}$, where $\infty \geq b > a \geq 0$ (without loss of generality) are the slopes of the two isotopy classes of arcs on Σ_0 . Figure 20 gives a complete list of bypasses that can be attached, without immediately giving rise to a destabilization. First assume $\infty \geq b > a \geq 1$, and that $\{a, b\} \neq \{1, \infty\}$. If d is a closed curve with slope a , then use $\Psi(d) \times I$ to obtain a bypass along Σ_0 , using the Imbalance Principle. If there is no destabilization, then the bypass attachment is of type A, and $\{a, b\} \mapsto \{a', b\}$, where $a > a' \geq 1$. If we repeat this procedure, we either arrive at $\{1, \infty\}$ or obtain a destabilization if a is already the smallest slope for which there exists an edge of the tessellation to b . Finally, if we take a closed curve d with slope 1, then $\Psi(d)$ has slope $\frac{2}{3}$. The bypass along Σ_0 with $\Gamma_0 = \{1, \infty\}$, which comes from $\Psi(d) \times I$, cannot be of type A or B, and hence we have a destabilization.

Next assume $\infty \geq b > p$ and $p > a \geq 0$. Define c to be the slope corresponding to $v_a + v_b$. If $c < p$, then use $\Psi(d) \times I$, where d has slope b .

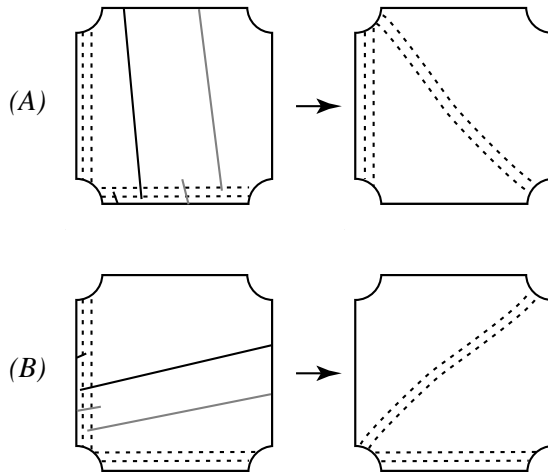


FIGURE 20. Allowable bypasses. The Legendrian arcs of attachment are shown. The bypasses are attached along solid black or grey lines from the front.

The resulting bypass is not of type A or B, since the slope of $\Psi(d) > c > 0$ while the bypasses of type A and B have slope less than c . Thus we can destabilize. If $c > p$, then use $\Psi(d) \times I$, where d has slope a . A type B move modifies $\{a, b\} \mapsto \{a, c\}$. We keep repeating, until eventually the new c satisfies $c < p$.

If $p > b > a \geq 0$, then we may assume $\frac{1}{2} \geq b > a \geq 0$. Assume $\{a, b\} \neq \{0, \frac{1}{2}\}$ already. The situation is similar to $\infty \geq b > a \geq 1$ above, and we eventually obtain $\{0, \frac{1}{2}\}$ or a destabilization. A bypass attachment of type A (which must exist if there is no destabilization) will put us into $\{0, 1\}$ which is already done. Therefore, Case II(a) always destabilizes.

Case III(a). Consider Figures 21, 22, 23 and 24, which list all the possible bypasses.

Figure 21 (just Type A) is for I(a) (Note: as discussed in the proof of Proposition 5.6 there are other possibilities for the slopes of the bypasses in I(a) but with the proper choice of basis for the torus we can always arrange to be in the situation shown in Figure 21). When two of the isotopy classes of arcs have more than one parallel copy, then we have Figure 22 and one of B, C, or D. When two isotopy classes have one copy each, then we have Figures 23 and 24 and one of E, F, G, H, I, J, K, L, or M. As before, represent a I(a) configuration by $\{a\}$ and a III(a) configuration by $\{a, b, c\}$. Let $\text{mult}(s)$ be the multiplicity (number of parallel copies) of the isotopy class with slope s .

Assume we have III(a) and the Legendrian curve does not destabilize. Notice that we may assume that two of the isotopy classes will have $\text{mult} = 1$ after repeated attachments of bypasses — this follows from the fact that if

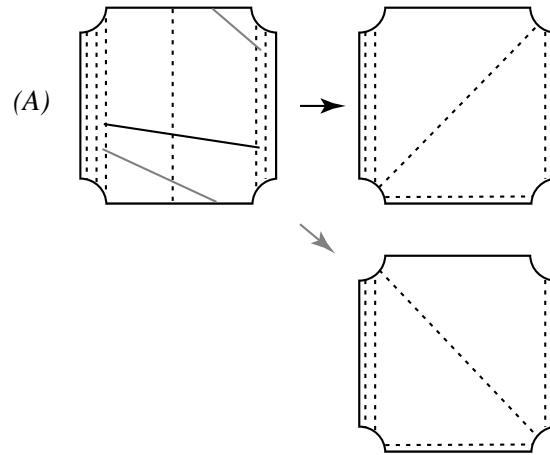


FIGURE 21. Allowable bypasses. Case I(a).

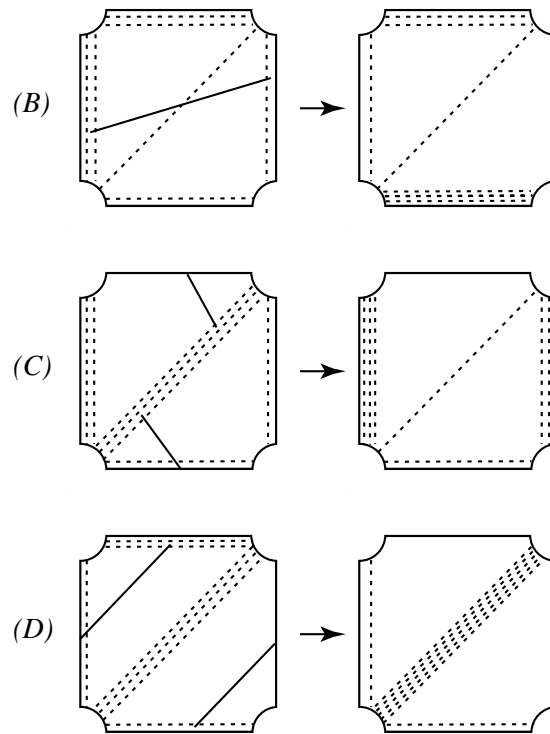


FIGURE 22. Allowable bypasses. Case III(a), and two of the isotopy classes have more than one parallel copy.

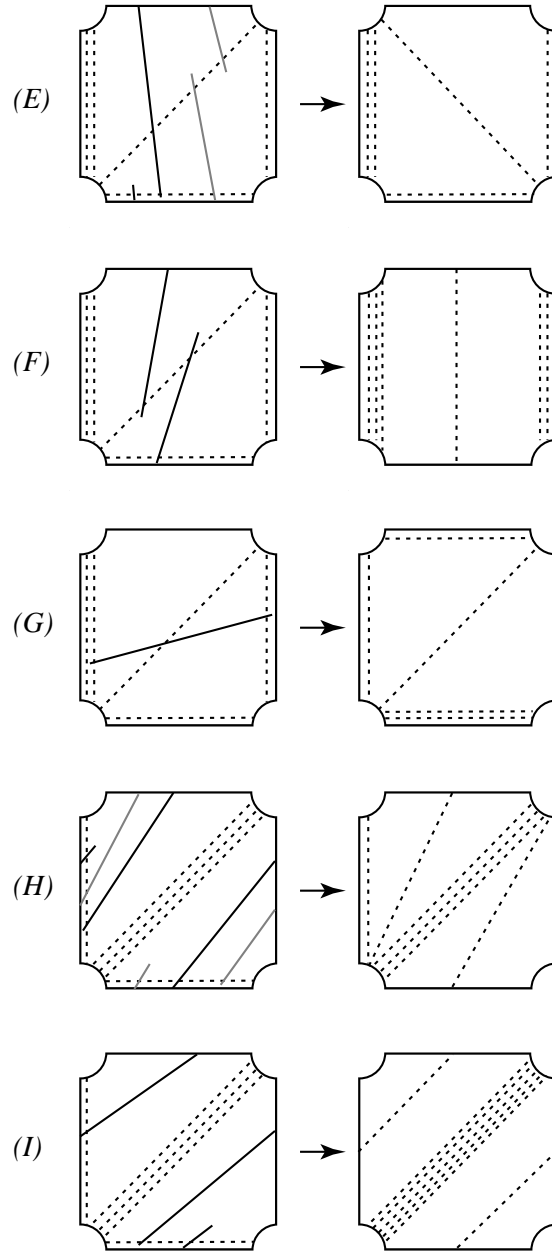


FIGURE 23. Allowable bypasses. Case III(a), and two isotopy classes have one copy each. Bypasses are attached along solid black or grey lines.

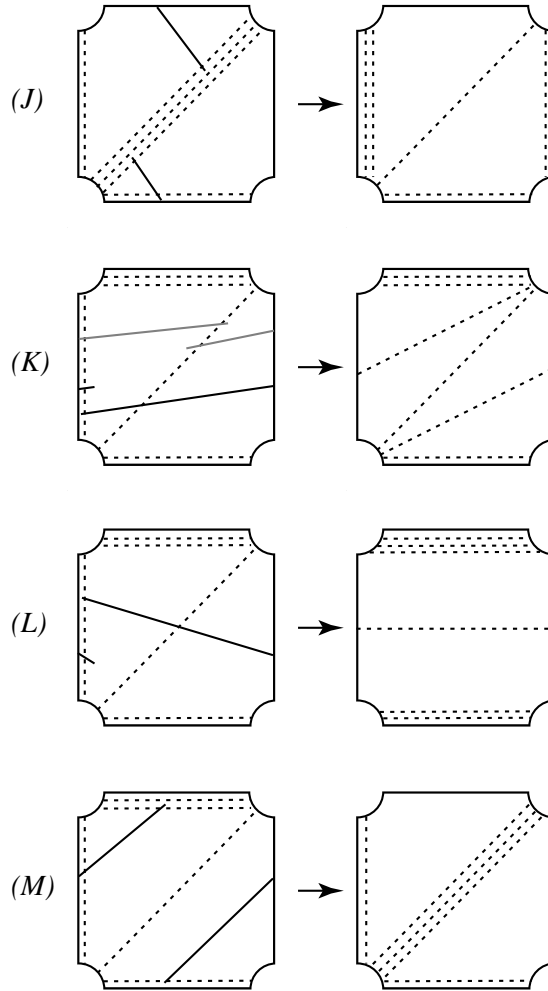


FIGURE 24. Allowable bypasses. Case III(a), and two isotopy classes have one copy each. Bypasses are attached along solid black or grey lines.

$mult > 1$ for 2 isotopy classes, then Figure 22 implies that there are only moves B, C, and D, and these yield two isotopy classes of $mult = 1$ after repeated application (note depending on the situation one can only find bypasses of type B, C or D and not some combination of these). Therefore we assume that two classes have $mult = 1$.

Now assume, in addition, that $\infty \geq b > c > a \geq 1$, and $\{a, b, c\}$ is not already $\{1, 2, \infty\}$. As before, take a closed curve d with slope a , and consider $\Psi(d) \times I$. Since $\Psi(d)$ intersects each curve of Γ_0 at least once, there exists a bypass of slope $\Psi(a)$ with $1 > \Psi(a) > p$. The only possibilities are then E,

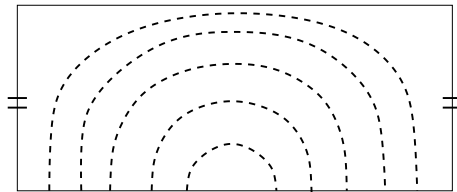


FIGURE 25. Nesting. The sides of the annulus are identified. The top is $d \times \{1\}$ and the bottom is $d \times \{0\}$.

G, J, and L. We also want to distinguish between $\Psi(a) > a'$ and $\Psi(a) < a'$, where a' corresponds to $v_a - v_b$. If $\Psi(a) > a'$, then we have G, J, or L. If $\Psi(a) < a'$, then we have E or J.

Assume $\Psi(a) < a'$. Suppose first $\text{mult}(a) = \text{mult}(b) = 1$. Then we have move J, and repeated application yields $\text{mult}(a) = \text{mult}(c) = 1$. Using move E, we obtain $\{a, b, c\} \mapsto \{a', a, b\}$ ($\infty \geq b > a > a' \geq 1$), $\text{mult}(a') = \text{mult}(a) = 1$ and $\text{mult}(b) > 1$. $\{a', a, b\}$ corresponds to an ideal triangle which lies further inward on D^2 — the kind on the right-hand side of Figure 14.

Assume $\Psi(a) > a'$. Suppose first $\text{mult}(a) = \text{mult}(b) = 1$. Then repeated application of move J yields $\text{mult}(a) = \text{mult}(c) = 1$. Next, repeated application of move G yields $\text{mult}(c) = \text{mult}(b) = 1$. Then we are now left with L which maps $\{a, b, c\} \mapsto \{a\}$. Taking d with slope a and annulus $\Psi(d) \times I$, we get $\{a\} \mapsto \{a'', a, b''\}$, where $a'' \geq 1$ is the smallest slope with an edge to a , $\text{mult}(a'') = \text{mult}(b'') = 1$, and either $a'' < b'' < a$ or $a'' < a < b''$. Thus, eventually the smallest slope a of $\{a, b, c\}$ goes to 1 and we get to $\{1, 2, \infty\}$ with $\text{mult}(1) = \text{mult}(2) = 1$, $\text{mult}(\infty) > 1$, or there exists a destabilization. Once we have $\{1, 2, \infty\}$, use $\Psi(d) \times I$, where d has slope 1, and repeatedly apply J and G until $\text{mult}(1) > 1$, $\text{mult}(2) = \text{mult}(\infty) = 1$, followed by L, which gives us $\{1, 2, \infty\} \mapsto \{1\}$. Since we may map via Ψ^{-1} to get $\{\infty\}$, we may use $\{\infty\}$ or $\{1, 2, \infty\}$ as we wish.

Up to now we made no use of the holonomy on $\partial\Sigma \times I$. We now mimic the proof of Proposition 5.8 and study the genus two handlebody $H = \Sigma \times I$. Σ_0 has Γ_0 consisting of n (odd) arcs and 1 closed curve, all of slope ∞ . Σ_1 has Γ_1 consisting of n arcs and 1 closed curve, all of slope 1. Take $d \times I$, where d has slope 1. The Legendrian realization $d \times \{1\}$ has twist number 0 on Σ_1 , whereas the Legendrian realization $d \times \{0\}$ has twist number $n + 1$ on Σ_0 . Since there is only one allowable bypass attachment onto Σ_0 with slope 1, all the dividing curves on $d \times I$ must be nested ‘in parallel’. See Figure 25. If we attach all the nested bypasses in succession to Σ_0 , we eventually obtain Σ'_0 , which is identical to Σ_1 , except for spiraling. See Figure 26. If we take the meridional disk $D = \delta \times I$, where δ is an arc with slope 1, then $|\partial D \cap \Gamma_{\partial H}| = n - 1$. As long as $n > 3$, there will exist a bypass along ∂D . In fact, since D is a disk, at least one bypass will lie entirely on Σ_0 . Since the bypass does not involve the closed dividing curve on Σ'_0 , it will have to lead

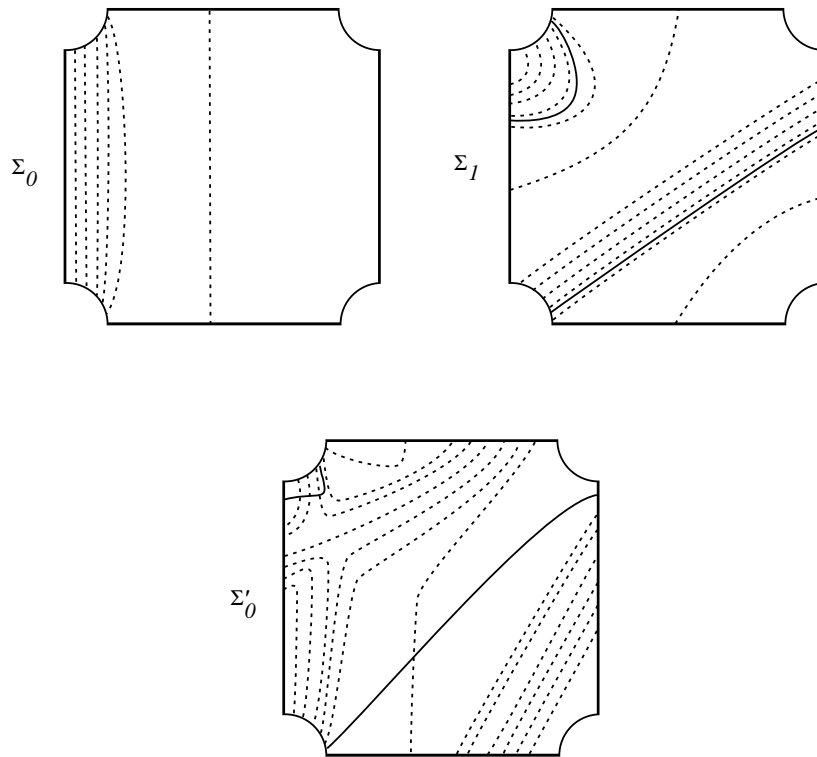


FIGURE 26. Σ_0, Σ'_0 and Σ_1 . The non-dashed curve is δ .

to a boundary-parallel dividing curve on Σ'_0 . Thus we have a destabilization of our knot.

Next assume $\frac{1}{2} \geq b > c > a \geq 0$. As above, there is a destabilization or we get to $\{0, \frac{1}{3}, \frac{1}{2}\}$, with $mult(0) = mult(\frac{1}{3}) = 1$, $mult(\frac{1}{2}) > 1$, or to $\{0\}$. In the former case, a bypass of type E then gives $\{0, \frac{1}{2}, 1\}$. In the latter case, a type A move gives $\{0, \frac{1}{2}, 1\}$.

Assume $p > a \geq 0$, $\infty \geq b > p$. First assume $b > c > p$. Then we let d have slope a , and consider $\Psi(d) \times I$. Here $\Psi(a) < p$. Then we have G, I, K, or M. I gives $\{c\}$ with $c > p$, which is done. If we have M, then the next bypass operation must be I, and we are done. If we have G, then the next step is M, and we are done again. We are left with K. Next assume $p > c > a$. Then let d have slope b , and consider $\Psi(d) \times I$. Here $\Psi(b) > p$. We will have F, H, J, or M. As before, F gives $\{b\}$ with $b > p$, which is done. J must be followed by F, and M must be followed by H or J, so we are left with H. At the end of the day, we are left with $\{\Psi^k(0), \Psi^k(1), \Psi^k(\infty)\}$ (or simply $\{0, 1, \infty\}$), and $mult(0) > 1$, $mult(1) = mult(\infty) = 1$. Now (see

Proposition 5.8), a handlebody H cut open along Σ with this configuration must give rise to a destabilization — use the disk $D = \delta \times I$, where δ has slope ∞ .

Case I(a). Given the configuration $\{a'\}$ with $\infty > a' > 1$, move A will put us into case III(a) with $\infty \geq b > c > a \geq 1$, which is done. Given $\{a'\}$ with $\frac{1}{2} > a' > 0$, move A will put us into case III(a) with $\frac{1}{2} \geq b > c > a \geq 0$, which is also done. \square

We have now proved that for \mathcal{K} , the isotopy class of the figure eight knot, there exists a unique maximal Thurston-Bennequin representative, and any Legendrian figure eight knot destabilizes to this unique representative. Now, using Lemma 2.5, we find that the classification easily extends to non-maximal tb Legendrian knots. This proves Theorem 5.1. \square

5.4. Transversal Figure Eight Knots. Theorem 5.1 implies that the figure eight knot is stably simple and thus we may use Theorem 2.10 to conclude

Corollary 5.10. *If K and K' are two transversal figure eight knots then they are transversally isotopic if and only if $l(K) = l(K')$. Moreover, the self linking numbers of transversal figure eight knots realize precisely the set of odd integers ≤ -3 .*

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REFERENCES

- [A] B. Aebischer, et. al., *Symplectic Geometry*, Progress in Math. **124**, Birkhäuser, Basel, Boston and Berlin, 1994.
- [Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Asterisque **107–108** (1983), 87–161.
- [BW] J. Birman and N. Wrinkle, *On transversally simple knots*, preprint 1999.
- [C] Y. Chekanov, *Differential algebras of Legendrian links*, preprint 1997.
- [E1] Y. Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, Vol. II (Ed. Donaldson and Thomas), Cambridge, 1990.
- [E2] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier **42** (1992), 165–192.
- [E3] Y. Eliashberg, *Legendrian and transversal knots in tight contact 3-manifolds*, Topological Methods in Modern Mathematics (1993), 171–193.
- [EF] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, in *Geometry, topology, and dynamics* (Montreal, PQ, 1995), 17–51, CRM Proc. Lecture Notes **15**, Amer. Math. Soc., Providence, RI, 1998.
- [Et] J. Etnyre, *Transversal torus knots*, Geometry and Topology **3** (1999), 253–268.
- [EH] J. Etnyre and K. Honda, *On the non-existence of tight contact structures*, to appear Annals of Math.

- [Ep] J. Epstein, *On the invariants and isotopies of Legendrian and transversal knots*, Dissertation, University of California, Davis (1997).
- [FT] D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997), 1025–1053.
- [Gi1] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helvetici **66** (1991), 637–677.
- [Gi2] E. Giroux, *Topologie de contact en dimension 3 (autour des travaux de Yakov Eliashberg)*, Séminaire Bourbaki, Vol. 1992/93, Astérisque **216** (1993), Exp. 760, 3, 7–33.
- [Gi3] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000), 615–689.
- [Go] R. Gompf, *Handlebody construction of Stein surfaces*, Annals of Math. **148** (1998), 619–693.
- [H1] K. Honda, *On the classification of tight contact structures I: lens spaces, solid tori, and $T^2 \times I$* , Geom. Topol. **4** (2000), 309–368.
- [H2] K. Honda, *On the classification of tight contact structures II*, to appear Jour. of Diff. Geom.
- [K1] Y. Kanda, *The classification of tight contact structures on the 3-torus*, Comm. in Anal. and Geom. **5** (1997), 413–438.
- [K2] Y. Kanda, *On the Thurston-Bennequin invariant of Legendrian knots and non exactness of Bennequin’s inequality*, Invent. Math. **133** (1998), 227–242.
- [KM] P. Kronheimer and T. Mrowka, *Monopoles and contact structures*, Invent. Math. **130** (1997), 209–255.
- [LM1] P. Lisca and G. Matić, *Tight contact structures and Seiberg-Witten invariants*, Invent. Math. **129** (1997), 509–525.
- [LM2] P. Lisca and G. Matić, *Stein 4-manifolds with boundary and contact structures*, Symplectic, contact and low-dimensional topology (Athens, GA, 1996), Topology Appl. **88** (1998), 55–66.
- [ML] S. Makar-Limanov, *Tight contact structures on solid tori*, Trans. Amer. Math. Soc. **350** (1998), 1013–1044.
- [Men] W. Menasco, *On iterated torus knots and transversal knots*, preprint 2000.
- [M] L. Moser, *Elementary surgery along a torus knot*, Pacific J. of Math. **38** (1971), 737–745.
- [Ro] D. Rolfsen, *Knots and Links*, Publish or Perish Press, Houston TX, 1990.
- [R] L. Rudolph, *An obstruction to sliceness via contact geometry and “classical” gauge theory*, Invent. Math. **119** (1995), 155–163.
- [Ta] S. Tabachnikov, *Estimates for the Bennequin number of Legendrian links from state models for knot polynomials*, Math. Res. Lett. **4** (1997), 143–156.

STANFORD UNIVERSITY, STANFORD, CA 94305
E-mail address: etnyre@math.stanford.edu
URL: <http://math.stanford.edu/~etnyre>

UNIVERSITY OF GEORGIA, ATHENS, GA 30602
E-mail address: honda@math.uga.edu
URL: <http://www.math.uga.edu/~honda>