CONSTRUCTIONS AND INVARIANTS OF HIGH-DIMENSIONAL LEGENDRIAN SUBMANIFOLDS

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CONSTRUCTIONS AND INVARIANTS OF HIGH-DIMENSIONAL LEGENDRIAN SUBMANIFOLDS

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Through the years, I have learned there is no harm in charging oneself up with delusions between moments of valid inspiration.

Steve Martin

For Sunetra and Asitava Roy, my parents

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SUMMARY

This thesis explores the question of understanding Legendrian submanifolds in contact manifolds of dimension greater than 3. There are two primary contributions. First, we explore two natural constructions of Legendrian spheres from supporting open book decompositions and show that these always yield the standard Legendrian unknot. Second, in joint work with Hughes, we explore the Legendrians obtained from the doubling construction in dimension 5, and represent them as Legendrian weaves. We show that a large family of pairwise non-isotopic Legendrians can be obtained by looking at doubles associated to torus links $\lambda(2, n)$. Further we also mention theorems in progress regarding the fillability of these doubled Legendrians.

CHAPTER 1
MAIN RESULTS

A contact manifold (M, ξ) is a smooth manifold M equipped with a nowhere integrable hyperplane field ξ. The construction and investigation of contact manifolds has historically been aided by studying distinguished submanifolds that interact suitably with the contact structure. In dimension 3, these submanifolds are either convex hypersurfaces, or Legendrian knots, which are 1-dimensional submanifolds. These in conjunction have helped achieve the classification of tight contact structures on several classes of 3-manifolds e.g. [1, 2, 3, 4]. A general theme in most of these results can be seen as follows: first understand Legendrian isotopy classes of a family of Legendrian knots in (S^3, ξ_{st}), then understand the contact structures that appear by performing contact surgery on these Legendrians. In higher dimensions, the first hindrance to carrying out this plan is the shortage of examples of Legendrian spheres. Some constructions of high dimensional Legendrian spheres are explored in [5, 6, 7, 8], in $\mathbb{R}^{2n+1}$ with the standard contact structure.

This thesis explores the construction of high-dimensional Legendrian submanifolds, and then computing their invariants and understanding their fillability properties. The main idea behind the constructions is to use a Legendrian in a $(2n-1)$-manifold, and use one or two exact Lagrangian fillings, to create a higher dimensional Legendrian in a $(2n+1)$-manifold. This idea was explored first to construct Legendrians in $(\mathbb{R}^{2n+1}, \xi_{st})$ by Ekholm [6] and called a ”doubling construction”. In Chapter 4, we generalise the doubling construction to arbitrary closed manifolds. In Chapter 5, which is based on joint work with Hughes, we construct a large class of Legendrian surfaces by doubling, compute their sheaf moduli invariants, and explore their exact fillability.

The material in Chapter 4 has appeared in print in [9], while the material in Chapter 5 will be part of an article [10], which is in preparation.

1.1 Constructing high-dimensional Legendrians from open books

We describe some general constructions of Legendrian spheres in $(2n+1)$-dimensional contact manifolds from Lagrangian disks in pages of supporting open books.

Construction 1.1.1. A natural way to construct Legendrian submanifolds in a contact manifold is via open book decompositions. Suppose (M, ξ) is supported by the open book (B, ν), where $\nu : (M - B) \to S^1$ is a fibration, and each page is symplectomorphic to (W, ω). Consider a properly embedded Lagrangian n-disk L on the page. Then consider two pages W, W' and two copies of the same Lagrangian, called L, L' on them. The disks can be individually perturbed to give Legendrian disks in M. Then, by Lemma 4.1.1, the open book can be perturbed so that Legendrian disks actually lie on the page of the open book as Lagrangian disks. We can further perturb L and L' so they can be smoothly joined to give the union, a Legendrian sphere $L \cup L'$. This now gives a closed Legendrian in (M, ξ). We will call this construction $S_{join}(L)$. Since L' is an isotopic copy of L, the notation suppresses L'. This will be described in more detail in Section 4.2.

Construction 1.1.2. Consider (M, ξ) and L similarly as above. Then, the open book can be stabilised, by modifying the page by attaching a Weinstein n-handle along ∂L, and then performing a positive Dehn twist along the resulting Lagrangian n-sphere, obtained by taking the union of L and the core of the handle. This resulting manifold is contactomorphic to (M, ξ). Also, the Lagrangian sphere in the new page can be perturbed to a

Legendrian sphere. We will call this Legendrian $S_{stab}(L)$.

Our main result is that these constructions give Legendrian isotopic spheres and are isotopic to the standard Legendrian unknot. The standard Legendrian unknot is defined to be the Legendrian realisation of the n-dimensional unknotted sphere in a Darboux neighbourhood, which is contactomorphic to a Darboux neighbourhood in (S^{2n+1}, ξ_{st}), which is the boundary of an exact Lagrangian disk in (B^{2n+2}, ω_{st}). Its front projection can be inductively constructed by starting from the unknot with maximal Thurston-Bennequin number in $(\mathbb{R}^3, \xi_{st})$ and successively spinning half of the front projection.

Theorem 1.1.3. *Consider a supporting open book decomposition of a contact manifold (M, ξ). Consider a Lagrangian disk L in the page. Then $S_{join}(L)$ is Legendrian isotopic to $S_{stab}(L)$, and they are both isotopic to the standard Legendrian unknot.*

A technical issue in the above proof is the following: S_{join} is defined with respect to a certain open book decomposition of (M, ξ), whereas S_{stab} is defined with respect to the stabilisation of the open book decomposition. While the open books support contactomorphic contact manifolds, to prove Legendrian isotopy, one needs to see stabilisation of the open book as an embedded operation. We will show the following in Section 4.1:

Theorem 1.1.4. *Assume (M^{2n+1}, ξ) admits the supporting open book (B, ν), whose stabilisation along the Legendrian disk L, which is a Lagrangian in the page, is (B', ν'). Then ν and ν' are obtained from each other by surgering out a $(2n+1)$-disk neighbourhood of L, and replacing it by another $(2n+1)$-disk.*

Construction 1.1.5. This construction of Legendrian spheres in $(\mathbb{R}^{2n+1}, \xi_{st})$ was introduced by Ekholm in [6]. Start with a Lagrangian disk L which is cylindrical near its boundary, in the symplectisation of $\mathbb{R}^{2n-1}$. Embed it in a hypersurface in $\mathbb{R}^{2n+1}$ transverse to the Reeb flow. Then join the Legendrian lifts of this disk and a reflection of the disk in the same hypersurface to obtain a Legendrian sphere $\Lambda(L, L)$. An example of this construction is given in Figure 1.1.

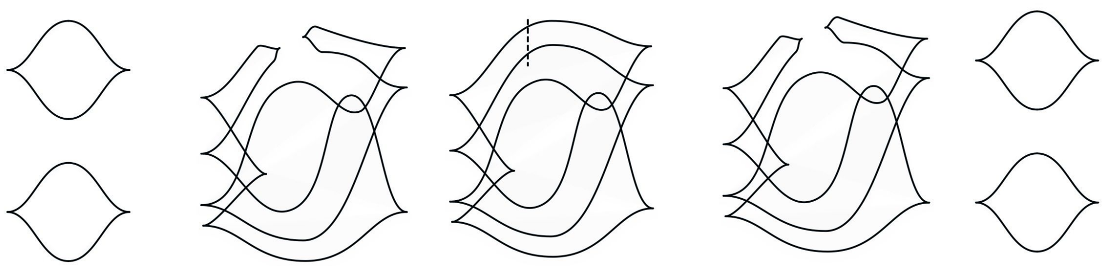


Figure 1.1: Front projection of a Legendrian sphere in $(\mathbb{R}^5, \xi_{st})$, where L is the Lagrangian disk described by a pinch move from the middle knot and going to two Legendrian unknots, which are then capped off.

In [11], Courte-Ekholm show that $\Lambda(L, L)$ is isotopic to the standard Legendrian unknot.

The contact manifold (S^{2n+1}, ξ_{st}) is supported by the open book where the pages are symplectomorphic to (D^{2n}, ω_{st}), and the monodromy is the identity. The binding is contactomorphic to (S^{2n-1}, ξ_{st}). Given a Lagrangian disk L in (B^{2n}, ω_{st}), one can construct all the three Legendrians as mentioned above. We can show that Courte-Ekholm's result is a particular case of Theorem 1.1.3.

Corollary 1.1.6. *(Originally proven in [11]) Given a Lagrangian disk L in (B^{2n}, ω_{st}), $\Lambda(L, L)$ is isotopic to the standard Legendrian unknot.*

The organisation is as follows: In Section 4.1, we prove Theorem 1.1.4. Then in Section 4.2, we prove the first half of Theorem 1.1.3, namely that the join and stabilisation constructions give isotopic spheres. Finally in Section 4.3, we prove that the constructions give the standard Legendrian unknot, completing the proof of Theorem 1.1.3 and also show how to recover Courte-Ekholm's result.

1.2 Understanding Legendrian surfaces via weaves

The work here is joint with Hughes. Let $\lambda = \lambda(\beta\Delta)$ be an $n-$component Legendrian link in $(\mathbb{R}^3, \xi_{st})$ given as the (-1)-closure of the positive braid β. Let L_1, L_2 be embedded exact Lagrangian fillings of λ of genus g given as the Lagrangian projection of a Legendrian weave with boundary λ.

Our first observation allows us to identify a weave description for the double $\Lambda(L_1, L_2)$. The *doubled N-graph* is defined by the graph obtained by gluing G_1 and G_2 along their boundary.

Proposition 1.2.1. *Let G_1, G_2, be N-graphs describing two exact Lagrangian fillings L_1, L_2 of a link K. Then the Legendrian $\Lambda(L_1, L_2)$ is the Legendrian weave corresponding to the doubled N-graph $(G_1 \cup G_2) \subset S^2$.*

The main technical tool for understanding isotopy classes of asymmetric doubles is the computation of the sheaf moduli of $\Lambda(L_1, L_2)$ in terms of the sheaf moduli of the fillings L_1 and L_2. Suppose the flag moduli invariant of Λ is $\mathcal{M}_1(\Lambda)$. Let $C_{L_1} \subseteq \mathcal{M}_1(\Lambda)$ and $C_{L_2} \subseteq \mathcal{M}_1(\Lambda)$ be the toric charts induced by L_1 and L_2 respectively.

Theorem 1.2.2 (with J. Hughes)**.** *The flag moduli $\mathcal{M}_1(\Lambda(L_1, L_2))$ is given by the intersection $\mathcal{C}_{L_1} \cap \mathcal{C}_{L_2}$.*

Theorem 1.2.3 (with J. Hughes)**.** *The symmetric double $\Lambda(L, L)$ is Legendrian isotopic to $\#^k \mathbb{T}^2_{std}$ where $k = 2g + n - 1$.*

Let L_1, L_2 be two embedded exact Lagrangian fillings as above. As a direct consequence of 1.2.2, we can conclude:

Corollary 1.2.4 (with J. Hughes). *The Legendrian double $\Lambda(L_1, L_2)$ is a non-loose Legendrian.*

The Legendrian torus link $\lambda(2, n)$, has at least C_n embedded exact Lagrangian fillings up to Hamiltonian isotopy, which can be represented by Lagrangian projections of Legendrian weaves [12, 13, 14]. These weaves are trivalent graphs obtained as the duals to certain triangulations of an $(n + 2)$-gon. Denote by L_{init} the filling of $\lambda(2, n)$ coming from the pinching sequence $(1, 2, \ldots, n)$. Denote the standard torus in $\mathbb{R}^5$ by T^2_{std} and the Clifford torus by T^2_c. These Legendrians are defined in Chapter 3.

Theorem 1.2.5 (with J. Hughes). *Let L_1 and L_2 be two exact Lagrangian fillings of $\lambda(2, n)$ obtained as above. Then,*

1. *The double $\Lambda(L_{init}, L_1)$ is Hamiltonian isotopic to $\#^k\mathbb{T}^2_{std}\#^l\mathbb{T}^2_c$ for some k and l such that $k + l = n - 1$*

2. *Given L_1, and j such that $0 \leq j \leq n-1$, there exists L_2 such that $\Lambda(L_1, L_2)$ is Hamiltonian isotopic to $\#^j\mathbb{T}^2_{std}\#^{n-1-j}\mathbb{T}^2_c$*

3. *The double $\Lambda(L_1, L_2)$ is Hamiltonian isotopic to $\#^{n-1}\mathbb{T}^2_{std}$ if and only if L_1 and L_2 are Hamiltonian isotopic*

From understanding the toric charts coming from fillings and how they behave under a procedure called *folding*, and the associated cluster structure, we expect to be able to prove the following results in [10]. We will talk about the relevant ideas in Section 5.3.

Theorem 1.2.6 (with J. Hughes). *[in progress] For decomposable exact Lagrangian fillings L, L' coming from a Legendrian weave with L and L' inducing distinct toric charts in $\mathcal{M}_1(\partial L)$, the Legendrian double $\Lambda(L, L')$ is not exact Lagrangian fillable.*

Let ϕ be a Legendrian loop of λ and consider the mapping torus $\Sigma_\phi(\lambda)$. See below for a precise construction. Let ρ denote the Kálmán loop on Legendrian $(2,n)$ torus links $\lambda(2,n) = \lambda(\sigma_1^{n+2})$.

Theorem 1.2.7 (in progress, with J. Hughes)**.** *There are at least $f(k)$ exact Lagrangian fillings of $\Sigma_{\rho^k}(\lambda(2,n))$ where*

$$f(k) = \begin{cases} C_{\frac{n-1}{3}} & k = \frac{2n}{3} \in \mathbb{N} \\ C_{\frac{n}{2}} & k = \frac{n+2}{2} \in \mathbb{N} \\ C_n & k = n+2 \\ 0 & \textit{otherwise} \end{cases}$$

where C_n is the nth Catalan number.

One can also obtain infinitely many exact Lagrangian fillings of various twist spuns from torus links:

Theorem 1.2.8. *[in progress, with J. Hughes] There are faithful $PSL(2,\mathbb{Z})$ and $Mod(\Sigma_{0,4})$ actions on the set of exact Lagrangian fillings of $\Sigma_{\rho^3}(\lambda(3,6))$ and $\Sigma_{\rho^4}(\lambda(4,8))$, respectively.*

Finally, we expect the following constraint for any asymmetric double coming from two embedded fillings which induce distinct charts on the sheaf moduli of λ.

Theorem 1.2.9. *The Legendrian double $\Lambda(L_1, L_2)$ is a non-loose Legendrian, when L_1 and L_2 induce distinct toric charts on the sheaf moduli of the boundary Legendrian.*

Conjecturally, it is expected that exact Lagrangian fillings distinct upto Hamiltonian isotopy will induce distinct toric charts.

CHAPTER 2

BACKGROUND - CONTACT AND SYMPLECTIC TOPOLOGY

2.1 Weinstein handlebodies

We briefly review the notion of Weinstein handle attachments here, which we will need to define stabilisation of open books. For more detailed exposition the reader is encouraged to consult [15].

A Weinstein domain is the symplectic analogue of a smooth handlebody. For a $2n$-dimensional domain, Weinstein k-handles can have index at most n, and are attached along isotropic $(k-1)$-spheres in the convex boundary. Recall that a submanifold S of a contact manifold is called *isotropic* if $T_xS \subset \xi_x$ for all $x \in S$.

Definition 2.1.1. A Weinstein handle of index k is $h^k = D^k \times D^{2n-k}$ with a symplectic structure so that $\partial_- h^k = (\partial D^k) \times D^{2n-k}$ is concave, and $\partial_+ h^k = D^k \times (\partial D^{2n-k})$ is convex. Moreover, $D^k \times \{0\}$ is isotropic and its intersection with $\partial_- h^k$ is an isotropic S^{k-1} in the contact structure induced on $\partial_- h^k$. Thus, the attaching sphere of a Weinstein k–handle is an isotropic S^{k-1}. Given an isotropic sphere S^{k-1} in the convex boundary of a symplectic manifold with a choice of trivialization of its conformal symplectic normal bundle, one can attach a Weinstein k–handle by identifying a neighborhood of the isotropic sphere with $\partial_- h^k$.

A Weinstein handle of index n is called a *critical* Weinstein handle, and is attached along a Legendrian sphere. It will be useful for us to understand the local model for attaching a critical Weinstein handle. Consider $\mathbb{R}^{2n}$ with the symplectic structure $\sum_{i=1}^n dx_i \wedge dy_i$. Now consider $H_{a,b} \coloneqq D_a \times D_b$, where D_a is the disk of radius a in the x_i subspace and

D_b the disk of radius b in the y_i subspace. Then, $H_{a,b}$ is a model for the Weinstein n-handle h^n. The expanding vector field $v = \sum_{i=1}^n -y_i dy_i + 2x_i dx_i$ induces contact structures on $\partial_- h^n = (\partial D_a) \times D_b$ and $\partial_+ h^n = D_a \times (\partial D_b)$.

2.2 Legendrian and Lagrangian submanifolds

Definition 2.2.1. Given a contact manifold (M^{2n+1}, ξ), an n-dimensional submanifold L is called a Legendrian if $T_x(L) \subset \xi_x$ for every $x \in L$.

It is well-known that a Legendrian sphere in a contact manifold always has a standard neighbourhood.

Lemma 2.2.2. *If S is a Legendrian n-sphere in (M^{2n+1}, ξ), then in any open set containing S there is a neighborhood N with boundary $\partial N = S^n \times S^n$ contactomorphic to an ϵ-neighbourhood N_ϵ of the zero section Z in the 1-jet space of S^n, denoted $J^1(S^n)$. We call N a standard neighbourhood.*

2.3 Legendrian surgery

The model we will use to describe Legendrian surgery is understood as what is happening on the boundary when a Weinstein handle is attached along the Legendrian sphere. This is called the *flat Weinstein model* and is described in Section 3 of [16] in more generality, for isotropic surgery along S^k for $k \leq n$. Our description follows the exposition there.

Notation: To make the notation less cluttered when we talk about $\mathbb{R}^{2n+2}$, we will write the coordinates $(z_1, w_1, \cdots, z_{n+1}, w_{n+1})$ as (z, w). The symplectic form $\omega_0 = \sum_{i=1}^{n+1} dz_i \wedge dw_i$ will be referred to as $dz \wedge dw$. Similar liberties will be taken with 1-jet space coordinates where $(z, p_1, q_1, \cdots, p_n, q_n)$ will be truncated to (z, p, q), and the contact structure there is $\ker(dz + pdq)$. Products between vectors should be thought of as dot products.

Consider the symplectic manifold $(\mathbb{R}^{2n+2}, \omega_0)$, where the coordinates are $(n+1)$ pairs of (z, w) coordinates, and $\omega_0 = dz \wedge dw$. The vector field $X = 2z\partial_z - w\partial_w$ is Liouville. The set $S_{-1} := \{(z, w) \mid |w|^2 = 1\}$ is transverse to X and inherits the contact form $\alpha = 2zdw + wdz$.

In S_{-1}, the sphere $\{z = 0, |w|^2 = 1\}$ describes a Legendrian sphere. Using $\psi_W : J^1(S^n) \to S_{-1}$ given by $(z, q, p) \mapsto (zq + p, q)$, we get a strict contactomorphism between S_{-1} and the standard neighbourhood described in Lemma 2.2.2. Thus S_{-1} can be regarded as the standard neighbourhood of a Legendrian sphere.

Now, Legendrian surgery along a Legendrian S will involve removing a neighbourhood of S identified with S_{-1} and gluing in another contact hypersurface of $(\mathbb{R}^{2n+2}, \omega_0)$. The contact hypersurface involved in that is called S_1 and we describe it here. Define functions f and g, described in Figure 2.1, to satisfy the following:

- f is increasing on $[1-\epsilon, \infty)$
- $f(w) = 1$ for $w \in [0, 1-\epsilon)$, $f(w) = w + \epsilon$ for $w > 1 - \frac{\epsilon}{2}$
- g is increasing on $(0, 1+\epsilon)$
- $g(z) = z$ for $z < 1$, $g(w) = 1 + \epsilon$ for $w > 1 + \epsilon$

Then, define the hypersurface $S_1 := \{(z, w) \mid f(w^2) - g(z^2) = 0\}$. As X is transverse to S_1, it inherits a contact structure. Then, Legendrian surgery along S is removing $\nu(S) \cong S_{-1}$ and gluing S_1 in its place. If $S \subset (M, \xi)$, and there is a symplectic manifold W obtained by attaching a Weinstein handle to part of the symplectisation

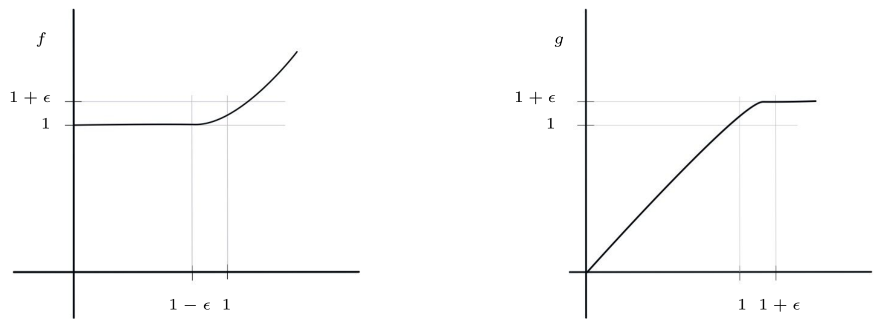


Figure 2.1: The functions f and g used to describe Legendrian surgery.

$(M\times[0,1], d(e^t\alpha))$, along S in $M\times\{1\}$, the Legendrian surgery along S can be understood as the upper boundary of W.

2.4 Open Book Decompositions

The background on contact open books is taken from the lecture notes by Van Koert [16]. The reader is referred to the same for more details.

Definition 2.4.1. An (abstract) contact open book (Σ, λ, ϕ), or $Open(\Sigma, \phi)$ if we suppress the Liouville form from the notation, consists of a compact exact symplectic manifold

(Σ, λ) and a symplectomorphism $\phi : \Sigma \to \Sigma$ with compact support, i.e., it is identity near $\partial\Sigma$.

Definition 2.4.2. An (embedded) supporting open book for a contact manifold (M, ξ) is a pair (ν, B), where B is a codimension-2 submanifold of M with trivial normal bundle, such that

- $\nu : (M - B) \to S^1$ is a fiber bundle, such that ν gives the angular coordinate of the D^2-factor of a neighbourhood $B \times D^2$ of B, and
- if α is a contact form for ξ, it induces a positive contact structure on B and $d\alpha$ induces a positive symplectic structure on each fiber of ν

The embedded open book constructed from Definition 2.4.1 is the manifold $\Sigma \times [0, 1]/\sim$, where the equivalence relation $\sim$ identifies all points (x, t) and (x, t') where $x \in \partial\Sigma$, and identifies points $(x, 0)$ with $(\phi(x), 1)$. For our purpose, we will employ another (but equivalent, up to contact isotopy) way of building a manifold from an abstract open book, where we will have something called the *thickened binding*. This construction will work as follows:

Definition 2.4.3. A manifold constructed from the abstract open book $Open(\Sigma, \phi)$ with *thickened binding* is the quotient of the disjoint union of the mapping torus $\Sigma \times [0, 1]/((x, 0) \sim (\phi(x), 1))$ and the thickened binding $\partial\Sigma \times D^2$, under the identification $(x, t) \sim (x, 1, t)$, where $x \in \partial\Sigma$, and $\{(x, r, \theta) \mid r \in [0, 1], t \in \mathbb{R}/\mathbb{Z}\}$ are the coordinates on $\partial\Sigma \times D^2$.

An open book with thickened binding can be given a compatible contact structure, as shown in Section 2.2 of [16]. Every contact manifold has a supporting open book decomposition, by work of Giroux. Further, the contact structure supported by an open book is unique upto isotopy, as said by the next theorem, due to Giroux.

Theorem 2.4.4 (Giroux). *If an open book (Σ, λ, ϕ) supports a contact structure (M, ξ_1), and ξ_2 is another contact structure on M supported by an open book whose pages are symplectomorphic to Σ and the monodromy is isotopic through symplectomorphisms to ϕ, then ξ_1 and ξ_2 are contactomorphic.*

An abstract open book defines a supporting open book for the corresponding contact manifold. This follows from work of Thurston-Winkelnkemper [17] and Giroux. The reader can refer to [16] for a proof (originally by Giroux), and more details. For open books, by a *page* we refer to Σ for abstract open books, and to the closure of a fiber of ν for embedded ones. In the manifold built from the abstract open book, the equivalence class $[(x,t)]$ for $x \in \partial\Sigma$ is the binding. In the embedded case, B is the binding. In the thickened binding case, $\partial\Sigma \times D^2$ is the binding. In the embedded case, as $M - B$ has the structure of a fibration over S^1, it makes sense to talk about the *monodromy* of an open book. In the abstract setting, ϕ is called the monodromy.

2.5 Relative Open Books

We will need the idea of open book decompositions of manifolds with boundary, and how to glue such decompositions together. The difference from Definitions 2.4.1 and 2.4.2 in this case is that the page topology can change. This definition is closely linked to the notions of *partial open books* [18] and *foliated open books* [19] in 3-dimensional contact geometry.

Definition 2.5.1. An (embedded) supporting relative open book for a contact manifold $(M, \partial M, \xi)$ is a pair (ν, B), where B is a codimension-2 submanifold of M with trivial normal bundle, such that

- $\nu : (M - B) \to S^1$ is a circle-valued Morse function. Further, ν gives the angular coordinate of the D^2-factor of a neighbourhood $B \times D^2$ of B, and

- if α is a contact form for ξ, it induces a positive contact structure on B and $d\alpha$ induces a positive symplectic structure on each fiber of ν

In our case, the change in page topology will only be effected by adding or removing a Weinstein n-handle. To help the reader's intuition, we will give the notion of an abstract relative open book, along the lines of abstract foliated open books [19] in 3 dimensions.

Definition 2.5.2. An (abstract) supporting relative open book for a contact manifold is a tuple $(\{S_i\}_{i=0}^{2k}, h)$ where:

- S_i's are Liouville domains for odd (or even) i such that S_{i+1} is obtained from S_i by either attaching a standard $2n$-ball or removing one, as a neighbourhood of a properly embedded Lagrangian n-disk, which happens alternately.

- The boundary of S_i is a smoothing of $B \cup \alpha_i$, where before smoothing, B is a convex boundary component with an outward pointing Liouville vector field, α_i is diffeomorphic to $(D^n \times S^{n-1})$, and for alternate i it is a convex or concave boundary component. The $2n$-ball can be regarded as $D^n \times D^n$ with the smoothed boundary having two convex parts, the attaching part ∂_- which is $D^n \times S^{n-1}$, and the upper part $S^{n-1} \times D^n$. At every stage, to go from S_i to S_{i+1} the ball is either attached along a concave α_i, or the convex α_i is the upper part of the boundary of the ball that gets removed.

- h is a symplectomorphism between S_{2k} and S_0.

In this case, we construct the manifold similarly as before by gluing together $S_i \times I$ pieces, then using h to create a "mapping torus", and then filling in the binding with

$B \times D^2$. There is a natural correspondence between the embedded and abstract descriptions. The schematic of the pages changing is shown in Figure 2.2. We should point that we expect the full generalisation of partial and foliated open books in high dimensions should involve more ways of changing the page topology, and also the boundary can possibly be enriched to be a convex hypersurface in the sense of Honda-Huang [20]. However, for the purpose of this paper, and understanding stabilisation, the simple notion described here suffices.

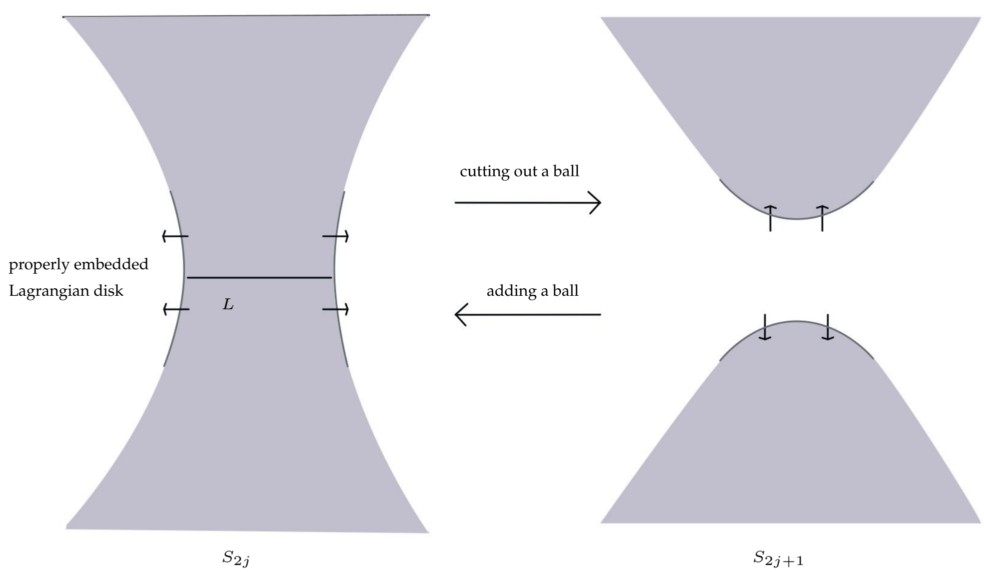


Figure 2.2: A schematic for how the page topology changes in a relative open book. The darkened part of the boundary is the region α.

Example 2.5.3. We may obtain a relative open book decomposition of a contact ball B^{2n+1}, obtained as the neighbourhood of a properly embedded Lagrangian disk in the page of an open book, as follows. A schematic of this decomposition is Figure 4.2. We derive this in detail in Section 4.1.

Here k = 4. The manifolds S_0, S_2, and S_4 are the disk cotangent bundles of the n-disk D^n, while the manifolds S_1 and S_3 are the disk cotangent bundles of the n-dimensional annulus $S^{n-1} \times I$. The map h is the identity. The Lagrangian disk in the page shows up in this neighbourhood as the core n-disk of S_0, S_2, and S_4.

Our main purpose to define these relative open books is to write down how to glue two such decompositions of manifolds together to obtain a closed manifold with an open book decomposition.

Lemma 2.5.4. *Consider two relative open book decompositions on $(M_1, \partial M_1)$ denoted $(\{S_{i,1}\}_{i=0}^{2k}, h_1)$, and on $(M_2, \partial M_2)$ denoted $(\{S_{i,2}\}_{i=0}^{2k}, h_2)$. Suppose the page boundaries are denoted $B_1 \cup \alpha_{i,1}$ and $B_2 \cup \alpha_{i,2}$. These induce relative open book decompositions on the thickened boundary of the manifolds $\partial M_1 \times I$ and $\partial M_2 \times I$. Suppose there is a contactomorphism $\phi : \partial M_1 \times I \to \partial M_2 \times I$, which restricts to each page boundary $\phi|_{(\alpha_{i,1} \times I)}$ as a local symplectomorphism that identifies the concave boundary at $\alpha_{i,1}$ with the convex boundary at $\alpha_{i,2}$. This extends the symplectic structures on $S_{i,1}$ and $S_{i,2}$ to a Weinstein structure on $S_{i,1} \cup S_{i,2}$. Then, $M_1 \cup M_2$ is a contact manifold with supporting open book decomposition where the pages are symplectomorphic to $S_{i,1} \cup S_{i,2}$, and the monodromy is $h_1 \cup h_2$.*

Proof. The fact that $M_1 \cup_\phi M_2$ is a contact manifold follows because the contact structures on each of them are identified by ϕ over the gluing region. The relative open book fibrations, by the given conditions, glue to give a fibration over the glued fibers. □

2.6 Generalised Dehn Twist

Suppose (W, ω) is a symplectic manifold with an embedded Lagrangian sphere $L \subset W$. A neighbourhood $\nu_W(L)$ is symplectomorphic to a neighbourhood of the zero section of the canonical symplectic structure on $(T^*S^n, d\lambda_{can})$, by the Weinstein neighbourhood theorem. The cotangent bundle of the n-sphere T^*S^n can be regarded as a submanifold of $\mathbb{R}^{2n+2}$ as the set $\{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid q \cdot q = 1, q \cdot p = 0\}$. In these coordinates, $\lambda_{can} = pdq$. Define an auxiliary map describing the normalised geodesic flow

$$\sigma_t(q,p) = \begin{pmatrix} \cos t & |p|^{-1} \sin t \\ -|p| \sin t & \cos t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Then define

$$\tau(q,p) = \begin{cases} \sigma_{g_1(|p|)} & p \neq 0 \\ -Id & p = 0 \end{cases}$$

g_1 is a smooth map as graphed in Figure 2.3. Since τ is identity outside a neighbourhood of the Lagrangian $\{p = 0\}$, it can be extended to all of (W, ω) by the identity and defines a symplectomorphism. The map τ is called the generalised Dehn twist about the Lagrangian sphere L.

2.7 Stabilisation of open books

Given a contact open book $M =$ Open (Σ^{2n}, ϕ), suppose L is an embedded Lagrangian n-disk in the page Σ whose boundary ∂L is a Legendrian sphere in the binding. Consider $\widetilde{\Sigma}$ to be the manifold obtained by attaching a Weinstein n-handle to Σ along ∂L. Then, call L_S the Lagrangian sphere in $\widetilde{\Sigma}$ defined by the union of L and the core of the n-handle.

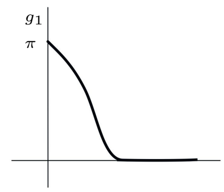


Figure 2.3: The function g_1 parametrising a Dehn twist.

Definition 2.7.1. The contact open book $\tilde{M} := \text{Open}\ (\tilde{\Sigma}, \phi \circ \tau_{L_S})$, where τ_{L_S} is the Dehn twist along L_S, is called the *stabilisation* of $\text{Open}(\Sigma, \phi)$ along L.

The following is a well-known statement due to Giroux. A proof can be found in [16].

Proposition 2.7.2. *The stabilisation of a contact open book Open(Σ, ϕ) along a Lagrangian disk L bounding a Legendrian sphere in $\partial\Sigma$ is contactomorphic to the contact manifold Open(Σ, ϕ).*

In Section 4.2, we use the following folklore theorem (refer [16] for details), that doing Legendrian surgery on a Legendrian sphere that lives on a page is the same as changing the monodromy by a Dehn twist about that sphere.

Theorem 2.7.3. *Let Open(Σ, ϕ) be a contact open book with a Legendrian sphere L_S, which is also a Lagrangian sphere in Σ. Denote the contact manifold obtained from Open(Σ, ϕ) by Legendrian surgery along L_S by $\widetilde{Open(\Sigma, \phi)}_{L_S}$. Then, the contact manifolds*

$$Open(\Sigma, \phi \circ \tau_{L_S}) \simeq \widetilde{Open(\Sigma, \phi)}_{L_S}$$

are contactomorphic.

Example 2.7.4. This is the higher dimensional analogue of Example 6.4 in [19]. The standard contact $(2n+1)$-sphere is supported by the open book $Open(D(T^*S^n), \tau_S)$. The page $D(T^*S^n)$ can be parametrised as $\{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |q|^2 = 1, p \cdot q = 0, |p|^2 \leq 1\}$, and τ_S represents a positive Dehn twist about the sphere $S = \{p_i = 0\}$ in the page. This open book decomposition can be obtained by gluing together two relative open book decompositions on $(2n + 1)$ balls. The first one is the relative decomposition in Example 2.5.3, but shift the indices on the pages by 1 (we can do that since the monodromy is identity). So the page $S_{i,1}$ is really the page S_{i-1} in Example 2.5.3. Here the Lagrangian is S. The second one is the complement of the neighbourhood of S_0. It can be denoted $(\{S_{i,2}\}_{i=0}^4, \tau_{S_0})$, where $S_{1,2}$ and $S_{3,2}$ are the disk cotangent bundles of n-disks, and $S_{0,2}$, $S_{2,2}$, and $S_{4,2}$ are disk cotangent bundles of the n-sphere. The schematic of the gluing is given in Figure 2.4. This example is understood in more detail in Section 4.1.

2.8 Doubling Construction of Legendrians

Given a Legendrian link $K \subset (\mathbb{R}^3, \xi_{st})$, and two exact Lagrangian fillings F, G, we recall the definition of the doubled Legendrian $\Lambda(F, G)$ from Section 2.1 of [6].

Let (x_1, y_1, x_2, y_2) be coordinates on $\mathbb{R}^4$ with the standard symplectic form $-d(y_1dx_1 + y_2dx_2)$, and note that the symplectic form on the symplectisation $\mathbb{R} \times \mathbb{R}^3$ is $d(e^t\alpha)$, where $\alpha = dz - ydx$ is the contact form on $(\mathbb{R}^3, \xi_{st})$.

Consider the exact symplectomorphism $\phi_T : (\mathbb{R} \times \mathbb{R}^3) \to \mathbb{R}^4$, defined by

$$\phi_T(x, y, z, t) = (x, e^t y, e^t - e^T - 1, z)$$

Suppose the Legendrian K is at $t = T$, and the exact Lagrangian filling F lives in $t \leq T$, and is a cylinder on K near T. The image of F under ϕ_T lands in $\{x_2 \leq -1\}$.

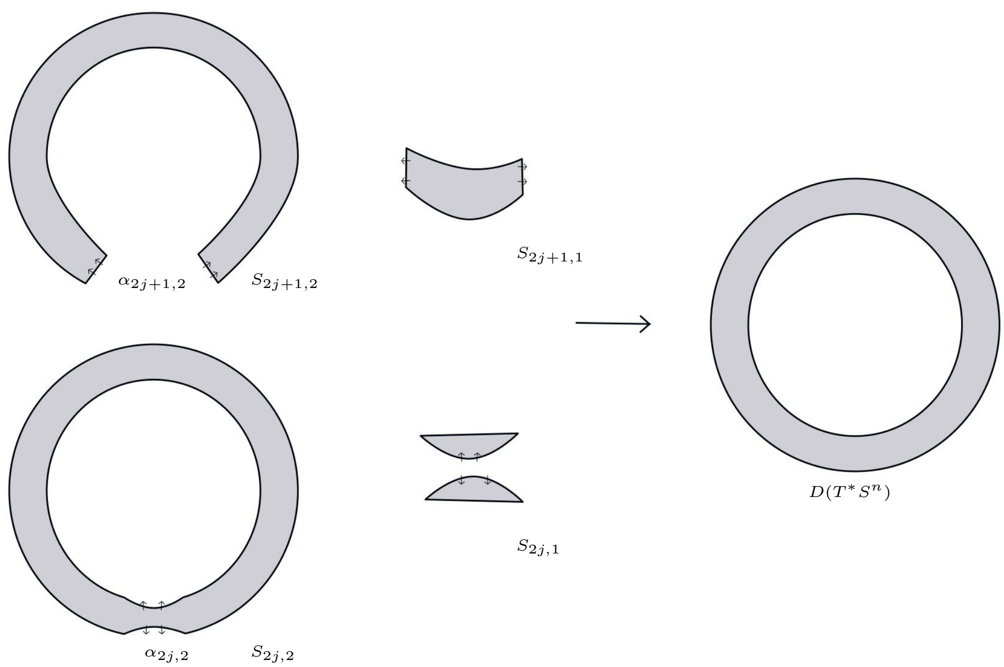


Figure 2.4: The schematic for gluing two relative open books to obtain S^{2n+1} as in Example 2.7.4. The concave and convex parts of the boundaries, that are relevant for gluing the open books, are labeled. The unlabeled boundaries constitute the binding and are convex.

Suppose further that F agrees with $\{t\} \times K$ on $(T-1) < t \leq T$, i.e. it has a conical end. Then, in that region, the coordinates for F in $(\mathbb{R} \times \mathbb{R}^3)$ look like $(x, dz/dx, z, t)$ where (x, z) are the coordinates for the front projection of K. Under ϕ_T, this maps to $(x, e^t(dz/dx), e^t - e^T - 1, z)$.

Consider this $\mathbb{R}^4$ as being the $\{w = 0\}$ slice in $\mathbb{R}^5$, and it has the standard contact structure $\xi_{st} = \ker(dw - y_1 dx_1 - y_2 dx_2)$. Exact Lagrangians in $\{w = 0\}$ can be Legendrian lifted via flowing along ∂_w.

The conical end of F can be Legendrian lifted to $(x, e^t(dz/dx), e^t - e^T - 1, z, e^t z)$ (the fifth coordinate is the w coordinate), whose front projection to the (x_1, x_2, z) coordinates is $(x, z, e^t z)$, i.e. the front in (x_1, w) plane scaled in the x_2 direction. The Legendrian lift of F thus is an embedded Legendrian with its boundary as just described.

Now, consider the reflection in $\mathbb{R}^4$ defined by $(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, -x_2, -y_2)$. Composing this reflection with ϕ_T, we can map G to an exact Lagrangian that lives in $x_2 \geq 1$. Similarly as above, we can lift it to a Legendrian with its boundary being a cylinder on K. Now, these two Legendrians with boundary can be glued by the Legendrian corresponding to the front $(x, sz, e^T z)$ where s goes from -1 to 1, where (x, z) are the coordinates in the front projection of K.

This is how the doubled Legendrian $\Lambda(F, G)$ is constructed. In this thesis, we will call a double *symmetric* if F and G are Hamiltonian isotopic, and *asymmetric* otherwise.

In [11], the authors show that the isotopy class of a symmetric double $\Lambda(L, L)$ is determined by the formal data associated to the filling L, which in turn determines the isotopy class of the Legendrian lift of L, by the h-principle. The following is the result they prove.

Theorem 2.8.1. *[11, Theorem 1] If $L \subset (\mathbb{R} \times \mathbb{R}^{2n-1})$ is an embedded exact Lagrangian submanifold with Legendrian boundary, then the Legendrian isotopy class of $\Lambda(L, L) \subset \mathbb{R}^{2n+1}$ is determined by the induced trivialisation of $TL \otimes \mathbb{C}$.*

CHAPTER 3

BACKGROUND - LEGENDRIAN WEAVES AND MICROLOCAL INVARIANTS

This chapter contains the relevant background for understanding Legendrian surfaces via weaves. The main theme is the following: certain Legendrian surfaces can be described via N-graphs, which is a graph with $(N-1)$ different coloured edges, with every colour being associated to a pair of consecutive integers between 1 and N, with single coloured trivalent vertices, and bicoloured hexagonal vertices (where two coloured edges corresponding to consecutive pairs interlace). This graph encodes how to "weave" the surface and draw its front diagram, analogous to how dots on a circle can encode a braid. The main reference for this is [21]. We will use the letters G and Γ to denote the N-graphs.

The invariants to distinguish these surfaces come from the microlocal theory of sheaves. We work in a simplified setting, as will be made clear in Section 3.3, so that the invariant $\mathcal{M}_1(\Lambda)$, the moduli space of microlocal rank 1 sheaves with singular support at the Legendrian, is essentially the data of a collection of flags in $\mathbb{C}^N$ for some N, with appropriate transversality conditions. The moduli space $\mathcal{M}_1(\Lambda)$ also contains information about the exact Lagrangian fillings of Λ. Every exact Lagrangian filling L induces a toric chart (chart isomorphic to $(\mathbb{C}^*)^{b_1(L)}$ on $\mathcal{M}_1$, and in certain nice cases, using a process called *mutations*, one can change the filling which changes the corresponding chart in a suitable way. This is the cluster structure on $\mathcal{M}_1$. The main reference for these sections is [22].

3.1 Positive braids and (-1)-closures

A braid with only positive crossings represents a Legendrian link in $J^1(S^1)$. By satelliting the braid over the standard Legendrian unknot, we get a corresponding Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ or equivalently in $(\mathbb{S}^3, \xi_{st})$. As a warm up to weaves, a Legendrian knot coming from an N-strand positive braid can be represented as a diagram with a sequence of (at most $(N-1)$) coloured points on a line, with every point representing a crossing. This sequence of points represents a unique Legendrian knot in $J^1(S^1)$, and the procedure of satelliting about the standard Legendrian unknot produces a Legendrian knot in $(\mathbb{S}^3, \xi_{st})$ which is obtained by adding a full negative twist Δ to the braid before closing the strands up. An example which gives the trefoil is shown in Figure 3.6. The knot thus obtained after satelliting is called the (-1)-closure of the braid β.

3.2 Legendrian Weaves

In this section, we describe Legendrian weaves, a geometric construction of Casals and Zaslow that can be used to combinatorially represent Legendrian surfaces Λ in the 1-jet space $J^1\mathbb{D}^2 = T^*\mathbb{D}^2 \times \mathbb{R}_z$ by the singularities of their front projection in $\mathbb{D}^2 \times \mathbb{R}_z$. In practice, one often considers the Lagrangian projection of Λ when Λ has a Legendrian link at its boundary in order to obtain an exact Lagrangian filling of $\lambda = \partial\Lambda$. Although the Legendrian surfaces we construct in this work do not have boundary, the Legendrian weaves we define in this section will generally be taken to have nonempty boundary, as we construct Legendrian doubles by gluing together two such (Lagrangian projections of) Legendrian weaves.

Let $\beta \in Br_n^+$ be a positive braid. The contact geometric description of a Legendrian weave surface with boundary $\lambda(\beta\Delta)$ is as follows. We construct a filling of a Legendrian

$\lambda(\beta\Delta)$ by first describing a local model for a Legendrian surface Λ in $J^1\mathbb{D}^2 = T^*\mathbb{D}^2 \times \mathbb{R}_z$. We equip $T^*\mathbb{D}^2$ with the symplectic form $d(e^r\alpha)$ where $\ker(\alpha) = \ker(dy_1 - y_2 d\theta)$ is the standard contact structure on $J^1(\partial\mathbb{D}^2)$ and r is the radial coordinate. This choice of symplectic form ensures that the flow of $e^r\alpha$ is transverse to $J^1\mathbb{S}^1 \cong \mathbb{R}^2 \times \partial\mathbb{D}^2$ thought of as the cotangent fibers along the boundary of the 0-section. The Lagrangian projection of Λ is then a Lagrangian surface in $(T^*\mathbb{D}^2, d(e^r\alpha))$. Moreover, since $\Lambda \subseteq (J^1\mathbb{D}^2, \ker(dz - e^r\alpha))$ is a Legendrian, we immediately obtain the function $z : \pi(\Lambda) \to \mathbb{R}$ satisfying $dz = e^r\alpha|_{\pi(\Lambda)}$, demonstrating that $\pi(\Lambda)$ is exact.

The boundary of $\pi(\Lambda)$ is taken to be a positive braid β in $J^1\mathbb{S}^1$ so that we regard it as a Legendrian link in a contact neighborhood of $\partial\mathbb{D}^2$. As the 0-section of $J^1\mathbb{S}^1$ is Legendrian isotopic to a max-tb standard Legendrian unknot, we can take $\partial\pi(\Lambda)$ to equivalently be the standard satellite of the standard Legendrian unknot. Diagramatically, this allows us to express the braid β in $J^1\mathbb{S}^1$ as the (-1)-framed closure of β in contact $\mathbb{S}^3$.

The immersion points of a Lagrangian projection of a weave surface Λ correspond precisely to the Reeb chords of Λ. In particular, if Λ has no Reeb chords, then $\pi(\Lambda)$ is an embedded exact Lagrangian filling of $\partial(\Lambda)$. In the Legendrian weave construction, Reeb chords correspond to critical points of functions giving the difference of heights between sheets.

N-Graphs and Singularities of Fronts

Given a graph $\Gamma \subset S^2$, to construct a Legendrian weave surface Λ in $J^1(S^2)$, we combinatorially encode the singularities of its front projection in a colored graph. Locally, $J^1(S^2)$ can be identified with $J^1(D^2)$ and we will describe the weaving in these local models. Local models for these singularities of fronts are given by Arnol'd [23, Section 3.2]; the three singularities that appear in our construction describe elementary Legendrian cobordisms and are pictured in Figure 3.1.

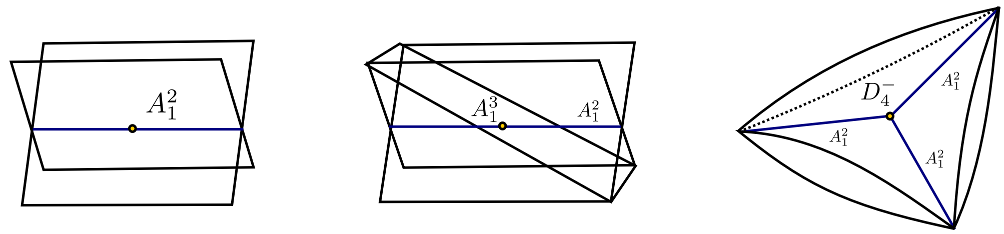


Figure 3.1: Singularities of front projections of Legendrian surfaces. Labels correspond to notation used by Arnold in his classification.

Since the boundary of our singular surface $\Pi(\Lambda)$ is the front projection of an N-stranded positive braid, $\Pi(\Lambda)$ can be pictured as a collection of N sheets away from its singularities. We describe the behavior at the singularities as follows:

1. The A_1^2 singularity occurs when two sheets in the front projection intersect transversally. This singularity can be thought of as the trace of a constant Legendrian isotopy in the neighborhood of a crossing in the front projection of the braid $\beta\Delta^2$.

2. The A_1^3 singularity occurs when a third sheet passes transversally through an A_1^2 singularity. This singularity can be thought of as the trace of a Reidemeister III move in the front projection.

3. A D_4^- singularity occurs when three A_1^2 singularities meet at a single point. This singularity can be thought of as the trace of a 1-handle attachment in the front projection.

Having identified the singularities of fronts of a Legendrian weave surface, we encode them by a colored graph $\Gamma \subseteq \mathbb{D}^2$. The edges of the graph are labeled by Artin generators of the braid and we require that any edges labeled σ_i and σ_{i+1} meet at a hexavalent vertex with alternating labels while any edges labeled σ_i meet at a trivalent vertex. To obtain a Legendrian weave $\Lambda(\Gamma) \subseteq (J^1\mathbb{D}^2, \xi_{st})$ from an N-graph Γ, we glue together the local germs of singularities according to the edges of Γ. First, consider N horizontal sheets $\mathbb{D}^2 \times \{1\} \sqcup \mathbb{D}^2 \times \{2\} \sqcup \cdots \sqcup \mathbb{D}^2 \times \{N\} \subseteq \mathbb{D}^2 \times \mathbb{R}$ and an N-graph $\Gamma \subseteq \mathbb{D}^2 \times \{0\}$. We construct the associated Legendrian weave $\Lambda(\Gamma)$ as follows [21, Section 2.3].

- Above each edge labeled σ_i, insert an A_1^2 crossing between the $\mathbb{D}^2 \times \{i\}$ and $\mathbb{D}^2 \times \{i+1\}$ sheets so that the projection of the A_1^2 singular locus under $\Pi : \mathbb{D}^2 \times \mathbb{R} \to \mathbb{D}^2 \times \{0\}$ agrees with the edge labeled σ_i.

- At each trivalent vertex v involving three edges labeled by σ_i, insert a D_4^- singularity between the sheets $\mathbb{D}^2 \times \{i\}$ and $\mathbb{D}^2 \times \{i+1\}$ in such a way that the projection of the D_4^- singular locus agrees with v and the projection of the A_2^1 crossings agree with the edges incident to v.

- At each hexavalent vertex v involving edges labeled by σ_i and σ_{i+1}, insert an A_1^3 singularity along the three sheets in such a way that the origin of the A_1^3 singular locus agrees with v and the A_1^2 crossings agree with the edges incident to v.

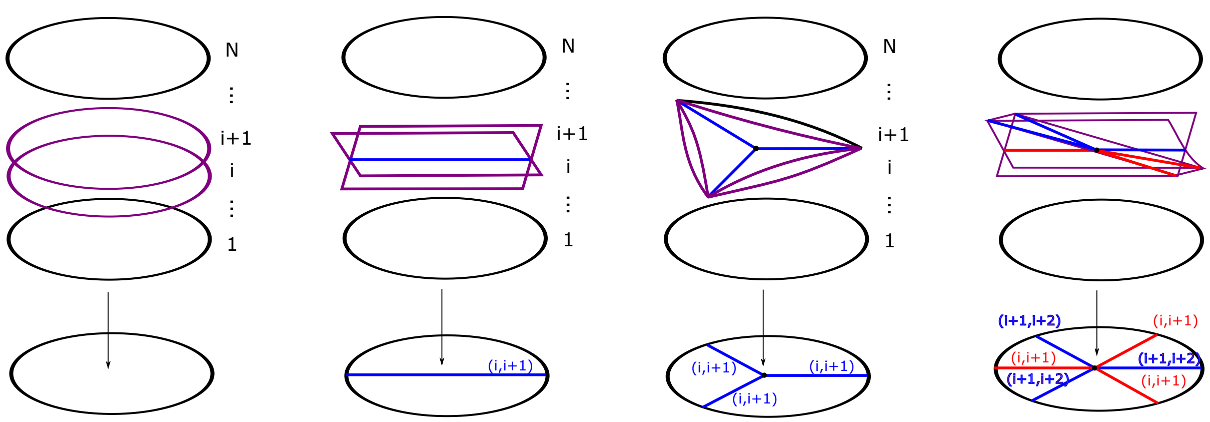


Figure 3.2: The weaving of singularities of fronts along the edges of the N-graph (courtesy of Roger Casals and Eric Zaslow, used with permission). Gluing these local models according to the N-graph Γ yields the weave $\Lambda(\Gamma)$.

If we take an open cover $\{U_i\}_{i=1}^m$ of $\mathbb{D}^2 \times \{0\}$ by open disks, refined so that any disk contains at most one of these three features, we can glue together the resulting fronts according to the intersection of edges along the boundary of our disks. Specifically, if $U_i \cap U_j$ is nonempty, then we define $\Pi(\Lambda(U_1 \cup U_2))$ to be the front resulting from considering the union of fronts $\Pi(\Lambda(U_1)) \cup \Pi(\Lambda(U_j))$ in $(U_1 \cup U_2) \times \mathbb{R}$.

Definition 3.2.1. The Legendrian weave $\Lambda(\Gamma) \subseteq (J^1\mathbb{D}^2, \xi_{st})$ is the Legendrian lift of the front $\Pi(\Lambda(\cup_{i=1}^m U_i))$ given by gluing the local fronts of singularities together according to the N-graph Γ.

Equivalence of N-graphs

We consider Legendrian weaves to be equivalent up to Legendrian isotopy fixing the boundary. Such Legendrian isotopies can also often be combinatorially understood through N-graphs. We can restrict our attention to specific isotopies, pictured in Figure 3.3 and refer to them as Legendrian Surface Reidemeister moves. From [21], we have the following theorem relating surface Reidemeister moves to the corresponding N-graphs.

Theorem 3.2.2 ([21], Theorem 4.2)**.** *Let Γ and Γ' be two N-graphs related by one of the moves shown in Figure 3.3. The Legendrian weaves $\Lambda(\Gamma)$ and $\Lambda(\Gamma')$ are Legendrian isotopic relative to their boundaries.*

The topology of the Legendrian surface

Given a graph G on $\mathbb{S}^2$, we can read the smooth topology of the surface $\Lambda(G)$ as follows. If G is an N-graph, the surface $\Lambda(G)$ is a branched (N)-fold cover of $\mathbb{S}^2$. The genus of

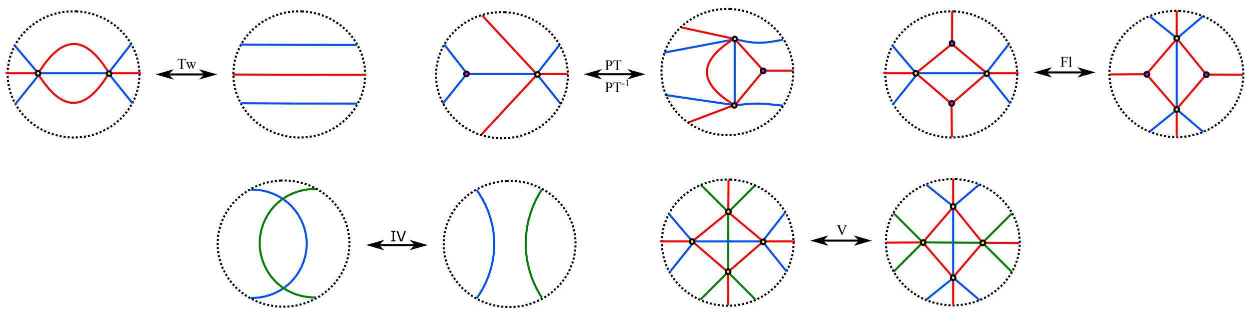


Figure 3.3: Legendrian Surface Reidemeister moves for N-graphs. Clockwise from top left, a candy twist, a push-through, a flop, and two additional moves.

$\Lambda(G)$ [21, Section 2.4] is given by:

$$g(\Lambda(G)) = \frac{1}{2}(v(G) + 2 - 2N)$$

3.2.1 Legendrians in $(\mathbb{R}^5, \xi_{st})$ and Lagrangian Fillings

Given an N-graph G that lives on $\mathbb{S}^2$, we will use $\Lambda(G)$ to denote the Legendrian obtained by satelliting the Legendrian in $J^1(\mathbb{S}^2)$ to the standard 2-dimensional Legendrian unknot. Figure 3.4 give some simple examples of N-graphs on S^2, and their corresponding Legendrians (obtained by satelliting about the standard unknot) are drawn in the front projection are shown in Figure 3.5.

Free N-graphs and Lagrangian fillings

This is described in Section 7.1.2 of [21]. Consider a graph G properly embedded in $\mathbb{D}^2$.

Definition 3.2.3. An N-graph $G \subset \mathbb{D}^2$ is said to be free if the Legendrian lift $\Lambda(G) \subset J^1(\mathbb{D}^2)$ contains no Reeb chords.

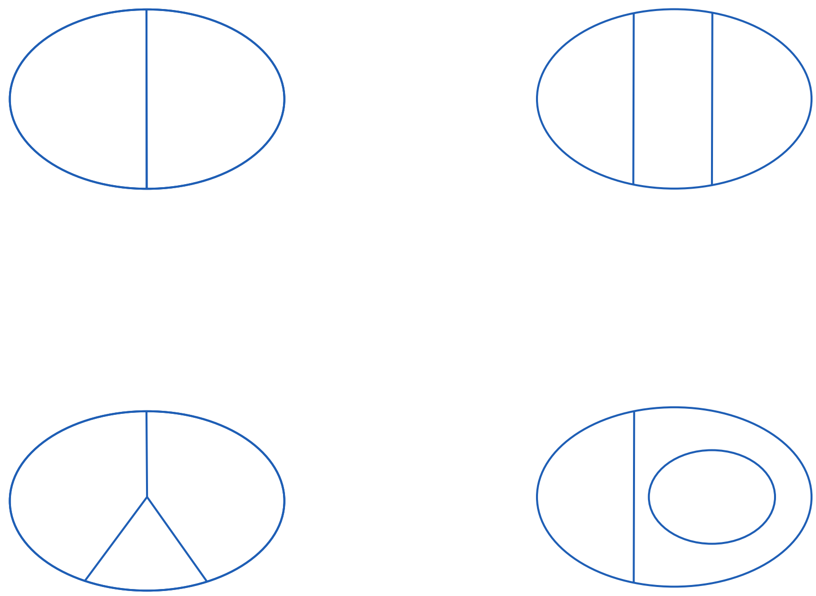


Figure 3.4: In clockwise from top left, N-graphs that describe the standard unknot, the standard torus T^2_{st}, the first in the infinite family of spheres found in [5], and the Clifford torus T^2_C.

When G is free, the Lagrangian projection of $\Lambda(G)$ is an embedded exact Lagrangian filling of $\Lambda(\partial G)$, which is the (-1)-closure of the braid β formed by the boundary of G.

Given a free N-graph G, the Lagrangian projection of $\Lambda(G)$ describes a Lagrangian filling of the knot the (-1)-closure of the boundary braid coming from ∂G.

Example 3.2.4. [14, 12] The torus knot $\lambda(2, n)$ admits at least $C_n = \frac{1}{n+1}\binom{2n}{n}$ fillings coming from 2-graphs, as pictured in Figure 3.6 for the trefoil $\lambda(2, 3)$. In Figure 3.6, the trefoil is $\lambda(2, 3)$ on the left drawn in its front projection, and as a 1-dimensional weave in the bottom. Satelliting this weave over the max-tb unknot produces the (-1)-closure of the

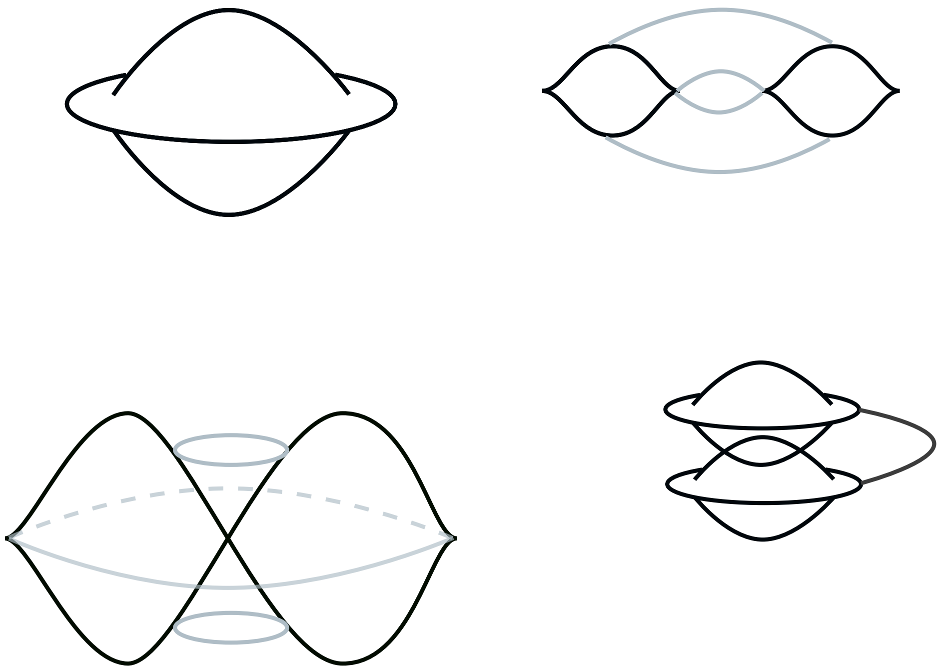


Figure 3.5: The fronts of some Legendrian surfaces in $(\mathbb{R}^5, \xi_{st})$. Clockwise from top left are the standard unknot, the standard torus T^2_{st}, a knotted sphere found in [5], and the Clifford torus T^2_C.

2-strand braid with 5 positive crossings. Its fillings are given by trivalent trees with 5-free boundary edges.

Example 3.2.5. Given any (-1)-closure of a positive braid, Section 3.3 of [22] gives an algorithm to produce a free N-graph filling, i.e., an exact Lagrangian filling.

Legendrian surgery moves on weaves

In [21], the authors describe some local graph moves that translate to certain surgeries on the associated Legendrians. In the original manuscript, these moves are described

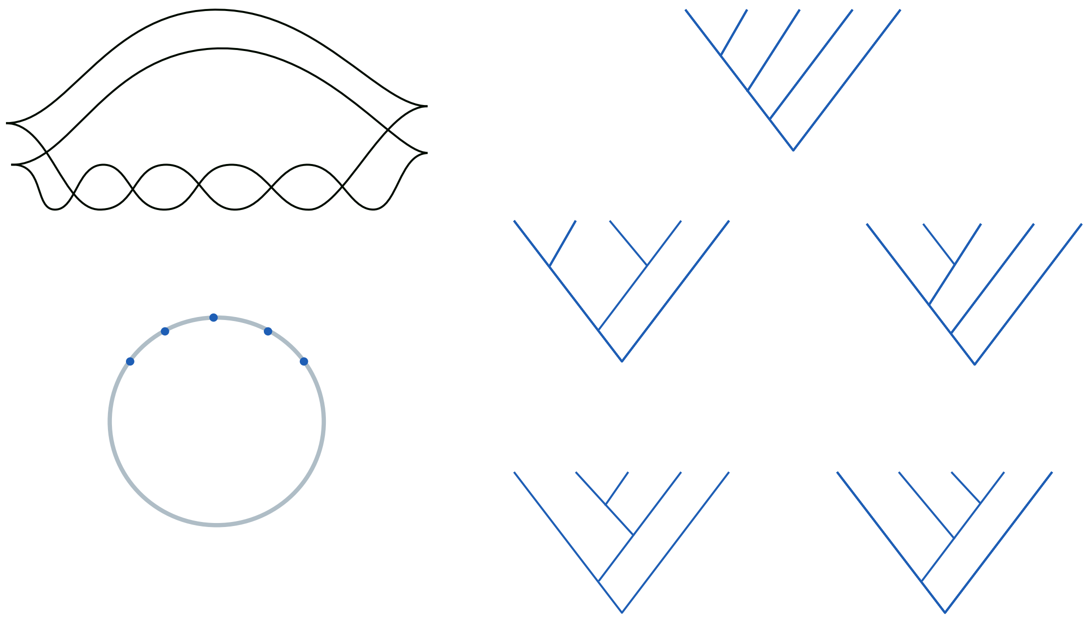


Figure 3.6: The trefoil $\lambda(2,3)$, pictured as the 1-weave on bottom-left satellited about the max-tb unknot, and its fillings, pictured as 2-graphs.

for Legendrians in $J^1(C)$ for arbitrary C and there are subtleties regarding where the resultant Legendrian lives. For our setting, everything lives in a $J^1(S^2)$ neighbourhood for a standard Legendrian unknot so these subtleties can be ignored. The local moves are described in Figure 3.7 and in the following theorem.

Theorem 3.2.6 (Theorem 4.10, [21])**.** *Given two N-graphs, the local modifications shown in Figure 3.7 correspond to the following:*

1. *A connect sum.*

2. *Connect summing with a T^2_{st}.*

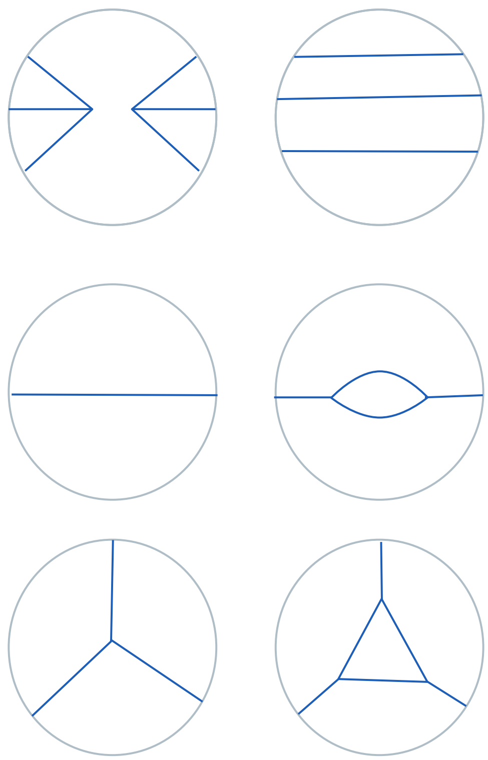


Figure 3.7: The surgery moves in Theorem 3.2.6.

3. *Connect summing with a* T^2_C.

3.2.2 Loose Legendrians

A subclass of Legendrian submanifolds, called *loose* [24], are classified by an h-principle. For Legendrians represented by weaves, there is a simple combinatorial criterion that characterises looseness. Let C be a surface (in our thesis we only use $C = S^2$). Let $\iota : C \to \mathbb{R}^5$ be a Legendrian embedding into the standard contact structure.

Definition 3.2.7. [21] An N-graph $G \subseteq C$ is said to have a bridge if there exists two disjoint 2-disks $D_1, D_2 \subseteq C$ such that the complement $G \setminus (G \cap D_1 \cup G \cap D_2)$ consists of

$(N-1)$ disjoint strands with labels $\tau_1, \tau_2, \ldots, \tau_{N-1}$ consecutive with respect to a transverse oriented curve in $C \setminus (D_1 \cup D_2)$.

Theorem 3.2.8. *[21] If $G \subset C$ is an N-graph with a bridge, then the satellite Legendrian $\iota(\Lambda(G))$ is a loose Legendrian.*

3.2.3 Mutations

The word *mutation* arises in symplectic geometry in various contexts (wall-crossing phenomena, Lagrangian mutations), and it is related to underlying connections with cluster varieties and quiver mutations. In [21] the authors define Legendrian mutations which is a related operation on Legendrian surfaces in $(\mathbb{R}^5, \xi_{st})$ – it amounts to performing a Lagrangian mutation on the Lagrangian projection and often (but not always) produces a new Legendrian not isotopic to the original. For the purpose of this thesis, the process of mutation will be defined for Legendrian surfaces represented by N-graphs.

Definition 3.2.9. [21] Let G be an N-graph and e an i-edge between two trivalent vertices. The graph produced by the local move shown in Figure 3.8 is called the mutation of G along e and denoted $\mu_e(G)$.

Definition 3.2.10. The Legendrians $\iota(\Lambda(G))$ and $\iota(\Lambda(\mu_e(G)))$ are said to be mutation-equivalent.

In the front projection, the local modification due to the mutation move is denoted in Figure 3.8.

Lagrangian fillings corresponding to two mutation-equivalent N-graphs are related by Lagrangian disk surgery [25].

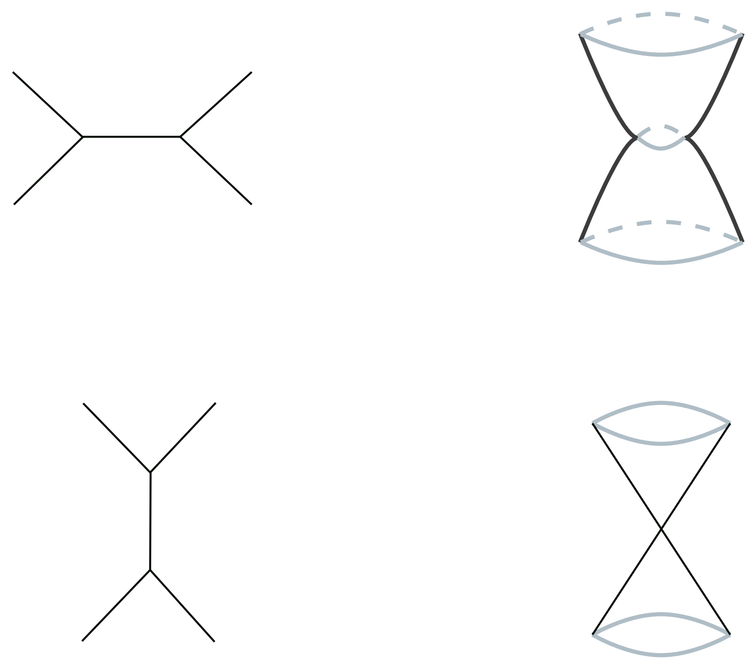


Figure 3.8: The mutation move in N-graphs (left), and the corresponding fronts (right).

3.3 Flag moduli Invariants of Legendrian knots and surfaces

Given a Legendrian $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$, the work of [26] shows that the moduli of constructible sheaves on $\mathbb{R}^{n+1}$ with singular support at Λ is a Legendrian isotopy invariant of Λ. For this thesis, we shall not define what *constructible sheaves* or *singular support* mean, but will use a combinatorial reformulation for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ due to [27], and for Legendrian weaves in $(\mathbb{R}^5, \xi_{st})$ due to [21]. These works recover the associated moduli of microlocal rank one sheaves as the quotient stack of a flag variety associated to the knot or weave. The invariant is denoted $\mathcal{M}_1(\Lambda)$.

First, we describe the general idea for how the flag moduli space arises. We caution the reader that several subtleties regarding algebraic geometry notions are suppressed in this section. The general discussion in [27] uses complexes of sheaves rather than sheaves, and [21] uses compatible local systems and defines invariants for Legendrians in $J^1(C)$ for a general surface that is not simply connected. However, since we only work in $J^1(S^2)$, and we do not have cusps in the front projections, we can work only with vector spaces and maps between them.

The front projection of Λ defines a stratification of $\mathbb{R}^{n+1} \setminus \Pi_F(\Lambda)$. One first associates a vector space to each component of $\mathbb{R}^{n+1} \setminus \Pi_F(\Lambda)$, or the top-dimensional strata, with 0 being associated to the unbounded component (this is the notion of a constructible sheaf). Then, across the lower-dimensional strata corresponding to Λ, the vector spaces on either side differ in dimension by one (this corresponds to *microlocal rank one*), and there are homomorphisms between them (the constraints on dimension and homomorphisms come from the singular support condition). We shall now make this precise for $n = 1$ and $n = 2$, in the context of (-1)-closures of positive braids, and for Legendrian weaves of N-graphs, respectively.

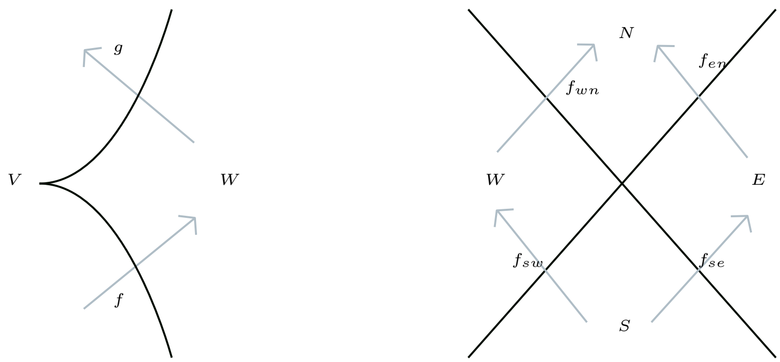
3.3.1 Invariants of Legendrian knots

The following theorem for (-1)-closures of positive braids is proven in [27, Section 3], but expressed as written here in [22, Proposition 4.1].

Theorem 3.3.1. *[27] For a Legendrian Λ which is the (-1)-closure of a positive braid with N strands, the invariant $\mathcal{M}_1(\Lambda)$ is the data, up to $PGL_N(\mathbb{C})$ equivalence, of $\mathbb{C}$-vector spaces at each connected component of $\mathbb{R}^2 \setminus \Pi_F(\Lambda)$ and linear maps between the vector spaces in adjacent components separated by an arc of $\Pi_F(\Lambda)$ such that:*

- *The vector space for the unbounded component is 0*
- *The dimensions of vector spaces in adjacent components, separated by an arc of $\Pi_F(\Lambda)$, differ by 1*
- *Given an arc of $\Pi_F(\Lambda)$, there is a linear map from the vector space for the lower component to the vector space in the higher component of $\mathbb{R}^2 \setminus \Pi_F(\Lambda)$*
- *At every cusp, the composition of the two maps involved is the identity, as shown in Figure 3.9*
- *At every crossing, the four linear maps involved form a commuting square which is exact, as shown in Figure 3.9*

Example 3.3.2. As an example, we show how to compute the sheaf moduli for the trefoil $\lambda(2,3)$ in Figure 3.10. The conditions given above say that the images of $\mathbb{C}$ in $\mathbb{C}^2$, under the homomorphisms f_i, should be complex lines that satisfy: $f_1 \neq f_2$, $f_2 \neq f_3$, $f_3 \neq f_4$, $f_1 \neq f_5$, $f_4 \neq f_5$ (here $\neq$ means the two should be 2-vectors which form a 2×2 matrix with determinant 1. Under the PGL_2 action, we can fix two of the lines $f_1 \neq f_2$ to be $(1,0)$ and $(0,1)$. Then $f_3 \neq f_2$ can be chosen to be $(-1,x)$ for some $x \in \mathbb{C}$, which means $f_4 \neq f_3$ should be $(y, -xy-1)$ for some $y \in \mathbb{C}$. It follows that $f_5 \neq f_4$ should be given by $f_5 = (1+yz, -x-z-xyz)$ for some $z \in \mathbb{C}$. Finally, $f_5 \neq f_1$ forces that $(x+z+xyz=1)$, which is the condition that defines all possible $\mathbb{C}$-vector space valued microlocal one constructible sheaves on $\mathbb{R}^2$ with singular support on $\lambda(2,3)$, and 0 on a neighbourhood of ∞.



$g \circ f = id_V$ $\qquad$ $0 \to S \xrightarrow{(f_{sw}, f_{se})} W \oplus E \xrightarrow{f_{wn} - f_{en}} N \to 0$

Figure 3.9: The conditions near a cusp and a crossing.

3.3.2 Invariants of Legendrian surfaces

We will need the notion of flags to define the invariants for surfaces. We will consider flags over fields, but in general it works for flags over commutative rings. Let K be the ground field.

Definition 3.3.3. A flag $\mathcal{F}$ is a sequence of nested linear subspaces in K^N, i.e. it is a collection of i-dimensional subspaces $\mathcal{F}^i$ of K^N. Equivalently, a flag is also a sequence of nested projective planes in PK^{N-1}.

There is a natural action of $GL_N(K)$, or simply GL_N, on the space of flags.

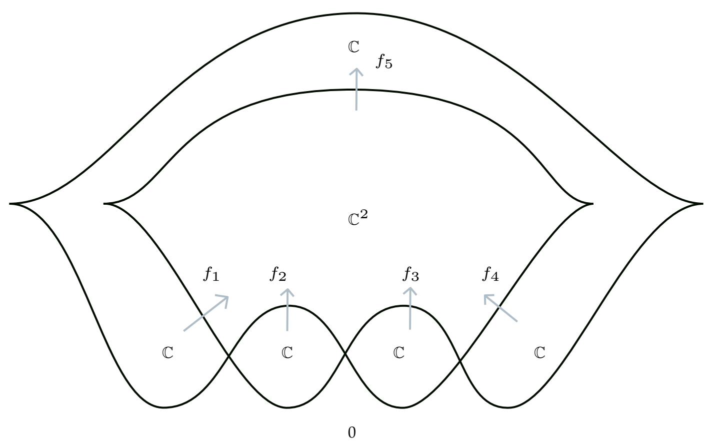


Figure 3.10: The trefoil $\lambda(2,3)$ as a rainbow closure, and computing its sheaf moduli.

Now we will describe the moduli of flags associated to an N-graph $G \subset C$ where C is simply connected. In [21] this space is denoted $\mathcal{M}(G, C)$ or $\mathcal{M}(G)$, and then it is shown that it is isomorphic to $\mathcal{M}_1(\Lambda(G))$. For our purpose, we will simply denote the moduli of flags as $\mathcal{M}_1(\Lambda(G))$, since that is the invariant of the geometric surfaces we consider. Depending on the context, $\mathcal{M}_1(\Lambda(G))$ will either be an invariant up to Legendrian isotopy for the Legendrian $\Lambda(G)$, or an invariant up to Hamiltonian isotopy for the Lagrangian filling $\pi(\Lambda(G))$ of the Legendrian knot $\Lambda(\partial G)$, when G is a free N-graph [21]. In this thesis we only consider $C = S^2$ and the Legendrians after satelliting about the standard unknot, thus the invariants are invariants up to Legendrian isotopy in $(\mathbb{R}^5, \xi_{st})$ for the non-free, Legendrian surface case.

Definition 3.3.4. [21, Definition 5.2] Let C be a connected, simply connected surface and let $G \subseteq C$ be an N-graph. Let S_N denote the symmetric group on N symbols. The *framed flag moduli space* $\widetilde{M}(C, G)$ associated to G is comprised of tuples of flags, specifically:

1. There is a flag $\mathcal{F}^\bullet(F)$ associated to each face F of the N-graph G.
2. For each pair of adjacent faces $F_1, F_2 \subset C \backslash G$, sharing an i-edge, their two associated flags $\mathcal{F}^\bullet(F_1)$, $\mathcal{F}^\bullet(F_2)$ are in relative position $\tau_i \in S_N$, i.e., they must satisfy

$$\mathcal{F}^j(F_1) = \mathcal{F}^j(F_2), 0 \leq j \leq N, j \neq i \mathcal{F}^i(F_1) \neq \mathcal{F}^i(F_2)$$

The *flag moduli space* of $\Lambda(G)$ is then the quotient stack

$$\mathcal{M}_1(\Lambda(G)) \coloneqq \widetilde{M}(C, G)/PGL_N$$

From the above, it is clear that the flag moduli vanishes for loose Legendrian weaves.

Lemma 3.3.5. *For a free N-graph G, the flag moduli space vanishes.*

Proof. The bridge edges force self-transversality conditions on a single flag, which cannot be satisfied. □

We will do some example computations, most of which have already been worked out in [21] and [12].

Example 3.3.6. First consider the flag moduli around an $(i, i+1)$-edge in an N-graph. By Definition 3.3.4, the data is of three flags in $\mathbb{C}^N$ – $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$, which only disagree at the i-th space, i.e., considering the two dimensional spaces $\mathcal{F}_1^{i+1}/\mathcal{F}_1^{i-1}$, $\mathcal{F}_2^{i+1}/\mathcal{F}_2^{i-1}$, and

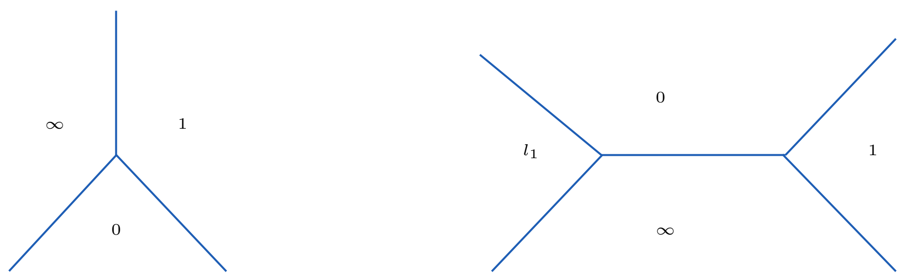


Figure 3.11: Computing the flag moduli around a trivalent vertex and a short I-cycle.

$\mathcal{F}_3^{i+1}/\mathcal{F}_3^{i-1}$, there are three mutually transverse lines l_1, l_2, and l_3. So the data is of three distinct points in $\mathbb{P}^1(\mathbb{C})$. However, by the PGL_N action, three mutually transverse lines can be mapped to $0, 1, \infty$. Thus the local flag moduli around a trivalent vertex is just a point.

Example 3.3.7. Following from Example 3.3.6, for a short I-cycle, we can fix the flag moduli around one of the vertices and choose the lines to be 0, 1, and ∞ as shown in Figure 3.11. Then, the local flag moduli depends on the choice of l_1, which needs to be distinct from 0 and ∞. Thus, the flag moduli around an I-cycle is exactly $\mathbb{C}^*$. This is what is called the *microlocal monodromy* and gives a cluster coordinate for the toric chart

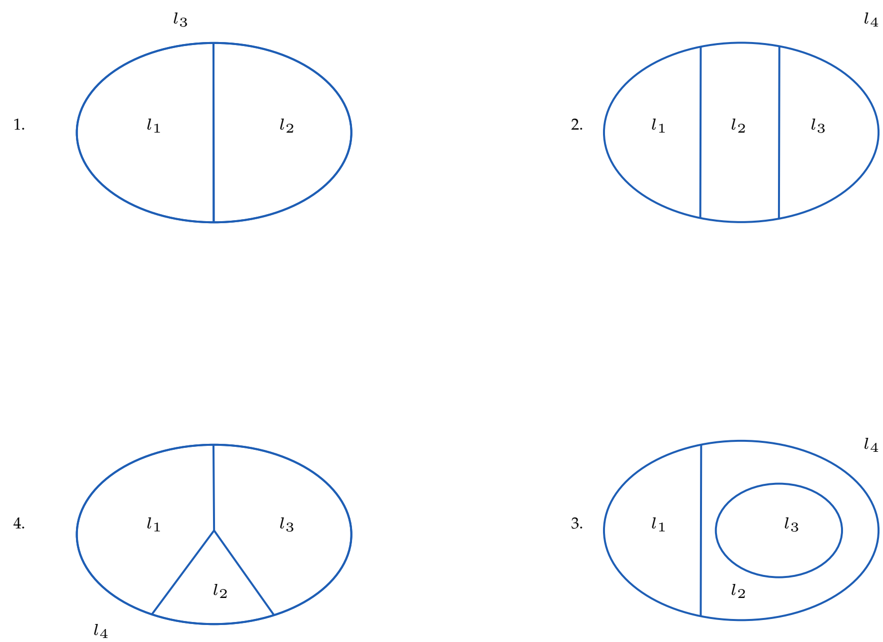


Figure 3.12: Computing the flag moduli for the examples in Figure 3.4

induced by a filling.

Example 3.3.8. Here we will work out the sheaf moduli, similar to above, for the examples in Figure 3.12.

1. For the unknot, we need three mutually transverse lines, hence the flag moduli is a point, similar to Example 3.3.6.

2. For T^2_{st}, similar to Example 3.3.7, we need $l_1 \neq l_2, l_1 \neq l_4, l_3 \neq l_4, l_2 \neq l_4$. Thus we can fix l_2, l_3, and l_4 to be 0, 1, and ∞ respectively. Thus the flag moduli depends on the choice of l_1 which cannot be 0 or ∞, hence it is $\mathbb{C}^*$.

3. For this sphere, we can fix l_1, l_4, and l_2 to be 1, 0, and ∞ respectively. Thus the flag moduli depends on the choice of $l_3 \neq l_2$, hence is isomorphic to $\mathbb{C}$.

4. For the Clifford torus T_c^2, all the four lines l_1, l_2, l_3, and l_4 have to be mutually transverse. Thus the flag moduli is isomorphic to $\mathbb{C} \setminus \{0, 1\}$.

The computations become significantly harder when one goes from 2-graphs to 3-graphs.

3.3.3 Point counts over finite fields

One extract easier to compute invariants for Legendrians Λ corresponding to 2-weaves. Following the description in [12], the points in $\mathcal{M}_1(\Lambda)$ can be counted over finite fields.

Proposition 3.3.9. *[12, Proposition 1.2] Let Γ be a simple, cubic planar graph, $\widehat{\Gamma}$ its dual graph. Let $P_{\widehat{\Gamma}}$ denote the chromatic polynomial, whose value $P_{\widehat{\Gamma}}(c)$ is the number of colourings of $\widehat{\Gamma}$ with c colours. Let F_q be a field with q elements. Then*

$$\#\mathcal{M}_1(\Lambda(\Gamma)) = \frac{1}{q^3 - q} . P_{\widehat{\Gamma}}(q + 1)$$

We can use point counts, as shown in the following examples, to distinguish cubic planar graph Legendrians.

Example 3.3.10.

1. First, we will see how the finite field counts work for some of the examples in 3.3.8. For T_{st}^2, the finite field point count is $(q - 1)$, while for T_c^2, it is $(q - 2)$.

2. We can use point counts to distinguish the two genus 3 surfaces in Figure 3.13. For the cube graph, the Legendrian on the left, the number of points in the flag moduli

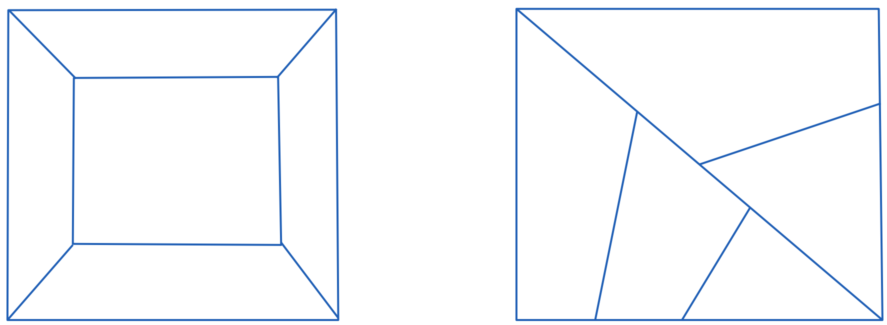


Figure 3.13: Two distinct genus 3 Legendrian surfaces represented by cubic planar graphs.

over F_q is $(q+1)^3 - 9(q+1)^2 + 29(q+1) - 32$, while for the one on the right, which can be seen to be $\#^3 T^2_c$ by Theorem 3.2.6, the point count is $(q-2)^3$.

3.3.4 Toric charts from exact Lagrangian fillings

Given a Legendrian $\Lambda \subset T^{*,\infty}M$, an exact Lagrangian filling L induces a ”nice” chart, isomorphic to $(\mathbb{C}^*)^{|b_1(L)|}$, on the sheaf moduli of Λ. We will cite the results that establish this, stated in the precise categorical language, even though we will not discuss the categorical background in depth. The result follows from [28, 29, 30], but is stated in full generality in Theorem B.20 in the appendix of [31], which is what we will paraphrase as follows.

Theorem 3.3.11. *Let $L \subset T^*M$ be an exact Lagrangian filling of a Legendrian submanifold $\Lambda \subset T^{*,\infty}M$ with zero Maslov class, whose primitive $f|_L$ is proper and bounded from below. Let $\widetilde{L} \subset J^1(M)$ be the Legendrian lift of L. Then*

1. *there exists a fully faithful functor taking local systems on $\widetilde{L}$ to constructible sheaves on $M \times \mathbb{R}$ supported on $\widetilde{L}$*
2. *There exists a fully faithful functor taking constructible sheaves on $M \times \mathbb{R}$ supported on $\widetilde{L}$ to constructible sheaves on M supported on Λ.*

Item (2) in the above theorem describes how exact Lagrangian fillings describe objects in the sheaf moduli of the boundary Legendrian – this amounts to imposing the additional singular support conditions coming from the exact Lagrangian. Item (1) describes how for the exact filling L, sheaves with singular support on the Legendrian lift $\widetilde{L}$ are in one-to-one correspondence with local systems on the L. Local systems can be thought of as $GL_1(\mathbb{C})$ representations of $\pi_1(L)$, which, by a general fact, are in one-to-one correspondence with locally constant sheaves on L (some literature in the field uses the term local system interchangeably for locally constant sheaves and representations). Also, for $\mathbb{C}^*$-valued representations, those can be abelianised and thus local systems on L can be identified with $(\mathbb{C}^*)^{b_1(L)}$.

Putting (1) and (2) together we get that L defines a chart on $\mathcal{M}_1(\lambda)$ isomorphic to $(\mathbb{C}^*)^{b_1(L)}$. Given this fact, one can obtain constraints to show that certain Legendrians cannot be exact Lagrangian fillable. For example:

Theorem 3.3.12. *[12, Theorem 1.3] Let $S \subset T^\infty\mathbb{R}^3$ be the genus-g Legendrian surface defined by a simple cubic planar graph Γ. Then S has no smooth oriented graded exact Lagrangian fillings in $\mathbb{R}^6$.*

3.4 Cluster structure on the space of exact Lagrangian fillings

Cluster algebras, first introduced by S. Fomin and A. Zelevinsky [32] in the context of Lie theory, are commutative rings endowed with a set of distinguished generators that have remarkable combinatorial structures. Cluster varieties, a geometric enrichment of cluster algebras introduced by V. Fock and A. Goncharov [33], are affine varieties equipped with an atlas of torus charts whose transition maps obey certain combinatorial rules. Cluster varieties come in dual pairs consisting of a cluster K_2-variety, also known as a cluster $\mathcal{A}$-variety, and a cluster Poisson variety, also known as a cluster $\mathcal{X}$-variety.

Combinatorially, a cluster variety can be described by starting with an initial seed, which is a quiver with functions associated to each vertex. The entire cluster variety is then produced by all possible mutations on the initial seed and its mutated seeds. The K_2 and $\mathcal{X}$ varieties are produced by two different but related quivers and the mutation formulas are slightly different.

In [22], Casals and Weng show that when λ is represented by a (-1)-closure of a positive braid, $\mathcal{M}_1(\lambda)$ is a cluster variety. They construct the initial seeds by constructing an initial weave filling – the quiver for the K_2-variety comes from the intersection quiver of relative cycles on the fillings, while the quiver for the Poisson variety comes from the intersection quiver of absolute cycles on the filling. The functions associated to the quiver, to produce the seed, come from *microlocal merodromies* in the first case, and microlocal monodromies in the second – it amounts to recording how the sheaf changes along a relative or absolute cycle, respectively, as in Example 3.3.7 for an absolute cycle. The term *microlocal merodromy* refers to how the sheaf changes while being parallel transported along a relative cycle in the exact Lagrangian filling, and is described in Section 4.6.1 of [22]. Mutating along the cycles in the weave result in mutations of the associated seed.

CHAPTER 4

CONSTRUCTIONS FROM OPEN BOOKS AND ISOTOPY

In this chapter, we deal with how to understand stabilisation of open books as a surgery. Then, we define two constructions of Legendrian spheres from Lagrangian disks in the page, and show that these are isotopic. We further show that these are isotopic to the standard unknot and generalise the result of Courte-Ekholm in the case of disk fillings.

4.1 Understanding stabilisation as a surgery

In this section, we will prove Theorem 1.1.4. We will need a lemma, proved in [16] that will allow us to assume that if L is a Lagrangian disk on the page of an open book, and L' is the associated Legendrian, the open book pages can be perturbed, without changing the page topology or the monodromy, so that L' is a Lagrangian on a page as well as a Legendrian in the manifold.

Lemma 4.1.1 (Lemma 4.2 in [16])**.** *Suppose (M, ξ) is a contact manifold of dimension greater than 3 with a supporting open book where the pages are Σ and the monodromy is ψ, i.e. $M = Open(\Sigma, \psi)$. If L is a Lagrangian sphere in the page, then we can isotope the contact structure on M and find a supporting open book with symplectomorphic page and isotopic monodromy such L becomes Legendrian in M.*

We will now discuss the proof of Theorem 1.1.4.

4.1.1 Outline of the idea:

We will describe stabilisation of an open book supporting (M, ξ) by removing and regluing a $(2n+1)$-ball in the manifold. The balls will have two different relative open book decompositions. Extending the induced relative open book fibration map of the complement of the first ball, over the new ball will show that after regluing, we get an open book decomposition of the manifold that has the same abstract open book that describes the stabilised manifold. We will glue the relative open books using Lemma 2.5.4. By Proposition 2.7.2, we will conclude that the supported contact structure on M by this open book is contactomorphic to ξ.

Now we discuss how we obtain the $(2n+1)$-ball to surger, as described in the above paragraph. The manifold M has the open book given by $\nu : (M - B) \to S^1$, where $\nu^{-1}(pt) \simeq \Sigma$. L is a Lagrangian disk in a page which is also a Legendrian in M. We can also assume that B here is a tubular neighbourhood of the binding.

Also, consider the abstract open book $Open(D(T^*(S^n)), \tau_{S_0})$ which is contactomorphic to (S^{2n+1}, ξ_{st}). Here $D(T^*(S^n))$ refers to the disk cotangent bundle of the sphere, which can be parametrised as $\{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |q|^2 = 1, p \cdot q = 0, |p|^2 \leq 1\}$, and τ_{S_0} represents a positive Dehn twist about the sphere $S_0 = \{p_i = 0\}$ in the page. The symplectic form on the page is given by $\omega = dp \wedge dq$. The Lagrangian disk $S \coloneqq \{(p, q) \mid q = (0, \dots, 0, 1)\}$ is a Legendrian in the manifold. Here also, we can assume that the binding is thickened.

By the Weinstein neighbourhood theorems, given two Legendrian disks in different contact manifolds, they have contactomorphic neighbourhoods. We will take a $(2n+1)$-ball neighbourhood of S and identify that with a neighourhood of L. We will call the neighbourhood of S as D_S, and that of L as D_L. Assume the pages in the open books are indexed by $S^1 = \mathbb{R}/\mathbb{Z}$. This neighbourhood D will be built by taking thickened neighbourhoods of S (or neighbourhoods of L) in pages $(0, 0.5)$, and neighbourhoods of ∂S in pages $(0.5, 1)$. This gives D the structure of a relative open book, and its intersection with the bindings of the ambient manifolds creates the binding. Hence, it makes sense to talk about pages of D, and other than two singular ones, the pages of D will have two kinds of topology. They will either be $T^*D^n = D^n \times D^n$, or $T^*(S^{n-1} \times I) = D^{n+1} \times S^{n-1}$. Thus $D_L = N_L \cup N_{\partial L} \cup B_L$, where N_L is the union of the neighbourhoods of L, $N_{\partial L}$ is the union of the neighbourhoods of ∂L, and B_L is the portion of the binding that comes within D_L. Similarly, $D_S = N_S \cup N_{\partial S} \cup B_S$. This is the same relative open book decomposition that was mentioned in Example 2.5.3.

Here we describe what we meant by *twisting the pages*. From the pages of M where *neighbourhood of* L were removed, we glue in the complements of ∂S neighbourhood, and the remaining pages, we glue in the complement of *neighbourhood of* S. Which is to say, ∂N_L glues to $\partial N_{\partial S}$, $\partial N_{\partial L}$ glues to ∂N_S.

This gluing is done so that for the *complement of* ∂S *neighbourhood*, which is essentially still $D(T^*S^n)$, a portion is identified with $D(T^*D^n)$, such that it seems as if L is identified with the core of this piece, and the remaining $D(T^*D^n) \sim D^n \times D^n$ is glued on like attaching a Weinstein n-handle along the isotropic sphere ∂L. In the sense of Section 2.5, the initial open book (B, ν) on M is split as the union of two relative open books on $M \setminus D_L$ and D_L. The abstract relative open book description of D_L is the same as in Example 2.5.3, denoted $(\{S_{i,1}\}_{i=1}^4, id)$ while the abstract relative open book description of $M \setminus D_L$ is $(\{S_{i,2}\}_{i=1}^4, \phi_\nu)$, where $S_{2j+1,2}$ is the complement of L, while $S_{2j,2}$ is the complement of ∂L. Similarly, the open book on (S^{2n+1}, ξ_{st}) is split into D_S and $S^{2n+1} \setminus D_S$, as in Example 2.7.4. Denote the relative open book on $S^{2n+1} \setminus D_S$ as $(\{S_{i,3}\}_{i=1}^4, \phi_\nu)$, where $S_{2j,3}$ is the complement of S, while $S_{2j+1,3}$ is the complement of ∂S. Then, the relative open

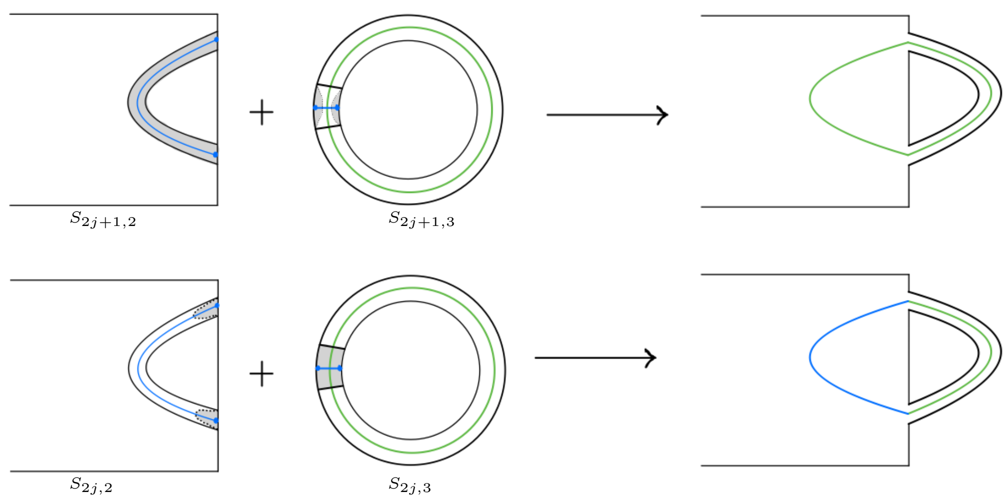


Figure 4.1: This is a schematic of how the pages are being modified under the stabilisation operation. On the left are pages of M, while the annuli represent $D(T^*S^n)$, their core being S_0. The blue arc in the annuli is S. In the page of M, the blue arc represents L. The shaded portion in the top row left represents a page in N_L, while that on the top row middle is a page in $N_{\partial S}$. For the gluing in the top row, the complement of the shaded region in the small rectangular portion of the annulus is first identified with the neighbourhood of L, then the remaining region glues on as a critical Weinstein handle. The gluing in the second row is similar, without the identification step.

books on $M \setminus D_L$ and $S^{2n+1} \setminus D_S$ are glued, matching the indices. The shift in indices from D_L to D_S is what we refer to as *twisting*. A schematic of what is going on is shown in Fig 4.1.

The rest of the section is devoted to working out the technical details of this gluing, and proving that this operation does indeed yield the stabilised open book as defined in Definition 2.7.1. For $n = 1$, this can be done using convex surfaces and foliated open books, and is mentioned in [19] by Licata-Vertesi. To help the reader's intuition, we

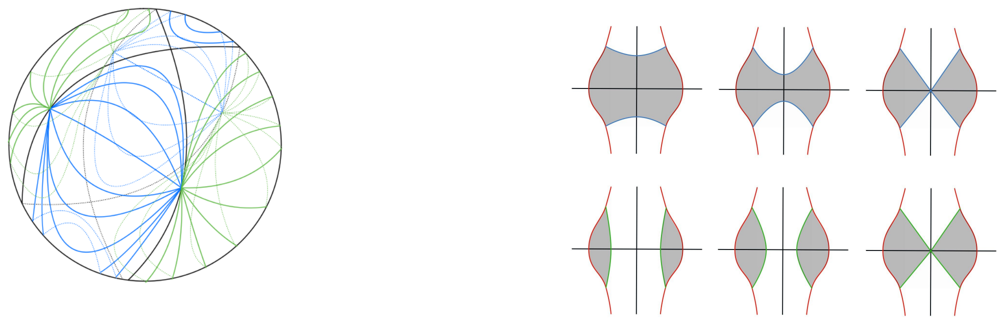


Figure 4.2: The smooth ball D_L that gets removed. The foliation is from its intersection with the pages, and the points where the arcs meet are the binding. On the right, is a schematic of the neighbourhoods whose intersection with D is shown in the left figure. The top three are the neighbourhoods of L, the leftmost corresponding to two blue arcs near the equator, the one to its right two blue arcs nearer the poles, and the rightmost being one of the neighbourhoods contributing the black arcs. Similarly, the lower three represent, from left to right, the neighbourhoods of ∂L contributing green arcs near the equator, ones that go higher (or lower), and the black arcs.

describe that argument first.

Proof of Thm 1.1.4 in the case $n = 1$ (Example 6.6, [19]): The Legendrian L is a properly embedded arc living on the page of an open book supporting (M, ξ), while S is a Legendrian arc in (S^3, ξ_{st}) is S, which is the Legendrian corresponding to a fiber in the $D(T^*S^1)$ page of the open book supporting (S^3, ξ_{st}). This is exactly the annulus open book for (S^3, ξ_{st}), and S is the co-core arc.

As discussed above, a neighbourhood of L in (M, ξ) is D_L is contactomorphic to D_S which is a neighbourhood of S. The balls D_S and D_L can be chosen so that their boundaries are convex, and the leaves of the characteristic foliation correspond to their intersection with the pages. This is pictured in Figure 4.2. The blue represent the intersection of ∂D_S with pages that have a neighbourhood of S inside D, i.e. pages in N_S, while the green represent the intersection with pages that have a neighbourhood of ∂S inside D, i.e., the pages in $N_{\partial S}$. The foliation on $\partial D_L = \partial(M - D_L)$ looks the same. Now glue $M - D_L$ to $S^3 - D_S$ by a rotation that maps the green leaves to the blue leaves. As the leaves of the characteristic foliation are matched up, the contact structure extends over the gluing and the new open book on M is modified exactly as described above. $\square$

4.1.2 Coordinates on the pages, and neighbourhoods of L and ∂L.

The pages $D(T^*D^n)$ intially are parametrised with coordinates from $\mathbb{R}^{2n+2}$, as

$$\{(p_1, p_2, \cdots, p_{n+1}, q_1, q_2, \cdots, q_{n+1}) \mid \sum p_i^2 \leq 1, \sum q_i^2 = 1, \sum p_i q_i = 0\}$$

The canonical primitive of the symplectic form is $\lambda = \sum p_i dq_i$. The disk S, with which the Lagrangian L in the page in M gets identified, is parametrised as: $S = \{q = (0, \cdots, 0, 1)\}$.

Consider $\psi : \mathbb{R} \times (T^*S^n \cap \{q_{n+1} > 0\}) \to \mathbb{R}^{2n+1}$, given by

$$\psi(z, p_1, p_2, \cdots, p_{n+1}, q_1, q_2, \cdots, q_{n+1}) = (z, x_1, \cdots, x_n, y_1, \cdots, y_n)$$
$$\text{where } x_i = q_{n+1} p_i, y_i = \frac{q_i}{q_{n+1}}$$

This allows us to put coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ on a neighbourhood of S on the page, such that $D(T^*S^n) \cap \{q_{n+1} > 0\}$ is identified with

$$\{(x, y) \in \mathbb{R}^{2n} \mid \left(\sum x_i^2 + \left(\sum x_i y_i\right)^2\right) \left(\sum y_i^2 + 1\right) \leq 1\}$$

and S becomes $\{y_i = 0\}$. The following calculation verifies this is a symplectomorphism:

$$
\begin{aligned}
\psi^*(\sum_{i=1}^{n} x_i dy_i) &= \sum_{i=1}^{n}(q_{n+1}p_i)d(\frac{q_i}{q_{n+1}}) \\
&= \sum_{i=1}^{n}(q_{n+1}p_i)\frac{q_{n+1}dq_i - q_i dq_{n+1}}{q_{n+1}^2} \\
&= \sum_{i=1}^{n} p_i dq_i - \frac{dq_{n+1}}{q_{n+1}}\sum_{i=1}^{n} p_i q_i \\
&= \sum_{i=1}^{n} p_i dq_i + p_{n+1}dq_{n+1} \\
&= \sum_{i=1}^{n+1} p_i dq_i
\end{aligned}
$$

The above computation uses that $\sum_{i=1}^{n+1} p_i q_i = 0$ implies $p_{n+1} = -\frac{1}{q_{n+1}}\sum_{i=1}^{n} p_i q_i$. The boundary of $D(T^*(S^n))$, which is given by $|p|^2 = 1$, now becomes

$$\{(x,y) \mid b(x,y) = 1\} \text{ where } b(x,y) = \left(\sum x_i^2 + \left(\sum x_i y_i\right)^2\right)\left(\sum y_i^2 + 1\right)$$

Figure 4.3 illustrates how this looks for $n = 1$ under the coordinate transformation by ψ. The red is the binding.

The blue in Figure 4.3 is the binding "rotated", i.e., $\{(x,y) \mid \{b(y,x) = 1\}$. The intersection of the blue and red is transverse. The yellow is the set of points "inside the page" that satisfy $\{b(x,y) = b(y,x)\}$. Consider the sets $\{b(x,y) - b(y,x) = t\}$, as t runs from 0 to $\frac{1}{2}$. Also, consider a smooth function $\rho(x,y)$ which is 0 at $\{b(x,y) = b(y,x) = 1\}$, and 1 on a large sphere inside the page away from the binding. Now, consider the sets

$$B_t := \{b(y,x) = \rho(x,y)(b(x,y) + t) + (1 - \rho(x,y))\}$$

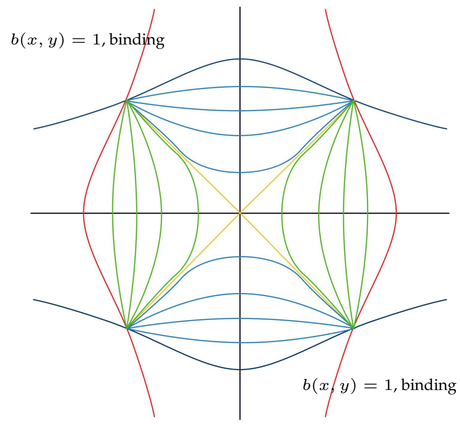


Figure 4.3: Coordinates on a piece of $D(T^*(S^n))$, which is inside the red

For t nonzero, they are indicated by the green in the schematic figure. Define a 1-parameter family of neighbourhoods of L by

$$\nu_t(L) := \{(x,y) \mid b(x,y) \leq 1, b(y,x) \leq \rho(x,y)(b(x,y)+t) + (1-\rho(x,y))b(x,y)\}$$

Similarly, define a 1-parameter family of neighbourhoods of ∂L by

$$\nu_t(\partial L) := \{(x,y) \mid b(x,y) \leq 1, b(x,y) \geq \rho(x,y)(b(y,x)+t) + (1-\rho(x,y))b(y,x)\}$$

Here, t belongs to the interval $(0, 0.5)$. An indicative picture for the 3 dimensional case is Figure 4.3.

Remark 4.1.2. The neighbourhoods $\nu_t(L)$ and $\nu_t(\partial L)$ are precisely the "pages" that build up N_L and $N_{\partial L}$, respectively, from 4.1.1 above.

4.1.3 Coordinates on D.

Now, we are in a position to exactly describe the parametrised smooth neighbourhood of the Lagrangian L that will be removed for stabilisation. Note that the above coordinates parametrise a neighbourhood of L, a Lagrangian disk, sitting on a page in an open book decomposition of M.

In M, suppose the open book is given by $\nu : (M - B) \to S^1$, while suppose the open book on S^{2n+1} is given by $\nu_1 : (S^{2n+1} - B') \to S^1$. Let us first focus on D as a neigbourhood of S on a page in S^{2n+1}, and obtain a parametrisation of it. For the Legendrian L which is Lagrangian in the page, a neighbourhood in M, call it D, can be given standard coordinates from $I \times \mathbb{R}^{2n}$ modulo some identifications and a standard binding piece, (we assume that the monodromy effects are localised in a place away from this neighbourhood). More precisely, if we assume that the page coordinate or S^1 factor of the open book on S^{2n+1} is given by $t \in \mathbb{R}/\mathbb{Z}$, then the monodromy affects the pages in the $t \in [0.7, 0.8]$ interval, and even in those places, it leaves substantial (as required by the next paragraph) neighbourhoods of ∂L unaffected, in the sense that they can be parametrised directly as standard pieces of $I \times \mathbb{R}^{2n}$. What we want is to ensure that the pieces D_L or D_S can be described without the monodromy of the open book affecting them.

Suppose in M, the pages of the open book decomposition are parametrised in the S^1 direction by $t \in \mathbb{R}/\mathbb{Z}$. Consider an interval's worth of L-neighbourhoods parametrised by t, given by:

$$N_L \coloneqq \{(x, y, t) \mid (x, y) \in \nu_{\beta(t)}(L), t \in [0, 0.5]\},$$

and an interval's worth of (∂L)-neighbourhoods parametrised by t, given by:

$$N_{\partial L} \coloneqq \{(x, y, t) \mid (x, y) \in \nu_{\beta(t)}(\partial L), t \in [0.5, 1]\}$$

Here $\beta(t) = (0.5)\sin(2\pi t)$. Also, we can assume that in this local scenario, the binding B is *thickened*, which allows us to choose the standard primitive of the symplectic form, $\sum x_i dy_i$ on the page, giving the contact form $dt + \sum x_i dy_i$ on $N_L \cup N_{\partial L}$ and match it up with an appropriate choice of contact form on the binding, for example as done in Section 2.2, page 4, of [16], to get an explicit description of the contact form restricted to $N_L \cup N_{\partial L} \cup B_L \subset M$. The thickened binding inside D_L can be parametrised as a quotient set $B = \{(x, y, r, t) \mid (x, y) \in \mathbb{R}^{2n}, b(x, y) = 1, r \in [0, 1], t \in \mathbb{R}/\mathbb{Z}, (x, y, 0, t) \sim (x, y, 0, t')\}$, and in D_L, the points in N_L (or $N_{\partial L}$) and ∂B overlap as: $\{(x, y, t) \in N_L \mid b(x, y) = 1\}$ are matched with $\{(x, y, 1, t) \in B\}$ (similarly for $N_{\partial L}$).

We claim that $D_L = N_L \cup N_{\partial L} \cup B_L$ is a smooth $(2n + 1)$-ball. For this, it suffices to check that the boundary $2n$-sphere ∂D_L is smooth. The boundary ∂D_L, in these coordinates, can be described as: $\partial N_L \cup \partial N_{\partial L} \cup \partial B_L$, modulo equivalences that go into the interior of B_L. Note that $\nu_{\beta(0)}(L) = \nu_{\beta(0)}(\partial L)$, and $\nu_{\beta(0.5)}(L) = \nu_{\beta(0.5)}(\partial L)$. ∂D_L can be described as:

$$\bigcup_{t\in[0,0.5]} \{(x, y, t) \mid b(y, x) = \rho(x, y)(b(x, y) + \beta(t)) + (1 - \rho(x, y))\}$$
$$\cup \bigcup_{t\in[0.5,1]} \{(x, y, t) \mid b(x, y) = \rho(x, y)(b(y, x) + \beta(t - 0.5)) + (1 - \rho(x, y))\} \cup \{(x, y, r, t) \in B \mid b(y, x) = 1\}$$

Let us focus on the region where $\rho = 1$ and check it is smooth. In that region ∂D_L can be described as:

$$\bigcup_{t\in[0,0.5]} \{(x,y,t) \mid b(y,x) = b(x,y) + \beta(t)\}$$
$$\cup \bigcup_{t\in[0.5,1]} \{(x,y,t) \in\mid b(x,y) = b(y,x) + \beta(t-0.5)\} \cup \{(x,y,r,t) \in B \mid b(y,x) = 1\}$$

modulo the identifications mentioned above, and $||(x,y)||$ everywhere sufficiently small. Other than the binding, the set can be described as a quotient of $\bigcup_{t\in[0,1]}\{(x,y,t) \mid b(y,x) - b(x,y) = (0.5)\sin(2\pi t)\}$. Thus this is smooth. The smoothness of the whole can be easily checked as it is the quotient under smooth equivalences of smooth pieces. The same holds for D_S.

4.1.4 The surgery step.

The above establishes coordinates on the boundary of D_L, and they work for both D_L and D_S. What we will do now is remove D_L from M and D_S from S^{2n+1}, then glue the complements after shifting indices.

We will surger D_L out and glue in $S^{2n+1} \setminus D_S$ by a map F which will give a contactomorphism between a neighbourhoods of $\partial(M \setminus D_L)$ and $(S^{2n+1} \setminus \partial D_S)$. The details follow.

Notation: Since both D_L and D_S are being given the same coordinates, we will drop the subscript from D and construct F from $\partial D \times I_1$ to $\partial D \times I_2$, for two intervals I_1 and I_2.

Coordinates on $\partial D \times I$. As mentioned above, ∂D can be described as a quotient of:

$$\bigcup_{t\in\mathbb{R}/\mathbb{Z}} \{(x,y,t) \in N_L \cup N_{\partial L} \mid b(y,x) = b(x,y) + \rho(x,y)\beta(t))\}$$

$$\bigcup \{(x,y,r,t) \in B \mid b(y,x) = 1, r \in [0,1]\}$$

where the points (x,y,t), when $b(x,y) = b(y,x) = 1$, are identified with $(x,y,1,t)$, and $(x,y,0,t) \sim (x,y,0,t')$.

On a collar neighbourhood of the boundary in $S^{2n+1} - D$, call it $\partial D \times [0,1]$, where s represents the I direction, a contact form is given by $dt + e^s \sum x_i dy_i$ on the page part, i.e., $(\partial D \times [0,1]) \cap (N_L \cup N_{\partial L})$, which is extended to the interior of the binding part, i.e., $(\partial D \times [0,1]) \cap B$, as $e^s h_1(r) \sum x_i dy_i + h_2(r)dt$, where h_1 and h_2 are described in the following figure. The $s = 1$ end represents the boundary of the manifold.

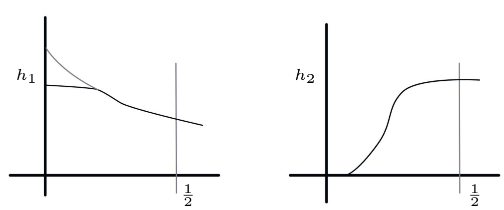


Figure 4.4: The functions h_1 and h_2 for the contact form near the binding.

4.1.5 Gluing the complements of D.

Extend the collar neighbourhood of $\partial(M-D)$ to look like $\partial D\times[-2,0]$, with the coordinate s representing the interval direction, and so, on $[-2,-1]$, the contact form on the page part is given by $e^s dt + \lambda_s$, and on the binding part by $e^s h_1(r)\lambda_s + h_2(r)dt$. Here $\lambda_s :=$ $(-1-s)(\sum x_i dy_i) + (s+2)(\sum(-y_i)dx_i)$, and the $s=0$ end represents the boundary of the manifold. On $s \in [-1,0]$, the contact structure is given by $e^s dt + \lambda_{-1}$ on the page part, and by $e^s h_1(r)\lambda_{-1} + h_2(r)dt$ on the binding. The $s=0$ end represents the boundary of the manifold.

Now, define the contactomorphism between a neighbourhood of $\partial(S^{2n+1}-D)$ and $\partial(M-D)$ as follows:

$$F : \partial D \times [0,1] \to \partial D \times [-1,0]$$

$$(x,y,t,s) \mapsto (-y,x,t+0.5,-s)$$

Using this, we will show in the following section that we can apply Lemma 2.5.4 to glue $S^{2n+1}-D$ to $M-D$, to get back M. We will see that this gives an open book decomposition of M.

4.1.6 An open book decomposition of M.

We would like to show that the above operation induces an honest open book decomposition of M that supports the contact structure on M. The fact that the operation smoothly does not change M is because we removed a ball and glued it back in by a map on the boundary (rotation) which is smoothly isotopic to the identity. We need to ensure that the conditions for Lemma 2.5.4 are met.

The contact manifold M originally has an open book fibration given by $\nu : (M - B) \to S^1$. In the neighbourhood of L we called D, the map ν was given on $D \setminus B$ by $(x, y, t) \mapsto t$. The standard $(2n+1)$-sphere S^{2n+1} originally has an open book fibration given by $\nu_1 : S^{2n+1} - B' \to S^1$, which in the neighbourhood $(D - B')$ was again given by $(x, y, t) \mapsto t$.

Call the fibers of ν to be all $2n$-dimensional Liouville domains symplectomorphic to X. Then, L is a Lagrangian n-disk in X, and ∂L is a Legendrian $(n-1)$-sphere in ∂X. To index the different pages, call $\nu^{-1}(t) = X_t$. For $t \in (0, 0.5)$, $X_t - D$ is a smooth manifold from which a neighbourhood of L has been removed. For $t \in (0.5, 1)$, $X_t - D$ is a smooth manifold from which a neighbourhood of the $(n-1)$-sphere ∂L has been removed. A neighbourhood of L that contains $X_t \cap D$ is parametrised as $\{(x, y) \in \mathbb{R}^{2n} \mid b(x, y) \leq 1\}$, where locally the symplectic form is $dx \wedge dy$ and L is given by $\{y = 0\}$.

The fibers of ν_1 are disk cotangent bundles of the $n-$sphere, call them Y and index them as Y_t as above. Recall that they can be described as subsets of $\mathbb{R}^{2n+2}$ in the following way: $Y = \{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |q|^2 = 1, p \cdot q = 0, |p|^2 \leq 1\}$ with the symplectic form $dp \wedge dq$. As in Section 4.1.2, we can parametrise a part of Y, i.e., where $q_{n+1} > 0$, as subsets of $\mathbb{R}^{2n}$ with the symplectic form $dx \wedge dy$, as $S = Y|_{q_{n+1} > 0} = \{(x, y) \in \mathbb{R}^{2n} \mid b(x, y) \leq 1\}$. This region contains the Lagrangian disk $S = \{q = (0, \ldots, 0, 1)\}$ which becomes $\{y = 0\}$ in S. For $t \in (0, 0.5)$, $Y_t \setminus D_S = Y_t \setminus \nu_{\beta(t)}(S)$ has a neighbourhood of S removed, thus is the unit cotangent bundle over the disk. For $t \in (0.5, 1)$, $Y_t \setminus D_S = Y_t \setminus \nu_{\beta(t)}(\partial L)$ is obtained from Y_t by removing a neighbourhood of the $(n-1)$-knot ∂S from ∂Y_t, and the core S^n stays. Thus it is still symplectic deformation equivalent to Y_t.

After gluing in $S^{2n+1} \setminus D_S$, we want to define an open book fibration on the manifold

$(M \setminus D_L) \cup (S^{2n+1} \setminus D_S)$. On the two pieces, we had the restrictions of the fibrations ν and ν_1, defined on the complements of the respective bindings B and B'.

Define the open book fibration on $((M \setminus D_L) \setminus B) \cup ((S^{2n+1} \setminus D_S) \setminus B')$, denoted ν_{new}, by ν on $((M \setminus D_L) \setminus B)$ and $(\nu_1 - 0.5)$ on $((S^{2n+1} \setminus D_S) \setminus B')$. The map extends smoothly. In the following we verify that it is a fibration. i.e., $\nu_{new}^{-1}(t)$ are symplectomorphic for all t. For that, we need to check that $X_t \cup_F Y_{t+0.5}$ are symplectomorphic for all t.

Consider $U_1 \subset X$ to be the region described by our coordinates as $\{b(x,y) \leq 1\} \cap \{b(y,x) \leq 1\}$. In Y identify a similar neighbourhood of S and call it $U_{1,S}$. Both U_1 and $U_{1,S}$ are diffeomorphic to $D^n \times D^n$, with symplectic form $\omega = \sum dx_i \wedge dy_i$, with Liouville forms $\sum x_i dy_i$ and $\sum(-y_i dx_i)$ respectively, and respective Liouville vector fields $\sum x_i \partial_{x_i}$ and $\sum y_i \partial_{y_i}$. The Liouville vector field points out of the boundary of U_1.

Now, $Y - U_{1,S}$ can be regarded as a Weinstein n-handle, with the same Liouville vector field, with its attaching region being $Y \cap \{b(y,x) = 1\}$.

Near $\partial(Y - U_{1,S})$ we have the coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$, the attaching sphere being given by $\{x_i = 0, \sum y_i^2 = 1\}$, and the attaching boundary of the handle is given by $\{x_i = 0, b(y,x) = 1\}$. Consider now $G : \partial(Y - U_{1,S}) \to \partial(X \cap U_1)$ given by $G(x,y) = (-y,x)$. Using G, one can attach the Weinstein n-handle $Y - U_{1,S}$ to X along ∂L. Call this manifold X_L.

Consider the following: $Y_{t+0.5} - D$ attaching to $X_t - D$. Suppose the gluing of $Y_{t+0.5} - D$ to $X_t - D$ happens in two steps. First the gluing under the restriction of F of $(Y_{t+0.5} - D) \cap U_{1,S}$ to $X_t - D$, and then the remainder of $Y_{t+0.5} - D$ is glued on. After the first step, for every t, we get back X. Then the second step is gluing $Y - U_{1,S}$ to it via G. Thus, for every t, we get X with a Weinstein handle attached along ∂L.

By construction, the contact structure on M post gluing is supported by this open book. The monodromy of the new open book decomposition, by construction, changes by a positive Dehn twist along the sphere $L \cup C$ in the page, where C is the core of the n-handle attached to the page. As this open book has the same pages and monodromy as the stabilisation of the open book ν supporting (M, ξ), the contact structure on M is still contactomorphic to ξ, by Theorem 2.4.4.

This finishes the proof of Theorem 1.1.4.

4.2 Constructions from open books and isotopies

In this section, we will first rigorously define the constructions S_{join} and S_{stab} in open books supporting (M, ξ). Then, we shall establish an isotopy between them, after first ensuring that there is a manifold contactomorphic to (M, ξ) where both the constructions make sense.

At various points in this section, we will do some perturbations by using Legendrians defined by piecewise functions. For that we need to carefully understand how to read the coordinates of such Legendrians in $J^1(S^n)$, which we will address in the following lemma.

Lemma 4.2.1. *Consider $J^1(S^n)$ parametrised as $\{(z, q, p) \mid q^2 = 1, p \cdot q = 0\}$. Given a function $f \in C^\infty_{\mathbb{R}}(S^n)$, $j^1(f)$ defines a Legendrian in $J^1(S^n)$, whose coordinates are given by (z, q, p) such that $z = f(q)$ and $p_i = -\frac{\partial f}{\partial q_i} + (df \cdot q)q_i$, where df is the vector given by $(df)_i = \frac{\partial f}{\partial q_i}$.*

Proof. The idea here is simply that when a function f is defined on the coordinates $(q_1, \ldots, q_{n+1}) \in \mathbb{R}^{n+1}$, the vector $df|_{\mathbb{R}^{n+1}} := (\frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_{n+1}})$ belongs to $T^*(\mathbb{R}^{n+1})$. Thus for $f \in C^\infty_{\mathbb{R}}(S^n)$ but defined on the coordinates $(q_1, \ldots q_{n+1})$, $df \in T^*(S^n)$ is given by the projection of $df|_{\mathbb{R}^{n+1}}$ from $T^*(\mathbb{R}^{n+1})$ to $T^*(S^n)$. $\square$

4.2.1 Making sense of join Legendrian.

The join Legendrian $L_1 \cup L_2$ is described in M (pre-stabilisation) in the following way. Suppose L_1 is a Lagrangian n-disk on the page, and is also a Legendrian in the manifold. By Lemma 4.1.1, this can be arranged. Consider two copies L_1 and L_2 in two pages. They can be glued through the binding. The details follow, and the schematic is given in Figure 4.5.

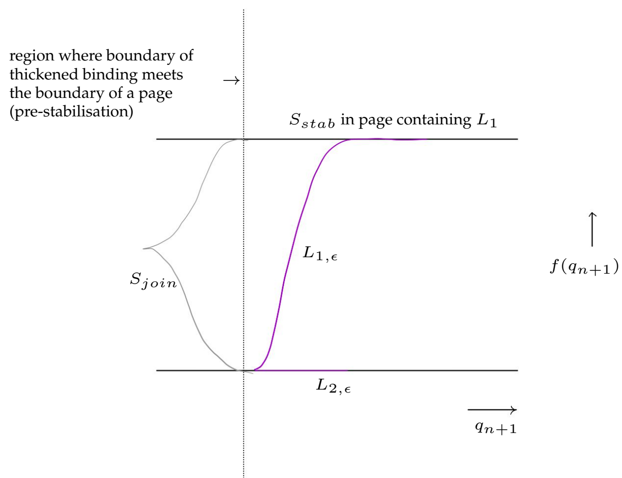


Figure 4.5: The above schematic will describe both pre- and post-stabilisation scenarios. Before stabilisation, the left of the vertical represents the thickened binding near $J^1(L) \subset M$. The horizontal lines represent $(z = c)$ slices, and the q_{n+1} coordinate is plotted horizontally. After stabilisation, the page extends to the left, and the picture represents $J^1(S_{Stab}(L))$.

The open book on M before stabilisation, by definition, has identity monodromy near the boundaries of the pages. In particular, if we consider all the pages and look near ∂L_1, we can parametrise it as $J^1(D^n)$, where $\{z = c\}$ represents the pages and the Legendrians $\{(z, q, 0) \mid z \text{ constant}\}$ represent copies of L near the boundary in different pages. Once two copies of L enter the binding, they can be joined up using the 1-jet of functions defined on q_{n+1} as depicted in Figure 4.5. Here we are assuming that the contact form on the binding is adjusted so that it still looks like $J^1(D^n)$ near the boundaries of the pages. This amounts to a careful choice of the functions h_1 and h_2 in Figure 4.4. This joined up sphere is $S_{join}(L)$. The portion to the left of the vertical dotted line in Figure 4.5 is the binding, while to the right are the pages. We will parametrise the n-disk as $\{q \in \mathbb{R}^{n+1} \mid q^2 = 1, q_{n+1} \geq 0\}$.

Now, the way stabilisation is described in Section 4.1, the pages are modified in a small neighourhood of ∂L. Thus, we can interpret in these coordinates as saying that as long as $\sum p_i^2$ is large enough, the part of the page is unchanged after stabilisation. In more precise terms, we make the assumption that $\{(z, q, p) \in T^*S^n \mid q \leq \sqrt{1+\epsilon}\}$ is the region that is affected. Also, we can assume that this ϵ is the same for the Legendrian surgery that happens during stabilisation, as described in Section 2.3.

This is the choice of ϵ that will be used for the remainder of this section.

Thus, we can modify $S_{join}(L)$ by choosing 1-jets of appropriate functions. At the final stage of this modification, we get $S_{join}(L)$ obtained by gluing two copies of L ”through the pages”. Since by Lemma 4.2.1 the p_i coordinates depend on this slope, by choosing a high enough slope for the function we can ensure that the p_i coordinates are large enough, and hence this perturbation of $S_{join}(L)$ lives completely outside the region which is cut out and replaced by stabilisation, and thus is well-defined after stabilisation as well. Also, to make it easier for us to define the isotopy later, we can ensure that $S_{join}(L)$ does not intersect the interior of the region which is identified with S_{-1} and replaced with S_1.

We can choose a Legendrian using a function that depends only on q_{n+1}. By Lemma 4.2.1, for a Legendrian $j^1(H(q_{n+1})) \subset J^1(S^n)$, $p_i = \frac{dH}{dq_{n+1}} q_{n+1} q_i$ for $i = 1, \ldots, n$, and $p_{n+1} =$

$\frac{dH}{dq_{n+1}}(-1 + q_{n+1}^2)$. Thus,

$$z^2+\sum_{i=1}^{n+1} p_i^2 = H^2+\left(\frac{dH}{dq_{n+1}}\right)^2 q_{n+1}^2(1-q_{n+1}^2)+\left(\frac{dH}{dq_{n+1}}\right)^2 (1-q_{n+1}^2)^2 = H^2+\left(\frac{dH}{dq_{n+1}}\right)^2 (q_{n+1}^2+1)$$

By choosing H defined on $\{q_{n+1} \geq \epsilon\}$ such that $H^2 + (\frac{dH}{dq_{n+1}})^2(q_{n+1}^2 + 1) > (1+\epsilon)$, and such that $H(\epsilon) = H'(\epsilon) = 0$, we can join it with the disk $\{q_{n+1} \geq \epsilon\}$ in the zero section to get a perturbed S_{join}. Such an H would need to satisfy

$$\left(\frac{dH}{dq_{n+1}}\right)^2 \geq \frac{\sqrt{1+\epsilon}(1-t)}{1+q_{n+1}^2} \text{ whenever } H = t\sqrt{1+\epsilon}$$

Such an H can be found, e.g. as shown in Figure 4.5. Consider a real valued function $H(x)$ on $[0,1]$ that starts at 0 with value $-\sqrt{1+\epsilon}$ and slope 0, then increases fast enough so that its slope beats the above inequality, and then reduces to 0 when the value of the function reaches $\sqrt{1+\epsilon}$ at $x = \frac{1}{2}$.

This perturbed S_{join} can be seen to be the union of two disks, all described in Figure 4.5, namely:

- $L_{1,\epsilon}$, the disk corresponding to $j^1(H)$, i.e., the top portion of Figure 4.5
- $L_{2,\epsilon} \coloneqq \{(c_2, q, 0) \mid q_{n+1} \geq \epsilon\}$ in page that contains L_2

The isotopy from S_{join} as originally defined to this can be done in M before stabilisation.

4.2.2 The stabilisation Legendrian

Our goal is to show that in the stabilised manifold, $S_{join}(L)$ is isotopic through smooth Legendrians to $S_{stab}(L)$. Recall that when M is stabilised, each page is modified by adding a Weinstein n-handle along ∂L, and the core of this handle glues to L to give a Legendrian sphere, which we call $S_{stab}(L)$.

First note that locally, a neighbourhood of $S_{stab}(L)$ can be identified with $J^1(S^n)$ with coordinates $\{(z, q, p) \mid q^2 = 1\}$, where $S_{stab}(L)$ is given by $(0, q, 0)$. By the description of stabilisation in Section 4.1, we can assume that the disk L is exactly the part of $S_{stab}(L)$ corresponding to $q_{n+1} \geq 0$, and that the page outside a small neighboourhood of the sphere $\{q_{n+1} = 0\}$ is exactly the page of the open book before stabilisation.

In Figure 4.5, $S_{stab}(L)$ built from the same page that contained L_1 (above), is seen as the union of the following two disks:

- $L_\epsilon := \{(c_1, q, 0) \mid q_{n+1} \geq -\epsilon\}$
- $C_\epsilon := \{(c_2, q, 0) \mid q_{n+1} \leq -\epsilon\}$

4.2.3 Proof of isotopy.

In the stabilised manifold, we now have the perturbed $S_{join}(L)$ and the $S_{stab}(L)$, as described in Sections 4.2.1 and 4.2.2 respectively. The perturbed $S_{join}(L)$ can be seen as the union of two disks as described in Section 4.2.1, while the perturbed $S_{stab}(L)$ can be seen as the union of two disks as described in Section 4.2.2. The proof of the isotopy will involve moving C_ϵ to $L_{2,\epsilon}$ and L_ϵ to $L_{1,\epsilon}$, while taking care that their overlaps move consistently. The following subsection is a reading guide for the rest of the proof.

4.2.4 Proof outline

1. We will be using different local models at different parts of the argument. As shown in Figure 4.6, a neighbourhood of $S_{stab}(L)$ can be thought of as $I \times T^*(S^n)$ with a piece cut out and replaced. The orange region is like a portal that represents the "surgery torus" that gets glued in. Recall from Section **??**, that Legendrian surgery is done by identifying a neighbourhood of the Legendrian sphere with S_{-1}, then replacing $S_{-1,c}$ by $S_{1,c}$. We will use a perturbed region called S_1^{st} to construct the first half of the isotopy, then move the pieces to $S_{1,c}$. What will be important is that $S_{-1} \cap S_1 = S_{-1} \cap S_1^{st}$.

   Using the contactomorphisms $\psi_W : J^1(S^n) \to S_{-1}$, and $\psi : S_1^{st} \to J^1(S^n)$, we will get local coordinates from $J^1(S^n)$ on the two pieces $(J^1(S^n) \setminus \psi_W^{-1}(S_{-1,c}))$, and $\psi(S_1^{st})$. We carefully constructed $S_{join}(L) = L_{1,\epsilon} \cup L_{2,\epsilon}$ so that $L_{1,\epsilon}$ lives on $\partial(J^1(S^n) \setminus \psi_W^{-1}(S_{-1,c}))$. The map $\psi \circ \psi_W$ and its inverse makes sense on $\psi^{-1}(S_{-1} \cap S_1$.

2. In Section 4.2.5 we will construct a family of Legendrian disks that live in S_1^{st}, with boundary on $S_{-1} \cap S_1^{st}$, that define an isotopy between $\psi_W(C_\epsilon)$ and $\psi_W(L_{2,\epsilon})$. We first construct the disks in $\psi(S_1^{st})$, and then bring them to S_1^{st} using ψ^{-1}. A nice observation is that in $\psi(S_1^{st})$, these two are disks that appear in the front projection as shown in Figure 4.7, and the isotopy is essentially twisting their front projections once to a point and then back to the opposite orientation. In some sense, these are a pair of Reidemeister 1-moves.

3. In Section 4.2.5 we show that the above disks can be used to get a family of Legendrian disks in $S_{1,c}$, with boundary on $S_{-1} \cap S_1$, that define an isotopy between $\psi_W(C_\epsilon)$ and $\psi_W(L_{2,\epsilon})$. One can imagine that at this point, in Figure 4.6, we have a

1-parameter family of disks indexed by $t \in [-1, 1]$, such that for $t \in (-1, 1)$, they live inside the orange portal, but we can see their boundaries at the boundary of the orange region. At $\{t = 1\}$ we get C_ϵ, while at $\{t = -1\}$ we get $L_{2,\epsilon}$.

4. The last step, done in Section 4.2.5, is to define a family of disks that will give an isotopy between the other halves of $S_{stab}(L)$ and $S_{join}(L)$, i.e., L_ϵ and $L_{1,epsilon}$. What we want is a family of disks, that in Figure 4.6, will have their boundaries match up with the boundaries of the disks living inside the orange, and their interiors will be disjoint. The way we do that is ensure we can define disks with the boundary condition of smoothly matching up with the boundaries of the disks found in Section 4.2.5, such that their interiors live in $(J^1(S^n) \setminus \psi_W^{-1}(S_{-1,c}))$.

4.2.5 Modelling the Legendrian surgery:

We will assume that in the stabilised manifold the Legendrian surgery corresponding to the Dehn twist about $S_{stab}(L)$ happens away from the other monodromy of the open book. Stabilisation can be thought of as a two step process happening on the top convex boundary of the symplectisation $(I \times M)$, where the first step is attaching an index (n) Weinstein handle along the isotropic sphere ∂L living in the binding, and the next step is attaching an $(n+1)$-handle along $L \cup C$. After the first step, on the boundary, we get a manifold M' presented by an open book ν_{int} (as in the open book in the intermediate step) whose pages have been modified by adding an n-handle to each, while the monodromy has been extended by identity over this handle. Then, the next step modifies the monodromy by adding a Dehn twist. We can call $L \cup C$ inside M' as $S'_{stab}(L)$. After the surgery, we get to M and the Legendrian sphere $S'_{stab}(L)$ becomes $S_{stab}(L)$.

Recall that the identification of S_{-1} with the standard neighbourhood of a Legendrian sphere, $\mathbb{R} \times T^*S^n$, in Section 2.3, is done via the following map:

$$\psi_W : \mathbb{R} \times T^*S^n \to S_{-1}$$
$$(z, q, p) \to (zq + p, q) \text{ which implies}$$
$$\psi_W^{-1}(z_1, w_1) = (z_1 \cdot w_1, w_1, z_1 - (z_1 \cdot w_1)w_1)$$

Locally, we can think of a neighbourhood $[-\sqrt{1+\epsilon}, \sqrt{1+\epsilon}] \times T^*S^n$ of $S'_{stab}(L) \subset M'$, across pages indexed by $t \in [0.4, 0.6]$ which is removed and reglued to achieve the above, i.e., under the contactomorphism between $\mathbb{R} \times T^*(S^n)$ and S_{-1}, $[-\sqrt{1+\epsilon}, \sqrt{1+\epsilon}] \times T^*S^n$ is the region that corresponds to $\{S_{-1} \mid z^2 \leq 1 + \epsilon\}$, and this corresponds to the neighbourhoods of L in pages in $[0.4, 0.6]$. As the same indexing of pages outside $t \in (0.4, 0.6)$ carries over to M, the following sentence makes sense. We will consider $S_{stab}(L)$ living in $t = 0.4$ and $S_{join}(L)$ formed by joining copies of L in $t = 0.4$ and $t = 0.6$.

Now, we will perturb both $S_{join}(L)$ and $S_{stab}(L)$ as described in 4.2.1 and 4.2.2 respectively, and see them both as unions of two pieces. We will construct the isotopy by constructing isotopies between the two pieces and matching them up along their overlaps. The main idea will be to construct families of functions whose 1-jets will build the isotopies piecewise and ensure they glue together.

Isotopy of C_ϵ to $L_{2,\epsilon}$

Here we first isotope half of $S_{join}(L)$ to half of $S_{stab}(L)$. These halves are seen described in Figure 4.5. C_ϵ is the part of $S_{stab}(L)$ that lives to the left of the picture. It is part of the core of the Weinstein handle that was attached to the page during the stabilisation procedure for the open book.

$$L_{2,\epsilon} = \{(z,q,p) \in (-\sqrt{1+\epsilon}, \sqrt{1+\epsilon}) \times T^*S^n \mid z = -\sqrt{1+\epsilon}, q_{n+1} \geq \epsilon, p \equiv 0\}$$

$$C_\epsilon = \{(z,q,p) \in [-\sqrt{1+\epsilon}, \sqrt{1+\epsilon}] \times T^*S^n \mid z = \sqrt{1+\epsilon}, q_{n+1} \leq -\epsilon, p \equiv 0\}$$

The idea will be to construct an isotopy between $L_{2,\epsilon}$ and C_ϵ inside the piece that is glued in during the Legendrian surgery. A schematic is shown in Fig 4.6.

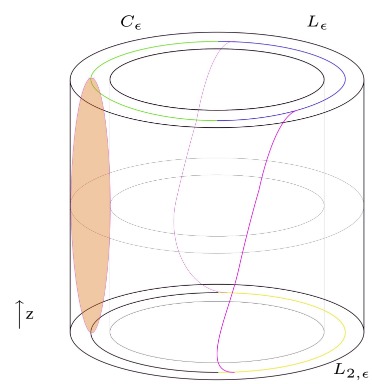


Figure 4.6: This is a schematic representing the region where the Legendrian surgery happens. This is a $J^1(S^n)$ neighbourhood of $S_{stab}(L)$, and is a 3-D representation of Figure 4.5. The orange is a cross section at $\{q = c\}$ of $\psi_W^{-1}(S_{-1,c})$, which is removed during Legendrian surgery. So only the closure of the complement of the orange and the solid torus it sweeps out is actually part of the manifold M. The cores of the annuli represent isotopic copies of the Legendrian $S_{stab}(L)$. The purple lines and their union with a part of the blue arc represents $L_{1,\epsilon}$. The purple is chosen to lie in the closure of the complement of the orange region, which represents the fact that this copy of $S_{join}(L)$ was constructed to lie out of the interior of the Legendrian surgery region.

We will follow the convention and notation from 2.3 for Legendrian surgery. First, we consider $S'_{stab}(L)$ lying on a page of M' with the open book ν_{int}. A neighbourhood of $S'_{stab}(L)$ is contactomorphic to $S_{-1} = \{(z,w) \in \mathbb{R}^{2n+2} \mid |w|^2 = 1\}$ with the contact form $2zdw + wdz$, induced by the symplectic dilation $X = 2z\partial_z - w\partial_w$. This is a contact hypersurface of $(\mathbb{R}^{2n+2}, \omega_{st})$. The surgery will be replacing S_{-1} by S_1, another contact hypersurface, given by $S_1 = \{(z, w) \in \mathbb{R}^{2n+2} \mid f(|w|^2) - g(|z|^2) = 0\}$, with a similarly induced contact form by X. The intersection of S_1 with S_{-1} is $\{(z, w) \in \mathbb{R}^{2n+2} \mid |z|^2 \geq 1 + \epsilon, |w|^2 = 1\}$. The surgery replaces $\{S_{-1} \mid |z|^2 \leq 1 + \epsilon\}$ with $\{S_1 \mid |z|^2 \leq 1 + \epsilon\}$. We will call these $S_{-1,c}$ and $S_{1,c}$ respectively.

The disks we are interested in are described as follows in S_{-1}:

$$\psi_W(L_{2,\epsilon}) = \{(-\sqrt{1+\epsilon})q, q) \mid q_{n+1} \geq \epsilon\}$$

$$\psi_W(C_\epsilon) = \{(\sqrt{1+\epsilon})q, q) \mid q_{n+1} \leq -\epsilon\}$$

Clearly, they can be seen as living in $S_{-1} \cap S_1 = \{(z, w) \in \mathbb{R}^{2(n+1)} \mid |z|^2 = (1+\epsilon), |w|^2 = 1\}$.

Finding isotopy inside S_1: Define $S_1^{st} \coloneqq \{(z, w) \mid |z|^2 = 1 + \epsilon\}$. Clearly, S_1^{st} is also transverse to X and hence inherits a contact structure. Now, consider the following contactomorphism

$$\psi : S_1^{st} \to \mathbb{R} \times T^*(S^n)$$

given by

$$\psi(z, w) = \left(\frac{2z.w}{\sqrt{1+\epsilon}}, -\frac{z}{\sqrt{1+\epsilon}}, w - \frac{(z.w)z}{1+\epsilon} \right) \tag{4.1}$$

Using this, we can see the disks we are interested in inside $J^1(S^n) = \mathbb{R} \times T^*(S^n)$.

$$\psi \circ \psi_W(L_{2,\epsilon}) = \{(-2, q, 0) \mid q_{n+1} \geq \epsilon\}$$

$$\psi \circ \psi_W(C_\epsilon) = \{(2, -q, 0) \mid q_{n+1} \leq -\epsilon\}$$

Remark 4.2.2. At this point, one could scale the z and q coordinates to get the above two isotopic as sets. However, for our purpose, we need the isotopy to respect the parametrisation and the induced orientation, i.e., under the isotopy, we want the point $(-2, q, 0) \in \psi \circ \psi_W(L_{2,\epsilon})$ to flow to $(2, -q, 0) \in \psi \circ \psi_W(C_\epsilon)$. So some more work needs to be done. The rest of the isotopy is essentially two Reidemeister twists in $J^1(S^n)$. A schematic is shown in Figure 4.7.

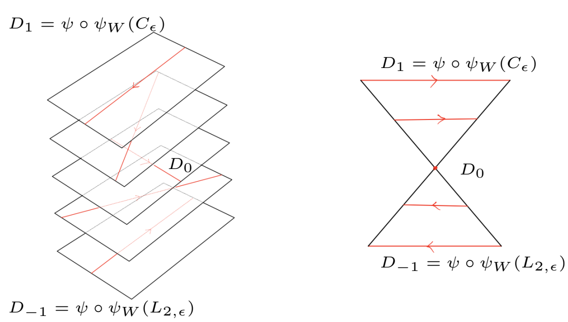


Figure 4.7: This is a schematic of what happens in $\psi(S_1^{std})$. The vertical direction is the z direction, and the figure represents what happens in the isotopy, and how it is really a Reidemeister 1-move. The right hand side represents the front projection, by projecting out the p coordinates.

Consider the unit n-disk $D_{model} \coloneqq \{(x_1, \ldots, x_n) \mid \sum x_i^2 \leq 1\}$.

For notational convenience, we will define the following

$$\epsilon_t \coloneqq (\epsilon - 1)|t| + 1$$

Also, for $t \in (0, 1]$, consider a sequence of smooth functions F_t defined on the 1-parameter family of disks $\{(2t, q, 0) \mid q_{n+1} \geq \epsilon_t\}$ in $[-2, 2] \times T^*S^n$, that satisfy the piecewise properties

$$
\begin{aligned}
F_t &\equiv 2t - \delta && \text{when } q_{n+1} > \epsilon_t + \delta \\
&= \frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)} \sum_{i=1}^{n} \frac{q_i^2}{2} + R_t && \text{near } q_{n+1} = \epsilon_t
\end{aligned}
$$

F_1 is the constant function 2. R_t is there to ensure that at the boundary, the value of F_t is exactly $2t$. $K_\epsilon(t)$ will be defined to satisfy a boundary condition for the disks. δ is small and arbitrary.

Now, define a 1-parameter family of maps $D_t : D_{model} \to [-2, 2] \times T^*S^n$ by

$$
q_i(D_t(x_1, \ldots, x_n)) \coloneqq x_i \sqrt{1 - \epsilon_t^2}
$$

for $i = 1, \ldots, n$ and

$$
q_{n+1}(D_t(x_1, \ldots, x_n)) \coloneqq \sqrt{1 - \sum_{i=1}^{n} q_i^2}
$$

$$
z(D_t(x_1, \ldots, x_n)) \coloneqq F_t(q)
$$

$$
\begin{aligned}
p_i(D_t(x_1,\ldots,x_n)) = -\frac{\partial F_t}{\partial q_i} + (dF_t.q)q_i \quad &\text{when } q_{n+1} \geq \epsilon_t \\
\implies \text{when } q_{n+1} = \epsilon_t& \\
p_i = -\frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)}q_i &+ \frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)}(1-\epsilon_t^2)q_i \\
= -\frac{\sqrt{1-t^2}}{K_\epsilon(t)}\epsilon_t^2 x_i& \\
\text{for } i = 1,\ldots,n, \text{ and}& \\
p_{n+1} = \frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)}(1-\epsilon_t^2)\epsilon_t&
\end{aligned}
$$

From 4.1, we can see that

$$\psi^{-1}(z,q,p) = \left((-\sqrt{1+\epsilon})q, p - \frac{zq}{2}\right) \tag{4.2}$$

So $\psi(S_1 \cap S_{-1})$ is exactly the set $\frac{z^2}{4} + \sum_{i=1}^{n+1} p_i^2 = 1$. Setting this condition for the disks at the extremes, i.e. when $q_{n+1}(D_t) = \epsilon_t$, we will derive the value of $K_\epsilon(t)$, as follows. (Note that at these points $\sum_{i=1}^{n} x_i^2 = 1$):

$$\frac{z^2}{4} + \sum_{i=1}^{n+1} p_i^2 = 1 \implies \sum_{i=1}^{n+1} p_i^2 = 1 - t^2$$

plugging in the values of p_i's, we get

$$\sum_{i=1}^{n}(-\frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)}q_i + \frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)}(1-\epsilon_t^2)q_i)^2 + (\frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)}(1-\epsilon_t^2)\epsilon_t)^2 = 1 - t^2$$

which implies $\frac{1-t^2}{(1-\epsilon_t^2)K_\epsilon(t)^2}(\epsilon_t^4 \sum_{i=1}^{n} q_i^2 + (1-\epsilon_t^2)^2\epsilon_t^2) = 1 - t^2$

which implies $\frac{1}{K_\epsilon(t)^2}(\epsilon_t^4 \sum_{i=1}^{n} x_i^2 + (1-\epsilon_t^2)\epsilon_t^2) = 1$ from which we see that $K_\epsilon(t)^2 = \epsilon_t^2$

By Lemma 4.2.1, the D_t's define a 1-parameter family of smooth Legendrian disks in $[-2,2] \times T^*S^n$. Further, if we define $D_0 : D_{model} \to [-2,2] \times T^*S^n$ by

$z(D_0) = q_i(D_0) = 0$ for $i = 1, \dots, n, q_{n+1}(D_0) = 1, p_i(D_0) = -x_i$ for $i = 1, \dots, n, p_{n+1}(D_0) = 0$

then we have a smooth family of disks D_t for $t \in [0,1]$. D_1 is exactly the disk $\psi \circ \psi_W(C_\epsilon)$.

The above thus defines an isotopy from $\psi \circ \psi_W(C_\epsilon)$ to D_0, which one can see in Figure 4.7 as being an isotopy from the top disk which is part of the core of T^*S^n, to the disk "transverse" to the core. One can think of this as one Reidemeister-1 move, as suggested in Figure 4.7. Now extend by an isotopy from D_0 to D_{-1} where D_{-1} is exactly $\psi \circ \psi_W(L_{2,\epsilon})$, defined as follows:

For $t \in (0,-1]$, define the functions as follows:

$$F_t = \begin{cases} 2t+\delta & \text{when } q_{n+1} > \epsilon_t + \delta \\ = -\frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2}K_\epsilon(t)} \sum_{i=1}^n \frac{q_i^2}{2} + R_t & \text{when } q_{n+1} = \epsilon_t \end{cases}$$

Now, analogous to above, for $t \in (0,-1])$ define a 1-parameter family of maps $D_t : D_{model} \to [-2,2] \times T^*S^n$ by

$$q_i(D_t(x_1,\ldots,x_n)) \coloneqq -x_i\sqrt{1-\epsilon_t^2}$$

for $i = 1,\ldots,n$ and

$$q_{n+1}(D_t(x_1,\ldots,x_n)) \coloneqq \sqrt{1-\sum_{i=1}^n q_i^2}$$

$$z(D_t(x_1,\ldots,x_n)) \coloneqq F_t(q)$$

$$p_i(D_t(x_1,\ldots,x_n)) = -\frac{\partial F_t}{\partial q_i} + (dF_t.q)q_i \qquad \text{when } q_{n+1} \geq \epsilon_t$$

This will define an isotopy from D_0 to $\psi \circ \psi_W(L_{2,\epsilon})$, and putting everything together, we have defined a smooth 1-parameter family of Legendrians that give an isotopy between $\psi \circ \psi_W(C_\epsilon)$ and $\psi \circ \psi_W(L_{2,\epsilon})$, with their boundaries on $\psi(S_1 \cap S_{-1})$. This family can be pulled back via ψ^{-1} to an isotopy in S_1^{std} between $\psi_W(C_\epsilon)$ and $\psi_W(L_{2,\epsilon})$, with their boundaries on $S_1 \cap S_{-1}$. We will denote this family of disks in S_1^{st} by D_t^{st}, where $D_1^{st} = \psi_W(C_\epsilon)$ and $D_{-1}^{st} = \psi_W(L_{2,\epsilon})$.

Contact isotopy from S_1^{std} to S_1

Consider a sequence of subsets of $\mathbb{R}^{2n+2}$ defined by $S_{1,t} = \{(z,w) \in \mathbb{R}^{2n+2} \mid f_t(|w|^2) - g_t(|z|^2) = 0, |z|^2 \leq 1+\epsilon\}$, where $f_1 = f$ and $g_1 = g$ are the functions pictured in Fig 4.4, $f_0(x) \equiv 1+\epsilon, g_0(x) = x$, and f_t and g_t are endpoint preserving smooth homotopies between f_0 and f_1, and g_0 and g_1, respectively, in the region $\{|z|^2 \leq 1+\epsilon\}$. Thus, $S_{1,0} = \{(z,w) \in \mathbb{R}^{2n+2} \mid |z|^2 = 1+\epsilon\} = S_1^{st}$, and $S_{1,1} = S_1$. As f_t and g_t have the same values when their arguments take the extremal values 1 and $1+\epsilon$ respectively, so all the $S_{1,t}$ have the same boundary at $S_1 \cap S_{-1}$. All the $S_{1,t}$ are transverse to the symplectic dilation X, thus are a smooth family of contact embeddings of $I \times D(T^*S^n)$ in $\mathbb{R}^{2n+2}$, that agree on the boundary $S^n \times S^n$. Pulling the contact form on $S_{1,t}$ back via the embedding to $S^n \times D^{n+1}$, we get a 1-parameter family of contact forms that agree on the boundary, call them α_t. Now, Moser's method constructs on $S^n \times D^{n+1}$ a vector field that is 0 where the forms agree, and flowing along this vector field gives a contactomorphism between $(S^n \times D^{n+1}, ker(\alpha_0))$ and $(S^n \times D^{n+1}, ker(\alpha_1))$, since the support of the vector field is compact. This gives a contact isotopy taking S_1^{st} to S_1, and we can use that to get a family of disks that live in S_1.

This means that the isotopy in S_1^{st} can be identified with an isotopy in S_1. Thus, we have a family of disks D_t^1 in S_1 such that $\partial D_t^1 = \partial D_t^{st}$ for all t. Moreover, $D_1^1 = D_1^{st} = \psi_W(C_\epsilon)$ and $D_{-1}^1 = D_{-1}^{st} = \psi_W(L_{2,\epsilon})$.

Our next step will be to build a family of disks to isotope L_ϵ to $L_{2,\epsilon}$, but ensuring that the boundaries of these disks match up with the boundaries of the $\psi_W^{-1}(D_t^1)$ we have found till now.

Isotopy of L_ϵ to $L_{1,\epsilon}$

In this subsection we will construct an isotopy between the remaining halves of $S_{join}(L)$ and $S_{stab}(L)$. We will have to be extra careful, as the family of Legendrian disks we will construct will have to join up with the family of Legendrian disks found above in subsections 4.2.5 and 4.2.5, to give a family of spheres between $S_{join}(L)$ and $S_{stab}(L)$.

We will first identify the boundaries of the disks D_t for $t \in [-1, 1]$. By the description above, we can see that for $t \in [0, 1]$,

$$\begin{aligned}
\partial(D_t) &= \{(z_t, q_t, p_t)\} \quad \text{where} \\
z_t &= 2t \\
q_{i,t} &= q_i(S_t(x_1, \ldots, x_n)) = x_i\sqrt{1-\epsilon_t^2}, \text{ for } i = 1, \ldots, n \\
q_{n+1,t} &= \epsilon_t \\
p_{i,t} &= p_i(S_t(x_1, \ldots, x_n)) = -\epsilon_t(\sqrt{1-t^2})x_i \text{ where } \textstyle\sum x_i^2 = 1, \text{ for } i = 1, \ldots, n, \\
p_{n+1,t} &= \sqrt{(1-t^2)(1-\epsilon_t^2)}
\end{aligned}$$

while for $t \in [-1, 0]$

$$\begin{aligned}
\partial(D_t) &= \{(z_t, q_t, p_t)\} \quad \text{where} \\
z_t &= 2t \\
q_{i,t} &= q_i(S_t(x_1, \dots, x_n)) = -x_i\sqrt{1-\epsilon_t^2}, \text{ for } i = 1, \dots, n \\
q_{n+1,t} &= \epsilon_t \\
p_{i,t} &= p_i(S_t(x_1, \dots, x_n)) = -\epsilon_t(\sqrt{1-t^2})x_i \text{ where } \sum x_i^2 = 1, \text{ for } i = 1, \dots, n, \\
p_{n+1,t} &= -\sqrt{(1-t^2)(1-\epsilon_t^2)}
\end{aligned}$$

We will want to build a family $(D^n)_t$ for $t \in [-1, 1]$ living in S_{-1}, such that $(\partial D^n)_t = \partial D_t$ as described above, $(D^n)_1 = \psi_W(L_\epsilon)$, $(D^n)_{-1} = \psi_W(L_{2,\epsilon})$. We will first see these boundaries in $\psi_W^{-1}(S_{-1})$, and for that we will see the images of the above via $\psi_W^{-1} \circ \psi^{-1} : \mathbb{R} \times T^*S^n \mid_{\frac{z^2}{4}+p^2=1} \to \mathbb{R} \times T^*S^n$, which is given by:

$$\psi_W^{-1} \circ \psi^{-1}(z, q, p) = \left(\frac{z\sqrt{1+\epsilon}}{2}, p - \frac{zq}{2}, -q\sqrt{1+\epsilon} - \frac{z\sqrt{1+\epsilon}}{2}(p - \frac{zq}{2}) \right)$$

The coordinates of $\psi_W^{-1}(\partial D_t)$ are as follows:

for $t \in [0, 1]$,

$$
\begin{aligned}
\psi_W^{-1} \circ \psi^{-1}(\partial D_t) &= \{(z_t, q_t, p_t)\} \quad \text{where} \\
z_t &= t\sqrt{1+\epsilon} \\
q_{i,t} &= -\epsilon_t(\sqrt{1-t^2})x_i - t(\sqrt{1-\epsilon_t^2})x_i \\
q_{n+1,t} &= \sqrt{(1-t^2)(1-\epsilon_t^2)} - t\epsilon_t \\
p_{i,t} &= (\sqrt{1+\epsilon})x_i\Big(- \sqrt{1-\epsilon_t^2} - t(-\epsilon_t(\sqrt{1-t^2}) - t(\sqrt{1-\epsilon_t^2}))\Big) \\
p_{n+1,t} &= (\sqrt{1+\epsilon})\Big(- \epsilon_t - t(\sqrt{(1-t^2)(1-\epsilon_t^2)} - t\epsilon_t)\Big)
\end{aligned}
$$

for $t \in [-1, 0]$,

$$
\begin{aligned}
\psi_W^{-1} \circ \psi^{-1}(\partial D_t) &= \{(z_t, q_t, p_t)\} \quad \text{where} \\
z_t &= t\sqrt{1+\epsilon} \\
q_{i,t} &= -\epsilon_t(\sqrt{1-t^2})x_i + t(\sqrt{1-\epsilon_t^2})x_i \\
q_{n+1,t} &= -\sqrt{(1-t^2)(1-\epsilon_t^2)} - t\epsilon_t \\
p_{i,t} &= (\sqrt{1+\epsilon})x_i\Big(\sqrt{1-\epsilon_t^2} - t(-\epsilon_t(\sqrt{1-t^2}) + t(\sqrt{1-\epsilon_t^2}))\Big) \\
p_{n+1,t} &= (\sqrt{1+\epsilon})\Big(- \epsilon_t - t(-\sqrt{(1-t^2)(1-\epsilon_t^2)} - t\epsilon_t)\Big)
\end{aligned}
$$

By choosing functions appropriately we can define an isotopy from L_ϵ to $L_{1,\epsilon}$, by constructing a family of disks whose boundaries match up with the above. First, note that we can extend the disks by a function that depends only on q_{n+1}. Now, note that L_ϵ lives on $\psi_W^{-1}(\{S_1 \mid |z|^2 = (1+\epsilon)\})$, while $L_{1,\epsilon}$ meets $\psi_W^{-1}(\{S_1 \mid |z|^2 \leq (1+\epsilon)\})$ only at $\partial L_{1,\epsilon}$. We will choose the family of disks going between them such that their interiors are disjoint from the interior of $\psi_W^{-1}(S_1)$. This will automatically ensure that when these disks are joined with D_t^1, we get a family of embedded Legendrians.

The boundary conditions above will ensure that the boundaries live on $\psi_W^{-1}(S_1)$, so in order that the interior of the disks do not, we need $\{z^2+p^2 \geq 1+\epsilon\}$ for (z, q, p) coordinates on the interiors of the disks.

By Lemma 4.2.1 and the discussion in 4.2.1, the required family of Legendrians can be described by $H_t(q_{n+1})$ defined on $\{q_{n+1} \geq q_{n+1,t}\}$ which satisfy $H_t(q_{n+1,t}) = t\sqrt{1+\epsilon}$, $H_t'(q_{n+1,t}) = \frac{\sqrt{1-t^2}}{\epsilon_t\sqrt{1-t^2}+|t|\sqrt{1-\epsilon_t^2}}$, $H_t^2 + (\frac{dH_t}{dq_{n+1}})^2(q_{n+1}^2+1) \geq (1+\epsilon)$. Also, $H_{-1} = H$ from 4.2.1, while $H_1 \equiv 0$, so as to give $L_{2,\epsilon}$ and L_ϵ, respectively.

These initial points are plotted in Figure 4.8 for $\epsilon = 0.1$, while the initial slopes are plotted in Figure 4.9.

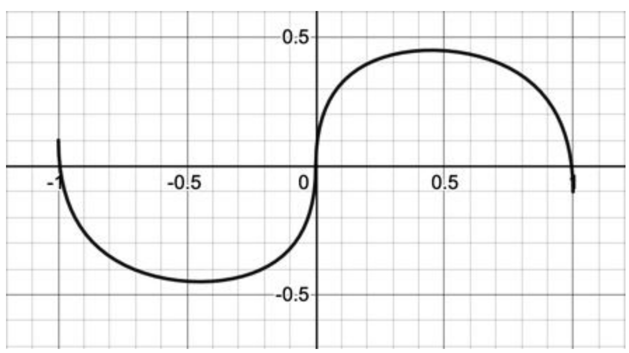


Figure 4.8: Plot of $q_{n+1,t}$ vs t when $\epsilon = 0.1$.
$q_{n+1,t} = \pm\sqrt{(1-t^2)(1-\epsilon_t^2)} - t\epsilon_t$

The choice of generating functions H_t that will now work, i.e. $j^1(H_t)$ will describe an isotopy between L_ϵ and $L_{2,\epsilon}$, can be described by the following method.

Similar to the function H that was chosen at the end of 4.2.1, consider a smoothly varying family $H_t(X)$ of functions defined on $[q_{n+1,t}, 1]$ that start with the specified slopes, that increase fast enough, beating the required inequality, and then reduce to 0 when the function value reaches $\sqrt{1+\epsilon}$.

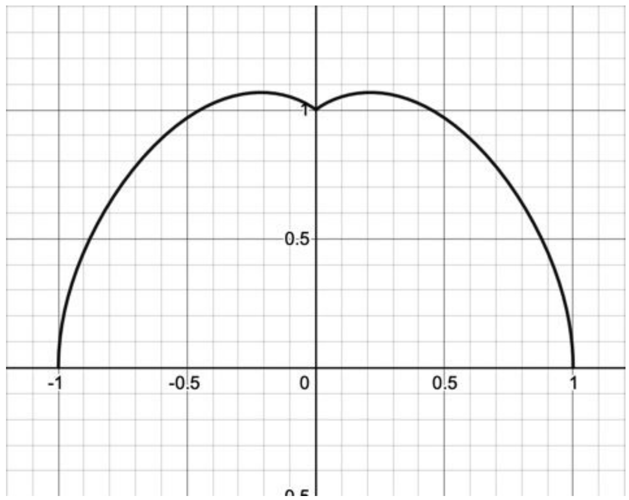


Figure 4.9: Plot of H_t' vs t when $\epsilon = 0.1$.
$H_t'(q_{n+1,t}) = \frac{\sqrt{1-t^2}}{\epsilon_t\sqrt{1-t^2}+|t|\sqrt{1-\epsilon_t^2}}$.

A schematic of the graphs of H_t can be seen in Figure 4.10.

4.3 The constructions give the standard unknot

In this section we show that $S_{join}(L)$ is isotopic to the standard Legendrian unknot, thus completing the proof of Theorem 1.1.3. Then we connect our constructions and results to Courte-Ekholm's work in [11] and prove Corollary 1.1.6. We will show that via a sequence of contactomorphisms and isotopies, $S_{join}(L)$ in (S^{2n+1}, ξ_{st}) can be identified with $\Lambda(L, L)$.

4.3.1 Isotoping S_{join} to the unknot

Proof of Theorem 1.1.3. In Section 4.2, we established that $S_{join}(L)$ and $S_{stab}(L)$ are isotopic. All that remains now is to show that S_{join} is the unknot. Using generating functions as described in Figure 4.5, $S_{join}(L)$ can be isotoped to $L_{2,\epsilon}$ joined with a pushoff arbitrarily close to it. In the figure, that amounts to bringing the top strand labelled $L_{1,\epsilon}$

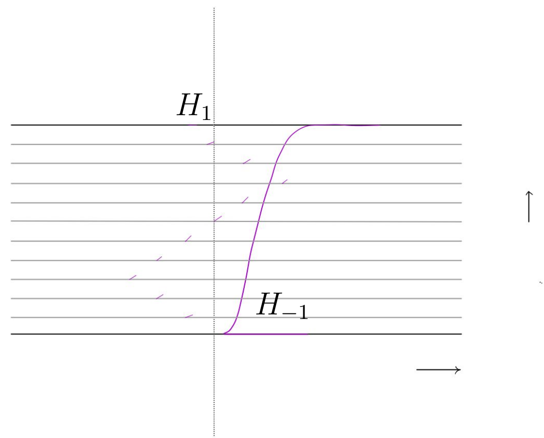


Figure 4.10: Here we can see the family H_t that will be used to describe the family of disks isotoping L_ϵ to $L_{1,\epsilon}$, drawn near their boundary so the picture does not get cluttered. Plotted are the initial points and slopes, following how $q_{n+1,t}$ and $p_{n+1,t}$ change as per the plotted graphs. Now, to describe the full disks, we can pick any family of homotopic functions that start at H_1, which is the constant 0 function at the top, then plot functions such that their slopes are large enough to ensure $z^2 + p^2 \geq 1$, and then H_{-1} is the purple graph plotted.

arbitrarily close to the lower one. Thus $S_{join}(L)$ is described in a standard neigbourhood of the Legendrian disk L, and hence the isotopy class of the sphere depends on the isotopy class of the disk. By Gromov's h-principle for Legendrian immersions, any two Legendrian disks are isotopic. Thus we can consider the same construction in a Darboux neighbourhood for the standard Legendrian disk, for which this construction gives the standard Legendrian unknot. This proves that $S_{join}(L)$ is isotopic to the unknot. □

4.3.2 Identifying with $\Lambda(L, L)$ in the (S^{2n+1}, ξ_{st}) case

We will quickly review Ekholm's [6] construction and Courte-Ekholm's proof strategy [11]. To start with, one considers a codimension 1 space $W_\rho = \{z = (\frac{\rho(x_n)}{\rho'(x_n)})y_n\} \cap \{0 < x_n < 1\}$ in $(\mathbb{R}^{2n+1}, \xi_{st})$ which is transverse to the Reeb flow. The function ρ can be considered a smoothing of the function $(1-|x|)$. A Lagrangian disk L with a cylindrical end can be embedded in W_ρ with its cylindrical end approaching $x_n = 0$. Reflecting the x_n, y_n coordinates, another copy of L, L^-, can be similarly embedded with its cylindrical end approaching that of L. Taking the Legendrian lift of these and joining along the ends gives the Legendrian sphere $\Lambda(L, L)$. Deforming the hypersurface W_ρ to $\{z = 0\}$, while staying transverse to ∂_z, recovers the construction of $\Lambda(L, L)$ as originally described in [6].

To show that this is the unknot, they describe the construction in $(\mathbb{R}^{2n+1}, ker(dz - \sum_{i=1}^{n-1} y_i dx_i + r_n^2 d\theta_n)$. Then, they modify the contact structure so that two halves of the sphere can be brought close to each other by flowing along ∂_{θ_n}, sketching out a pre-Lagrangian $(n+1)$-disk foliated by Legendrian disks in the process.

Proof of Corollary 1.1.6. By a contactomorphism (refer Example 2.1.10 in [34]), we can identify a hemisphere of (S^{2n+1}, ξ_{st}) with $(\mathbb{R}^{2n+1}, ker(dz + \sum r_i^2 d\theta_i))$. Under this, the modified $S_{join}(L)$, as in the proof above, which is the Legendrian lift of L in a page, joined with a pushoff, is identified with the Legendrian lift of a disk in $\{z = 0\} \cap \{0 < x_n < 1\}$, joined with a pushoff. (Note that in the proof above we have modified the open book pages so that the Lagrangian in the page is a Legendrian, but for the contactomorphism we want to respect the open book structure given by θ_n, and hence have to use Legendrian lifts.)

Then, by a further sequence of contactomorphisms, we can get to $(\mathbb{R}^{2n+1}, ker(dz - \sum_{i=1}^{n-1} y_i dx_i + r_n^2 d\theta_n)$, where the Legendrian sphere is still given by the Legendrian lift of a disk in $\{z = 0\}$ joined with its pushoff. Now deforming the hypersurface $\{z = 0\} \cap \{0 < x_n < 1\}$ to W_ρ, and tracing back through Courte-Ekholm's proof, it is clear that this sphere is in fact isotopic to $\Lambda(L, L)$. Since it is the image under a contactomorphism of the unknot, $\Lambda(L, L)$ is in fact the unknot. This completes the proof of Corollary 1.1.6. $\square$

Remark 4.3.1. Courte and Ekholm also use the h-principle for their proof. Hence, the proof here does not use a radically different argument. The point is to see their result as a part of a more general picture.

CHAPTER 5
LEGENDRIAN SURFACES AND THEIR INVARIANTS

All the results and observations in this chapter are joint work with J. Hughes. In this chapter, we describe how to obtain weave descriptions of doubles obtained from fillings described by weaves. Then, we discuss specific results about doubles of certain torus links, and also some work in progress.

5.1 Doubling Legendrian weaves

Definition 5.1.1. Consider two properly embedded N-graphs $G_1 \subset \mathbb{D}^2$ and $G_2 \subset \mathbb{D}_2$ with the same boundary. Then the doubled N-graph $G_1 \cup G_2$ is defined to be the N-graph obtained on S^2 by gluing the two disks via their boundaries and identifying the boundaries of G_1 and G_2.

Consider the following elementary observation. Note that we did not need to assume embedded to define the doubled Legendrian in Section 2.8.

Proposition 5.1.2. *Let G_1, G_2, be N-graphs describing two exact Lagrangian fillings $\pi(\Lambda(G_1), \pi(\Lambda(G_2))$ of a link K. Then the Legendrian $\Lambda(\pi(\Lambda(G_1)), \pi(\Lambda(G_2)))$ is the Legendrian weave corresponding to the doubled N-graph $(G_1 \cup G_2) \subset S^2$.*

Proof. This is easy to observe after choosing the right $D^2 \subset \mathbb{R}^5$ to satellite the weave along. Consider the Legendrian disk (call it D_1) whose front projection is the left half the Legendrian unknot pictured in the top left of Figure 3.5 (by Gromov's h-principle, there is a unique Legendrian D^2 in $\mathbb{R}^5_{st}$ but this helps in identifying with the setup in

Section 2.8). Suppose the boundary of the disk is along $x_2 = -1$, and the interior of disk extends into $\{x_2 \leq -1\}$. Similarly choose the other half of the Legendrian unknot (call it D_2) and translate it to live in $\{x_2 \geq 1\}$. Interpolate between them with a Legendrian annulus whose front is the cylinder in the x_2 direction of the front for the 1-dimensional Legendrian unknot in (x_1, z). Now, considering the Legendrians $\Lambda(G_i)$ after satelliting over D_i, and gluing them as in the doubling construction, is exactly the same as considering the Legendrian weave in $J^1(S^2)$ over the graph $G_1 \cup G_2 \subset S^2$, and satelliting it over the standard unknot in $\mathbb{R}^5$. □

For simplifying notation, henceforth we will denote the double coming from weaves by $\Lambda(G_1, G_2)$, instead of $\pi(\Lambda(G_1), \pi(\Lambda(G_2))$. In the following, we show a refinement of Theorem 2.8.1 of [11] for weaves. In particular, for an exact Lagrangian filling $\Lambda(G)$ of a link given by a Legendrian weave G, we describe exactly the symmetric double $\Lambda(G, G)$.

Theorem 5.1.3. *For an N-graph G, the double $\Lambda(G, G)$ is Legendrian isotopic to $\#_k T^2_{st}$, where k is the number of trivalent vertices in G.*

Proof. The proof will apply Theorem 3.2.6 to G to decompose G into bigons, but we might have to do a number of candy twists (refer, Figure 3.3) to simplify the graph first and remove hexagonal vertices. It is clear that any hexagonal vertex near ∂G (i.e. 3 of its edges end at ∂G) gives rise to a local picture like the left picture corresponding to the candy twist move and can be undone. Any trivalent vertex near the boundary, i.e. two of its edges end at ∂G, contribute a bigon to the doubled graph $G \cup G$, and hence a T^2_{st}. Inductively removing the bigons results in a simplified graph and proves the result. □

The main technical tool for understanding isotopy classes of asymmetric doubles is the computation of the sheaf moduli of $\Lambda(L_1, L_2)$ in terms of the sheaf moduli of the fillings L_1 and L_2. Suppose the flag moduli invariant of Λ is $\mathcal{M}_1(\Lambda)$. Let $C_{L_1} \subseteq \mathcal{M}_1(\Lambda)$ and $C_{L_2} \subseteq \mathcal{M}_1(\Lambda)$ be the toric charts induced by L_1 and L_2 respectively.

Theorem 5.1.4. *The flag moduli $\mathcal{M}_1(\Lambda(L_1, L_2))$ is given by the intersection $\mathcal{C}_{L_1} \cap \mathcal{C}_{L_2}$.*

Proof. The flag moduli $\mathcal{M}_1(\Lambda(L_1 \cup L_2))$ is obtained by imposing the additional singular support conditions coming from both L_1 and L_2 on $\mathcal{M}_1(\lambda)$. However, as the discussion in appendix B of [31] makes clear, especially in Theorem B.20, the toric charts C_{L_i} are obtained by imposing the singular support conditions coming from L_i. □

5.2 Doubles of torus link fillings

Call L_{init} the weave filling of $\lambda(2, n)$ corresponding to the graph with all trivalent vertices on the "outside" edge; analogous to the topmost graph on the right of Figure 3.6. The Legendrians obtained by doubling these fillings of $\lambda(2, n)$ correspond to 2-graphs that were characterised in Theorem 2 of [35] by Whitney, and also called "two-tied trees" by Kauffmann [36].

5.2.1 Doubles of $\lambda(2, n)$ fillings

Theorem 5.2.1. *Let L_1 and L_2 be two exact Lagrangian fillings of $\lambda(2, n)$ obtained as above. Then,*

1. *The double $\Lambda(L_{init}, L_1)$ is Hamiltonian isotopic to $\#^k\mathbb{T}^2_{std}\#^l\mathbb{T}^2_c$ for some k and l such that $k + l = n - 1$*

2. *Given L_1, and j such that $0 \leq j \leq n-1$, there exists L_2 such that $\Lambda(L_1, L_2)$ is Hamiltonian isotopic to $\#^j\mathbb{T}^2_{std}\#^{n-1-j}\mathbb{T}^2_c$*

3. *The double $\Lambda(L_1, L_2)$ is Hamiltonian isotopic to $\#^{n-1}\mathbb{T}^2_{std}$ if and only if L_1 and L_2 are Hamiltonian isotopic.*

Proof. The proof of items (1) and (2) will be using Theorem 3.2.6.

For (1), start with the "highest" or "interiormost" trivalent vertex in the weave representation of L_1 (if there are none, then it has to be L_{init} and $l = 0$). The two edges of this vertex that go to the boundary of the graph continue into the weave for L_{init} and meet the outside edge at two trivalent vertices, thus creating a triangle. This corresponds to a connect summed T_c, doing the reverse operation as in the third row of Figure 3.7 takes that summand out. The resulting 2-graph is the double of the initial filling of $\lambda(2, n-1)$ and some other weave filling. By induction, (1) is proved.

For (2), observe that given any filling L_1 of $\lambda(2, n)$, there are $(n-1)$ short I-cycles visible in the 2-graph. Mutating along one of them keeps the others intact. If we take $\Lambda(L_1, L_1)$, this is isotopic to $\#^{n-1}T^2_{st}$. Now, mutate along an I-cycle connected to the "highest" trivalent vertex of L_1, call that L_1'. Locally, the mutation operation causes the bigon in $\Lambda(L_1, L_1)$ corresponding to the highest trivalent vertex to become a triangle in $\Lambda(L_1, L_1')$, thus $\Lambda(L_1, L_1')$ is isotopic to $\#^{n-2}T^2_{st}\#T^2_c$. However, we can see the remaining $(n-2)$ I-cycles of L_1 still in L_1', call them the "surviving cycles", along with the newly created I-cycle. Mutating along the surviving I-cycles successively in order of their height allow us to produce up to $(n-1)$ T^2_c summands, by changing all bigons to triangles (the triangles might become visible only after removing triangles in their interior).

For (3), note that by Theorem 5.1.4, the sheaf moduli of $\Lambda(L_1, L_2)$ is isomorphic to $(C^*)^{n-1}$ iff $L_1 = L_2$, since otherwise $\mathcal{M}_1(\Lambda)$ is a proper subvariety of $\mathcal{C}_{L_i}$, as L_1 and L_2 induce distinct toric charts in $\mathcal{M}_1(\lambda(2, n)$. □

5.2.2 Some computations for doubles of $\lambda(3,6)$ fillings

The torus link $\lambda(3,6)$ admits infinitely many exact Lagrangian fillings, which was shown by Casals-Gao [37]. The fillings are described by Legendrian weaves coming from 3-graphs, consisting of two parts, an initial filling of $\lambda(3,6)$, call it L_0 and a cobordism from $\lambda(3,6)$ to itself, call it C, stacked on top of it a number of times. These pieces are drawn as weaves in Figure 5.1. Schematically, if the infinite family of fillings is $\{L_i\}_{i=1}^{\infty}$, then $L_i = L_1 \cup C^{i-1}$, i.e., the initial filling and $(i-1)$ copies of the cobordism on top.

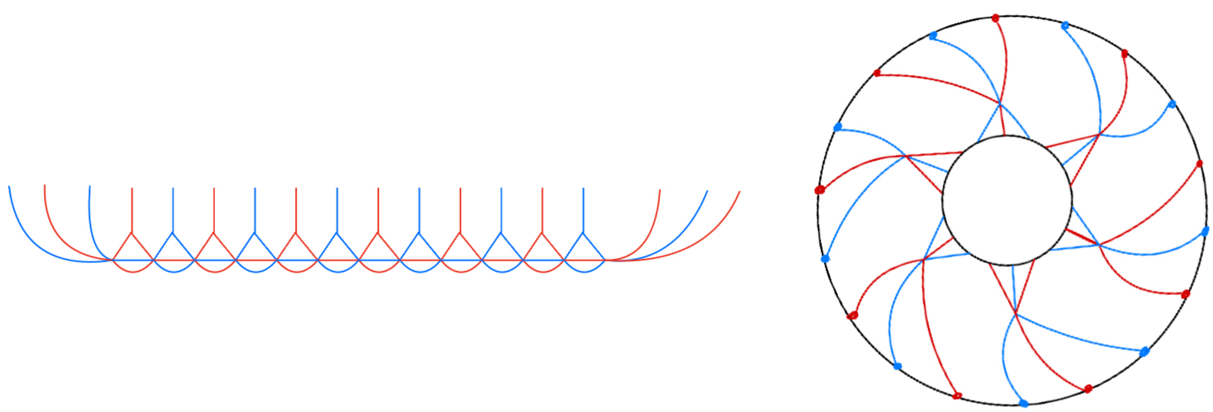


Figure 5.1: The initial filling of $\lambda(3,6)$ on the left, and the infinite order self-cobordism on the right.

A natural question to ask is, what are the isotopy classes of the doubles $\Lambda(L_1, L_i)$? Using the first three moves for 3-graphs in Figure 3.3, we can show the following.

Theorem 5.2.2. *The Legendrian surface $\Lambda(L_1, L_2)$ is isotopic to the connect sum of standard and Clifford tori with a knotted link.*

Proof. The proof is purely diagrammatic. Some of the steps are shown in Figure 5.2. The first arrow is after applying some push-throughs along the top and bottom. The second arrow shows a flop, and some more flops and push-throughs after that lead to a triangle showing up, which is a Clifford torus summand. Taking this summand out to simplify the picture, we continue applying the equivalence moves. Every time a bigon (as shown in the fifth picture in the sequence) or Clifford torus shows up, it is taken out. Ultimately, we are left with the knotted link represented in the sixth picture. □

Similarly, by working with these diagrams, it can be shown that $\Lambda(L_1, L_i)$ for $i = 2, 3, 4$ are all isotopic to cubic planar Legendrian surfaces connect summed with the same knotted sphere. Also, the cubic planar Legendrians for $i = 1, 2, 3, 4$ are all non-isotopic.

We expect that the family $\{\Lambda(L_1, L_i)\}_{i=1}^{\infty}$ will all be pairwise non-isotopic, which should follow from explicit computations of point counts of their flag moduli, along the lines of Section 6.4 in [21]. This is work in progress.

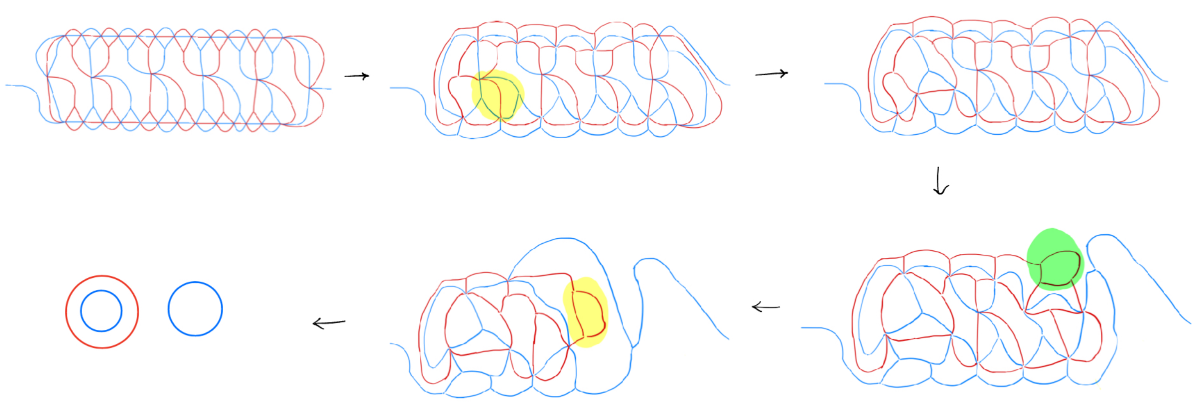


Figure 5.2: Decomposing a double of two fillings of $\lambda(3, 6)$ into a connect sum of a link of spheres with standard and Clifford tori.

5.3 Future directions

In this section we will mention the ideas behind work in progress, to prove Theorems 1.2.6, 1.2.9, 1.2.8, and 1.2.7.

5.3.1 Fillings and Invariants of Twist-spun Legendrians

Let $\lambda \subseteq (\mathbb{R}^3, \xi_{st})$ be a Legendrian link and φ be a Legendrian loop of λ. That is, $\varphi = \{\varphi_\theta\}_{\theta\in[0,1]}$ is a Legendrian isotopy with φ_1 fixing λ pointwise. The image $\{\varphi_\theta(\lambda)\} := \{\lambda_\theta\}$ is an S^1 family of Legendrians, and we can form the mapping torus of φ to obtain a Legendrian surface in contact $(\mathbb{R}^5, \xi_{st})$. More explicitly, we define $\Sigma_\varphi(\lambda)$ by observing that an S^1 family of Legendrians $\{\lambda_\theta\} \subseteq T^*\mathbb{R}_x \times S^1 \times \mathbb{R}_z$ lifts uniquely to a Legendrian $\Sigma_\varphi(\lambda) \subseteq T^*\mathbb{R}_x \times T^*S^1 \times \mathbb{R}_z$ with contact structure $dz - p_\theta d\theta - p_x dx$. We can then canonically identify $T^*\mathbb{R} \times T^*S^1$ with $T^*\mathbb{R}^2$ by the map $\mathbb{R} \times S^1 \to \mathbb{R} \times \mathbb{R}\backslash\{0\}$ defined by $(x, \theta) \mapsto e^x\theta$, giving a (strict) contact embedding $(T^*\mathbb{R}^2 \times \mathbb{T}^*S^1 \times \mathbb{R}_z, dz - p_\theta d\theta - p_x dx) \hookrightarrow (\mathbb{R}^5, \xi_{st})$. See [38, Section 1.3] for more details. In this sense, we can give the following definition.

Definition 5.3.1. Given a Legendrian loop φ of a Legendrian link λ, the twist-spun Legendrian $\Sigma_\varphi(\lambda)$ is the union of Legendrian tori

$$\lambda \times [0, 1]/(\varphi(\lambda) \times \{0\} \sim \lambda \times \{1\})$$

embedded in $(\mathbb{R}^5, \xi_{st})$.

Considering a Lagrangian filling L of λ, it gives rise to an exact filling of $\Sigma_\varphi(\lambda)$ if it is "well-behaved" under the loop φ. On the sheaf moduli and cluster side, starting with a (-1)-closure, the cluster variety of $\mathcal{M}_1(\Sigma_\varphi(\lambda))$ is obtained by *folding* the cluster variety of λ along ϕ.

Understanding this procedure for the cases mentioned will allow us to prove Theorems 1.2.8 and 1.2.7.

5.3.2 General doubles of N-graph Legendrians

Similar to Theorem 3.3.12 by Treumann-Zaslow, we expect to show that the nontrivial intersection of two toric charts is a variety that is not large enough to accomodate a toric charts. This will allow us to show that an asymmetric double is never exact Lagrangian fillable, which is Theorem 1.2.6. The precise proof will need to be argued using point counts over some finite fields, which is work in progress.

We will sketch the proof of Theorem 1.2.9. The sheaf moduli of the associated link is the braid variety, which is known to be an irreducible variety. Any two exact Lagrangian fillings induce toric charts for the variety, which are Zariski open subsets of the braid variety, and are hence dense. Thus, their intersection will always be non-empty, hence the double will be non-loose, since it has non-vanishing sheaf moduli, by Lemma 1.2.2.

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