Braids

&

Contact Topology

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Outline

I Intro to Contact Structures

II Braids and Bennequin

III Positivity in the Braid group

IV Contact Geometry and Positivity

V Open Book Decompositions

VI Monoids and Geometry

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I Intro To Contact Structures

a hyperplane field ξ^{2n} on a manifold M^{2n+1} is called a **contact structure** if there is (at least locally) a 1-form α such that

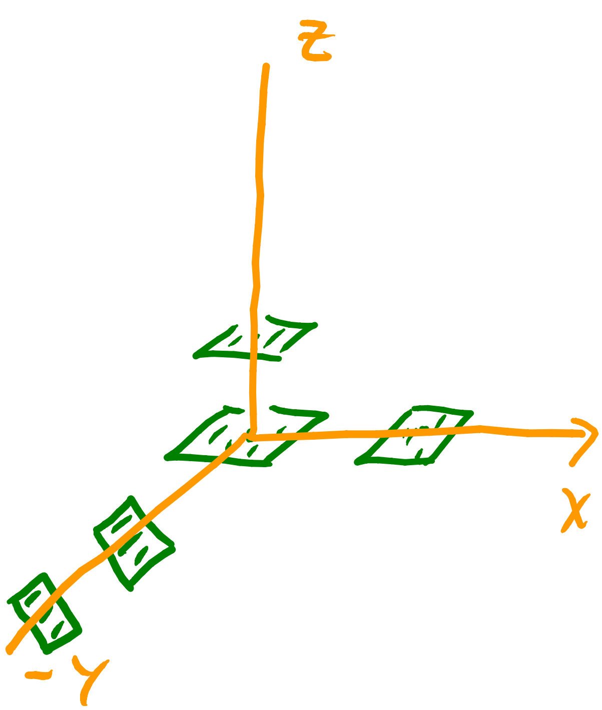
$$\boxed{\begin{gathered}\xi = \ker \alpha \\ \alpha \wedge \underbrace{d\alpha \wedge \ldots \wedge d\alpha}_{n \text{ times}} \neq 0\end{gathered}}$$

example:

on $\mathbb{R}^{2n+1}$ let $\alpha = dz - \sum_{i=1}^{n} y_i dx_i$

and $\xi_{std} = \ker \alpha$

$= \operatorname{span}\left\{ \frac{\partial}{\partial y_i}, y_i \frac{\partial}{\partial z} + \frac{\partial}{\partial x_i} \right\}$

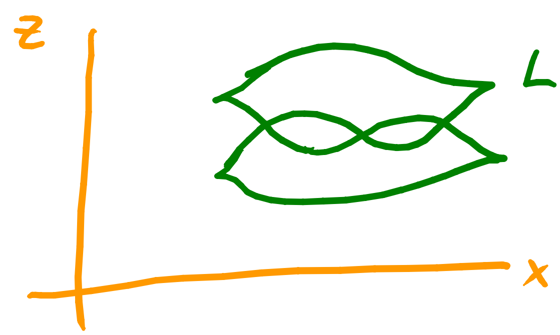


Darboux: all contact structures look locally like this one let (M, ξ) be a contact manifold

$L^n \subset M^{2n+1}$ is a <u>Legendrian submanifold</u> if $\boxed{T_x L \subset \xi_x}$ $\forall x \in L$

<u>example:</u> in $(\mathbb{R}^3, \xi_{std})$ project L to the xz-plane

<u>note:</u> $y = \frac{dz}{dx}$ on L

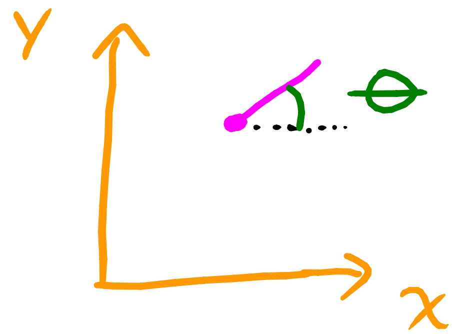


this is called the front projection (notice resemblance to wave fronts)

<u>$Th^{\underline{m}}$:</u> any arc in a contact 3-manifold can be C^0-approximated by a Legendrian curve (rel end pts)

A natural occurance of contact structures

consider the configuration space of a skate (or front wheel of a car)



(x,y) determine the position of the skate in the plane

θ determines the angle it forms with the x-axis

so the configuration space is

$$\boxed{W = \mathbb{R}^2 \times S^1}$$

note: 1) At a fixed point the skate can point in any direction

2) Skate can only move in the direction it is pointing (we assume it does not scrape)

So if $\gamma(t) = (x(t), y(t), \theta(t))$ is a motion of the skate then

$$\boxed{\frac{y'(t)}{x'(t)} = \tan\theta(t)}$$

if we set $\xi = \ker(\cos\theta\, dy - \sin\theta\, dx)$ then ξ is a contact structure on W and

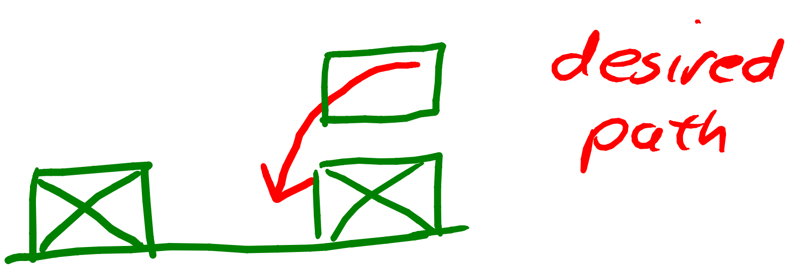
γ is the motion of a skate

$\Longleftrightarrow$

γ parameterizes a Legendrian curve

<u>Application:</u>

You can always (in theory) parallel park your car.



Other Occurrences of Contact Structures

- **PDE** (Sophus Lie 1872)

  given $F : \mathbb{R}^{2n+1} \to \mathbb{R}$
  finding $z : \mathbb{R}^n \to \mathbb{R}$ solving

  $$\boxed{F\left(x_1 \ldots x_n, \frac{\partial z}{\partial x_1} \ldots \frac{\partial z}{\partial x_n}, z\right) = 0}$$

  is equivalent to finding
  $u : \mathbb{R}^n \to \mathbb{R}^{2n+1}$ s.t.

  $$\boxed{\begin{array}{l} F \circ u = 0 \\ \text{Image}(u) \text{ is Legendrian} \end{array}} \text{ in } \xi_{std}$$

- **Riemannian Geometry** g metric on M

  $$\begin{array}{ccc} TM & \underset{g}{\cong} & T^*M \\ \cup\!\! \mid & & \cup\!\! \mid \\ S(TM) & & S(T^*M) \end{array}$$

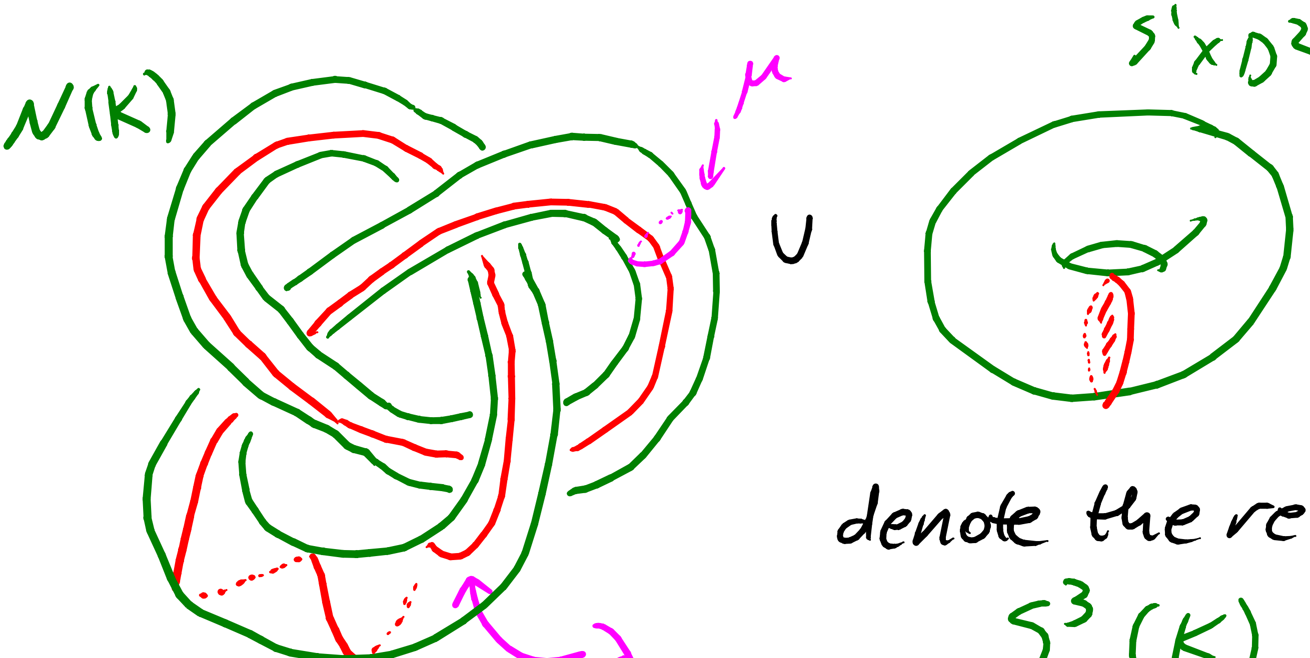
  Geodesic flow $\qquad\qquad$ Reeb flow

- **Optics** via Huygen's Principle
- **Thermodynamics** by work of Gibbs
- **Low-dimensional Topology**

we will see lots of examples later, but we note a few results here

recall: Dehn Surgery on a knot $K \subset S^3$

- remove a nbhd $N(K)$ of K from S^3
- glue back $S^1 \times D^2$ so that $\{pt\} \times \partial D^2$ goes to a curve $\lambda \subset \partial(S^3 \setminus N(K))$

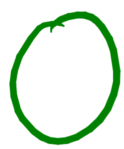
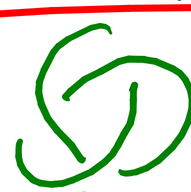
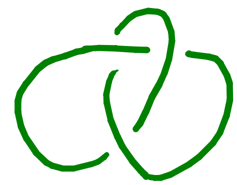


denote the result $S^3_\lambda(K)$

- Kronheimer-Mrowka proved

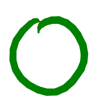
$$\boxed{\begin{array}{l}\text{Non trivial knots have } \underline{\text{property } P} \\ \text{i.e. } \pi_1(S^3_\lambda(K)) = 1 \\ \quad \Rightarrow \lambda = \mu \text{ or } K = \text{unknot}\end{array}}$$

- Ozsvath and Szabo proved

If L is ,  or 

then $S^3_p(K) \cong S^3_p(L)$ for any p

implies $K = L$

the $L =$  was conjectured by Gordon in the 1970's and was originally proven by Kronheimer-Mrowka-Ozsvath-Szabo

II Braids & Bennequin

Recall $\mathbb{R}^3$ has a contact structure

$$\xi_{std} = \ker(dz + r^2 d\theta)$$

Question: Is there another?

consider $\xi_{ot} = \ker(\cos r\, dz + r \sin r\, d\theta)$

this looks "different" but how do we know there is no diffeom.

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3$$

taking ξ_{std} to ξ_{ot}?

Bennequin's Answer: Use braid theory!

(and in the process give birth to contact topology and initiate tools in Braid theory independently developed more fully! by Birman-Menasco)

Outline Of Proof:

a knot K in a contact manifold (M, ξ) is called transverse if

$$T_x K \pitchfork \xi_x \quad \forall \, x \in K$$

Step 1: such a K in $(\mathbb{R}^3, \xi_{std})$ can be braided

Step 2: define the self-linking number of K: $sl(K)$

compute $sl(K)$ using

- Seifert surface Σ for K
- braid representation of K

Step 3: analyze the Birman–Menasco braid foliations on Σ to prove

$$\boxed{sl(K) \leq -\chi(\Sigma)}$$

← Bennequin inequality

Step 4: Notice in $(\mathbb{R}^3, \xi_{ot})$ there is a transverse unknot with $sl = 1$

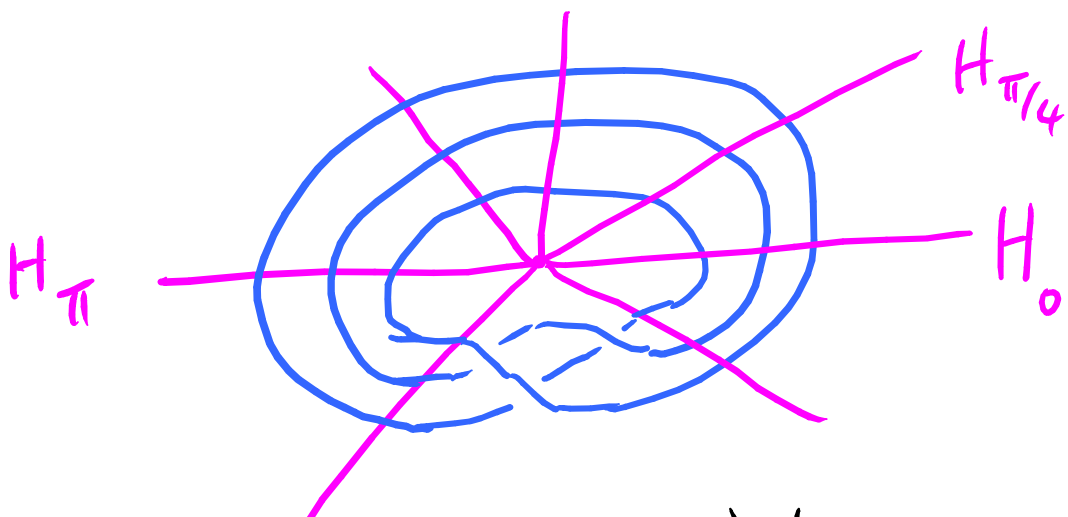
Some Details of Proof

Step 1: notice that for large r the contact planes

$$\xi_{std} = \text{span}\left\{ \frac{\partial}{\partial r},\ r^2\frac{\partial}{\partial z} - \frac{\partial}{\partial \theta} \right\}$$

are almost tangent to the constant θ half-planes $H_{\theta_0} = \{(r,\theta,z) : \theta = \theta_0\}$

Since braids are $\pitchfork$ to the H_θ's



we see (closed) braids naturally give $\pitchfork$ knots

Th$^{\underline{m}}$ (Bennequin '82)

Contacting Alexander →

$\boxed{\text{Any transverse knot in } (\mathbb{R}^3, \xi_{std}) \text{ can be isotoped through } \pitchfork \text{ knots to a closed braid.}}$

one can prove this by observing that the proof that any knot can be braided can easily be adapted to this situation

Fact (Orevkov - Shevchishin, Wrinkle '03)

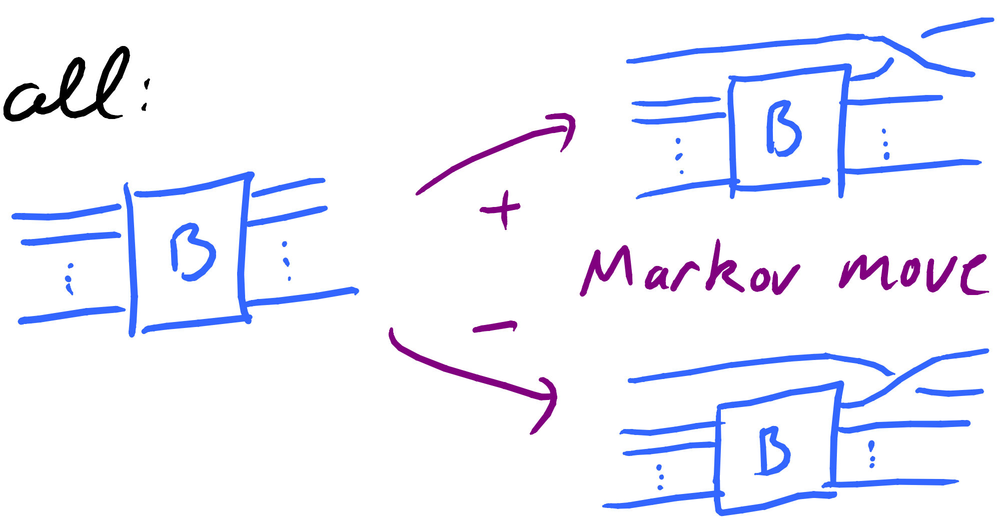
Contacting Markov

to closed braids representing in $(\mathbb{R}^3, \xi_{std})$ are isotopic as transverse knots $\iff$ they are related by

1) conjucation in braid group (i.e. braid isotopy)

2) positive Markov moves

recall:



so classifying transverse knots in $(\mathbb{R}^3, \xi_{std})$ can be done purely in terms of braids!

Step 2: The Self-linking number

Note: there is a never zero vector field v in ξ_{std}. To see this notice $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \mapsto (x,y)$

gives $d\pi|_{\xi_p} : \xi_p \to T_{\pi(p)}\mathbb{R}^2 = \mathbb{R}^2$

an isomorphism for all p

so $\exists!$ vector field v in ξ such that $d\pi(v) = \frac{\partial}{\partial x}$

constant vector field in $\mathbb{R}^2$

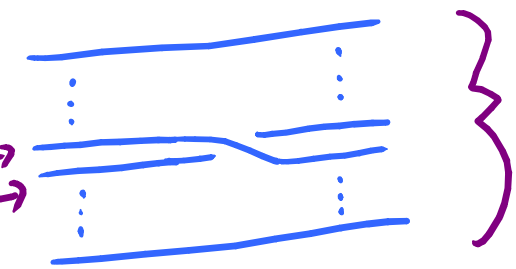
Now if K is a transverse knot let K' be $K + \varepsilon v$ for some small ε

The self-linking number of K is

$$\boxed{sl(K) = \text{linking}(K, K')}$$

exercise: $sl(K)$ is independent of v

Recall: any n-strand braid can be represented by a word in $\sigma_1 \dots \sigma_{n-1}$ and $\sigma_1^{-1} \dots \sigma_{n-1}^{-1}$ where



σ_i

Lemma:

If the transverse knot K is given as the closure of the n-braid

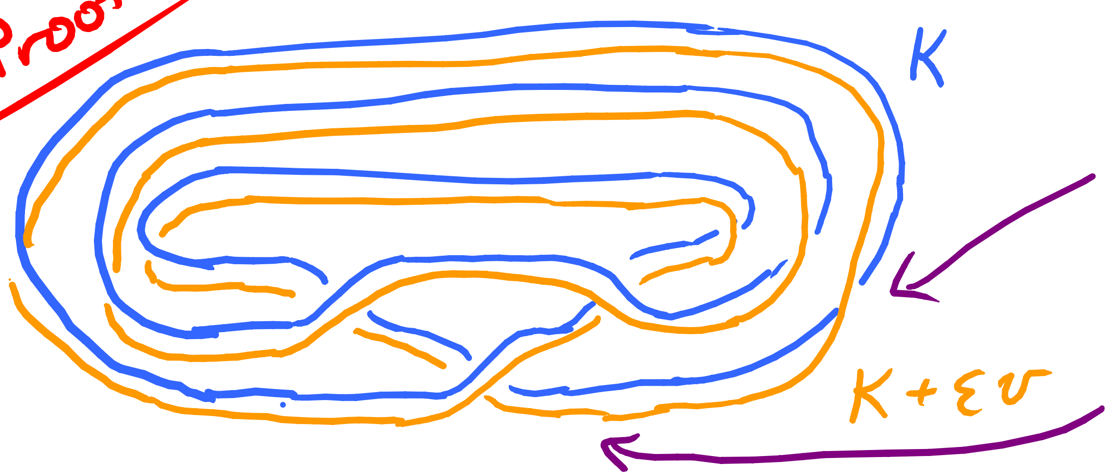
$$B = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$$

where $\varepsilon_i = \pm 1$

Then $sl(K) = a(B) - n(B)$

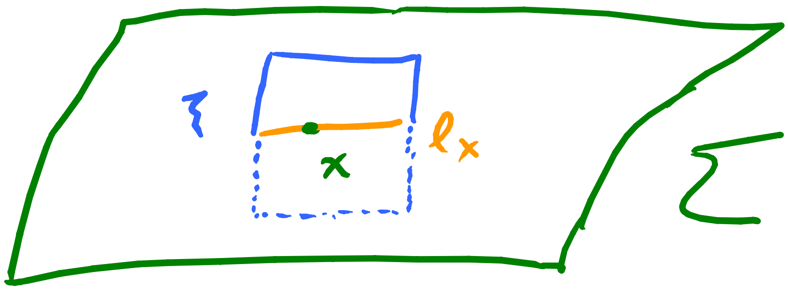
algebraic length $\sum \varepsilon_i$ — braid index

Proof



let Σ be an oriented surface such that $\partial\Sigma = K$

for each $x \in \Sigma$ let $l_x = \xi_x \cap T_x\Sigma$

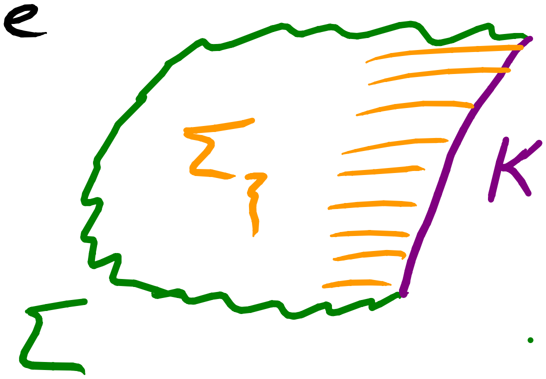


at most points l_x is a line but at some points $l_x = T_x\Sigma = \xi_x$

these are called singular points

<u>exercise:</u> There is a singular 1-dimentional foliation Σ_ξ whose tangents are given by l_x. We call Σ_ξ the <u>characteristic foliation</u>

note that since $K = \partial\Sigma$ is $\pitchfork$ to ξ we see



exercise: Can perturb Σ, rel $\partial\Sigma$, so that all singular points "look like"

elliptic hyperbolic

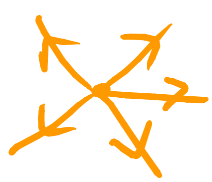
ξ is oriented (by $2r\,dr \wedge d\theta$) and so is Σ so to each singular point we can assign a sign

+ if orientations agree

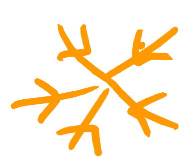
− if they disagree.

We get an induced orientation of l_x and Σ_ξ

exercise: if an elliptic point is + then



if − then



given Σ let $e_\pm = \#\begin{pmatrix}\pm \text{ elliptic pts} \\ \text{in } \Sigma_\xi\end{pmatrix}$

$h_\pm = \#\begin{pmatrix}\pm \text{ hyper. pts} \\ \text{in } \Sigma_\xi\end{pmatrix}$

Lemma:

if $K = \partial\Sigma$ then

(1) $\qquad sl(K) = -(e_+ - h_+) + (e_- - h_-)$

exercise: prove this

Hint: let w be a vector field along Σ that is tangent to Σ_ξ and points out of Σ along $\partial\Sigma$

note $\text{link}(K, K+\varepsilon w) = 0$

show the difference between v and w along K is given in terms of $e_\pm, h_\pm$

notice that w is a vector field tangent to Σ and pointing out $\partial\Sigma$ so by Poincaré-Hopf we have

$$(2) \quad \boxed{\chi(\Sigma) = e_+ + e_- - h_+ - h_-}$$

if we add (1) and (2) we get

$$sl(K) + \chi(\Sigma) = 2(e_- - h_-)$$

thus if we can show that Σ can be isotoped, rel $\partial\Sigma$, so that $e_- = 0$ then we will have shown

<u>Thm (Bennequin '82):</u>

$$\boxed{sl(K) \leq -\chi(\Sigma)}$$

to do this we need...

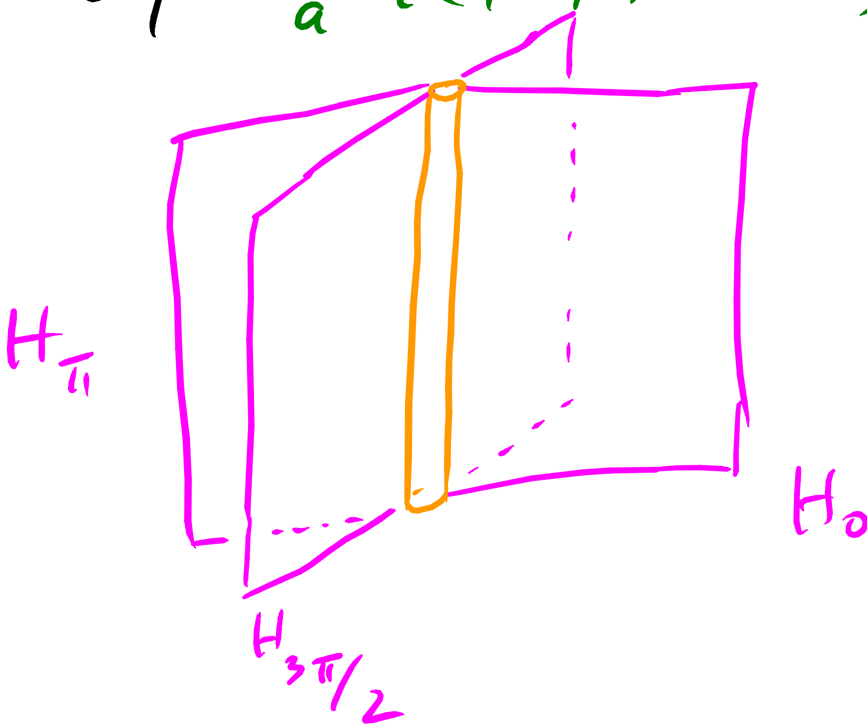
Step 3: Braid Foliations

let K be a closed braid

Σ a surface s.t. $\partial\Sigma = K$

recall $\mathbb{R}^3 - (z\text{-axis})$ is foliated

by $H_a = \{(r, \theta, z) : \theta = a\}$ $a \in [0, 2\pi)$



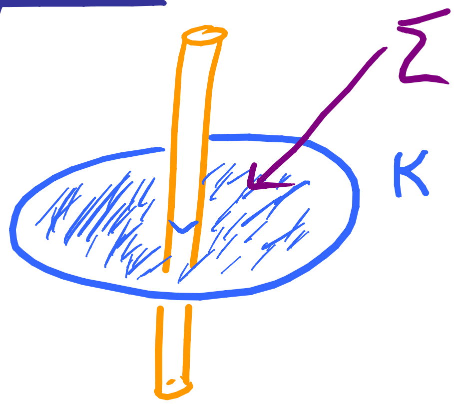
this induces a singular foliation on Σ

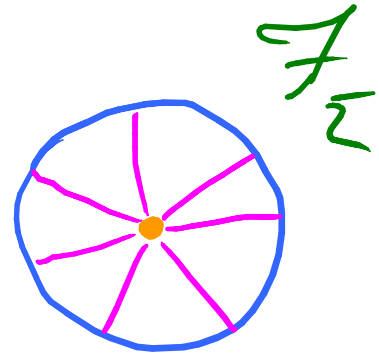
called the braid foliation

and denoted $\mathcal{F}_\Sigma$

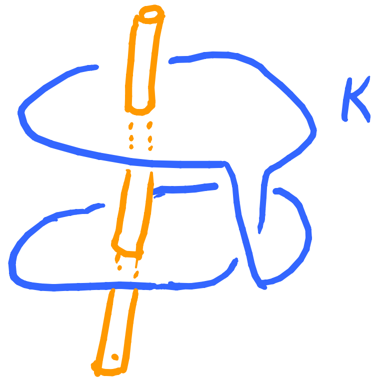
examples:

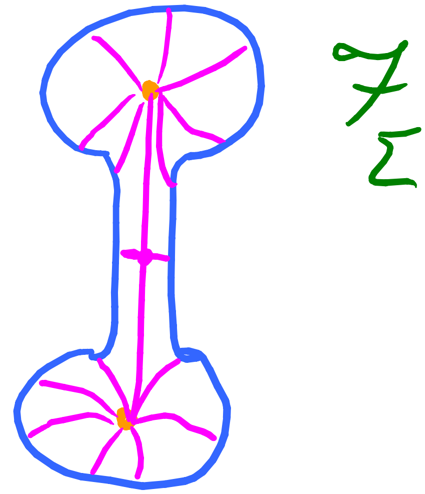
1)





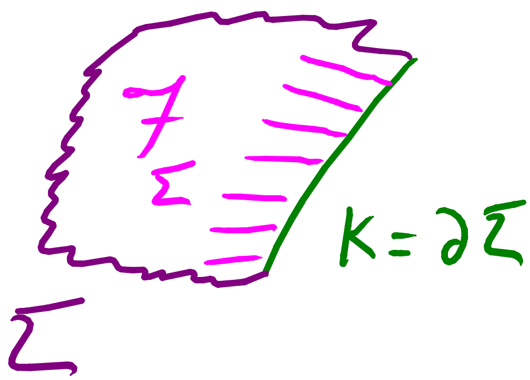
2)





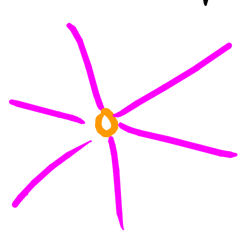
note/exercise:

1) near $\partial\Sigma$



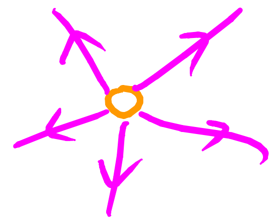
2) Can assume $\Sigma \pitchfork$ (z-axis)

so finitely many points like

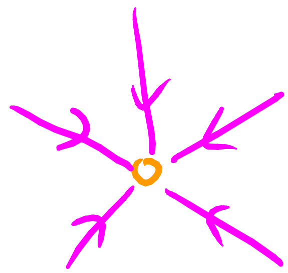


3) Can orient $\mathcal{F}_\Sigma$ (as we did Σ_ξ)

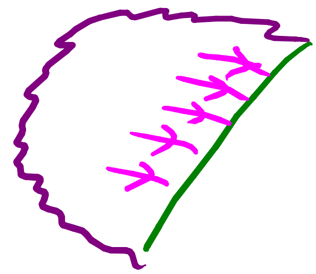
then if z-axis positively transverse to Σ then



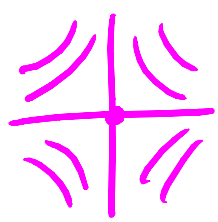
otherwise



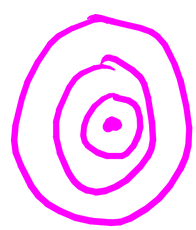
also near $\partial\Sigma$



4) Can perturb Σ so that the only singularities of $\mathcal{F}_\Sigma$ (away from z-axis) are

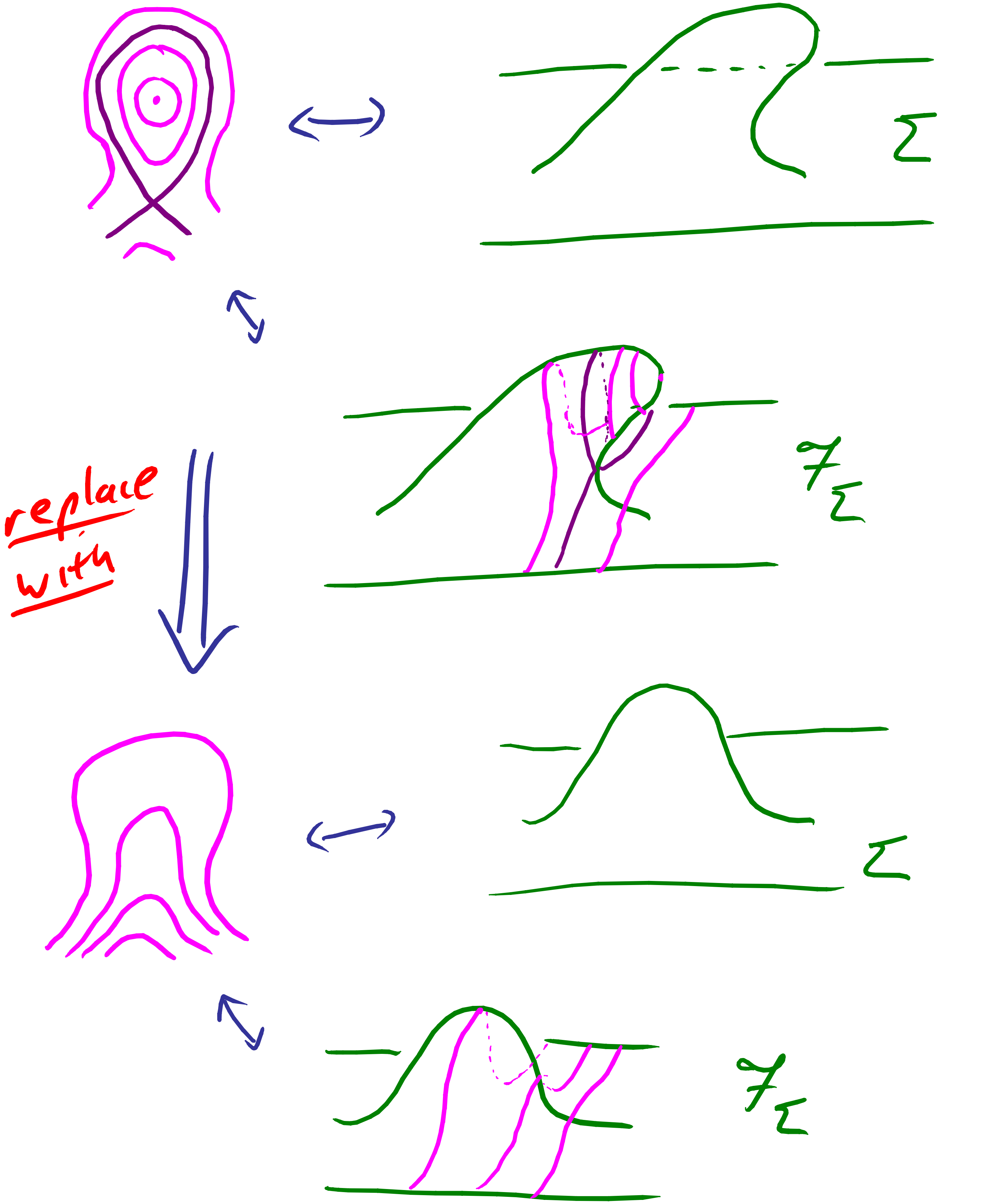


hyperbolic



center

5) if you keep expanding a center singularity you see (more or less)



so we can isotop Σ so that there are no center singularities!

6) You can isotope Σ so that

$$\boxed{\mathcal{F}_\Sigma \text{ is very close to } \Sigma_\xi}$$

thus you can read off $e_\pm$ $h_\pm$ from $\mathcal{F}_\Sigma$

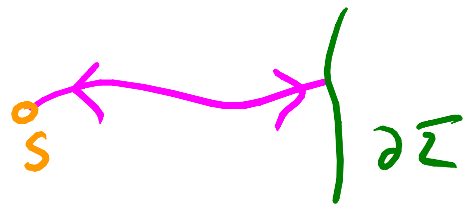
and prove the inequality by isotoping Σ so that it does not $\cap$ z-axis negatively!

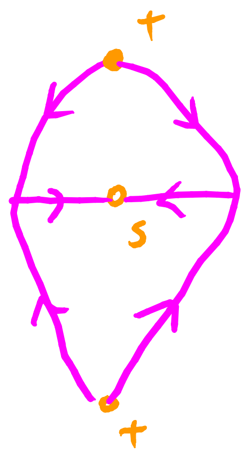
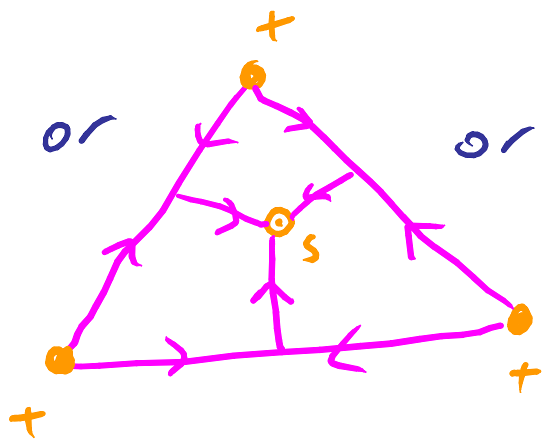
From now on let K be an unknot and Σ be a disk it bounds

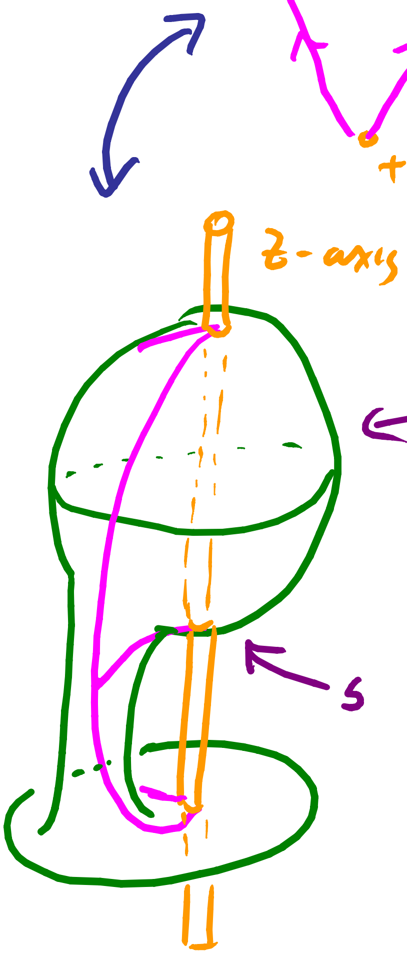
consider a negative singularity $s \in \mathcal{F}_\Sigma$

let $B_s = \overline{\{\text{all leaves of } \mathcal{F}_\Sigma \text{ limiting to } s\}}$

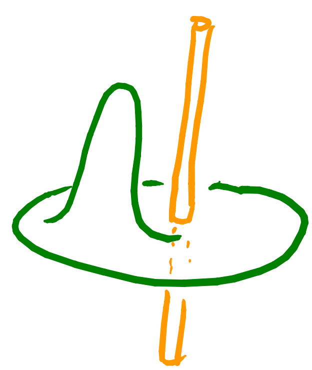
note: B_s is disjoint from $\partial\Sigma$

(since we can't have )

so B_s is  or  or ...

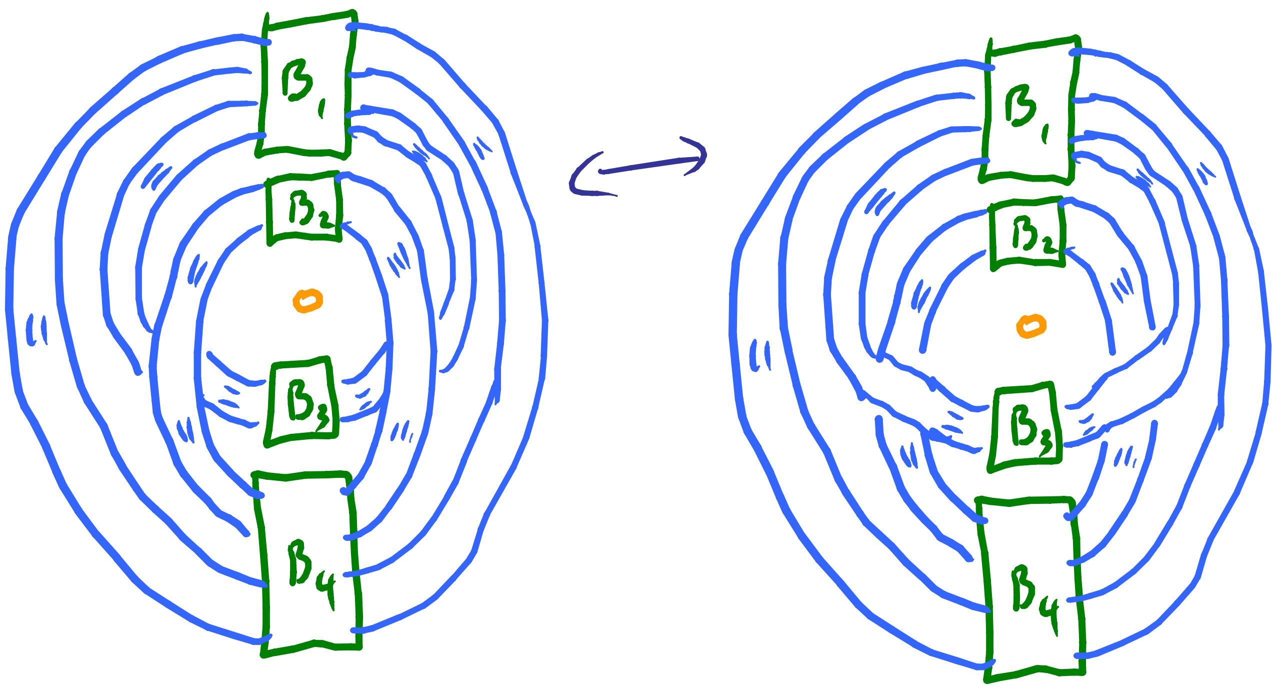


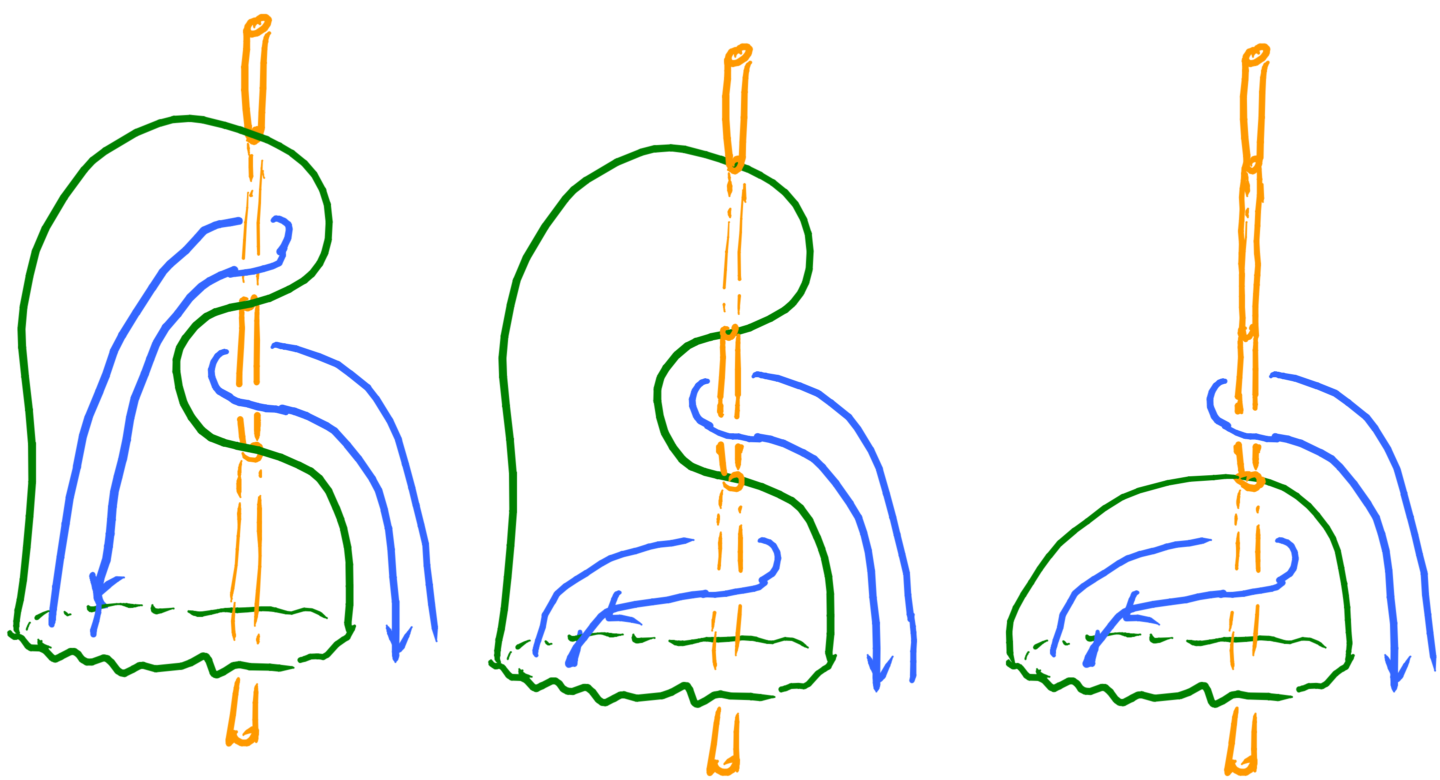
if the "sack" is empty then replace Σ with



ellimінating s!

If the "sack" is not empty then you can empty it via exchange moves





<u>note</u>: this does not change
the braid index or algebraic length
of the braid.
so does not change $sl(K)$

<u>exercise</u>: show this can be done
without changing ξ_Σ
(we can then elliminate s)

<u>exercise</u>: think about more
general B_s (can always reduce
to the above)

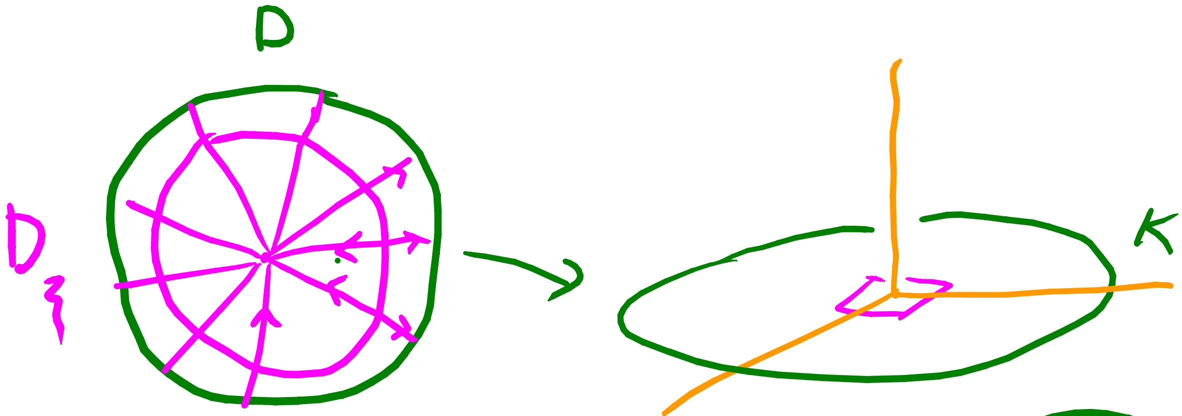
Continuing this analysis we will prove
that for any transverse unknot K
we can isotope a disk D with $\partial D = K$
so that $e_- = 0$
thus $sl(K) \leq -\chi(D) = -1$

Note: let $D = \{(r,\theta,z) : \begin{smallmatrix}\theta \le \pi+\varepsilon \\ z=0\end{smallmatrix}\}$

$K = \partial D$

In $\xi_{ot} = \ker\{\cos r\, dz + r \sin r\, d\theta\}$

the knot K is transverse



Purturb D to get

so $sl(K) = 1$

Thus we see ξ_{std} and ξ_{ot} are indeed different contact strs!

with more work you can extend the Bennequin inequality for unknots in ξ_{std} to any knot (or link)

There are 2 ways to continue the above work:

I. Contact geometry

tight vs. overtwisted (Eliashberg)

II. Braid theory (Birman–Menasco)

we first discuss contact geometry

<u>Definition:</u> We call a contact structure ξ overtwisted if there is a disk D such that $T_x D = \xi_x$ for all $x \in \partial D$ otherwise call ξ tight (this def$^{\underline{n}}$ due to Eliashberg)

Note: If ξ is overtwisted then (by extending D as above) we get a transverse unknot K with $sl(K) = 1$ so Bennequin bond does not hold.

Theorem (Eliashberg + ε, '92):

let (M, ξ) be a contact manifold the following are equivalent

I. ξ is tight

II. $sl(K) \leq -\chi(\Sigma)$ for all $\pitchfork$ K and $\partial\Sigma = K$

III. $sl(K)$ is bounded above for all $\pitchfork$ knots

IV. $sl(K)$ is bounded above for all $\pitchfork$ knots in a fixed topological knot type (for example $\pitchfork$ unknots)

In braid theory Birman–Menasco have proven many things useing the braid foliation analysis

a sample of results:

<u>3-braids:</u> A link L that is the closure of a 3-braid has a unique 3-braid representative (upto conjugation) except for

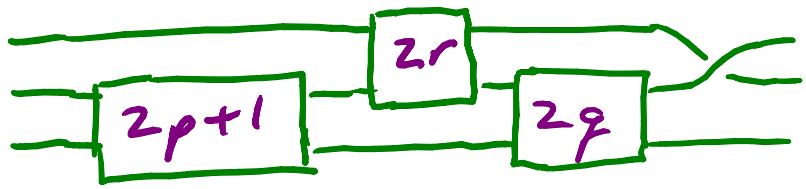
1) Unknot: $\sigma_1\sigma_2, \sigma_1^{-1}\sigma_2^{-1}, \sigma_1\sigma_2^{-1}$
2) $(2,k)$ torus knot: $\sigma_1^k\sigma_2, \sigma_1^k\sigma_2^{-1}$
3) Links that are the closure of $\sigma_1^p\sigma_2^q\sigma_1^r\sigma_2^\delta, \sigma_1^p\sigma_2^\delta\sigma_1^r\sigma_2^q$ where p,q,r distinct, abs val ≥ 2, $\delta = \pm 1$

<u>Markov Theorem Without Stabilization:</u>

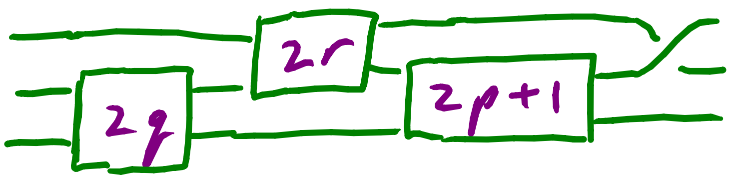
"For each n there are a finite set of "moves" s.t. if two braids of index $\leq n$ represend the same link then you can get from one to the other with these moves"

Transverse "Non-simple" knots:

let K_1 and K_2 be the closures of the 3-braids



and



where $p+1 \neq q \neq r$, $p,q,r > 1$

then K_1 and K_2 are transverse knots that are

1) topologically isotopic
2) have same sl
3) are <u>not transversely isotopic</u>

<u>Note</u>: these were the first such examples!
(see also E-Honda...)