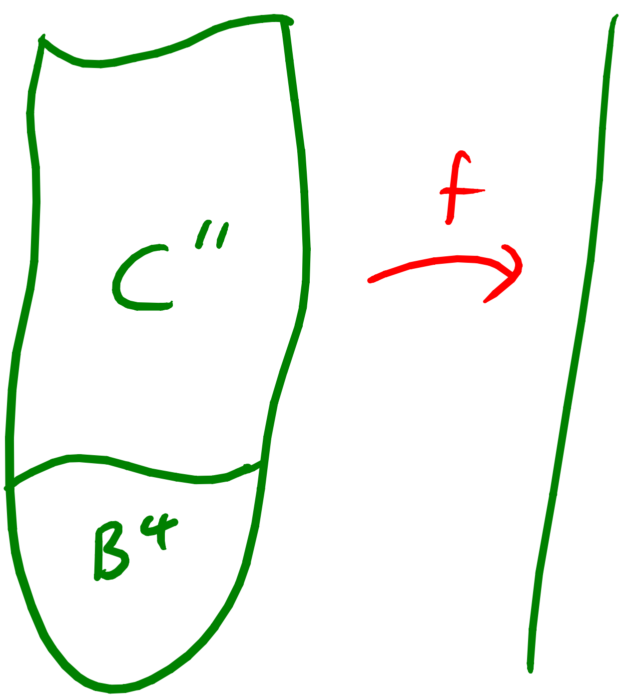
Contact Structures on 5-manifolds



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I. Introduction

a (co-oriented) contact structure on a manifold M^{2n+1} is a hyperplane field $\xi^{2n} \subset TM$ such that $\exists$ a 1-form α satisfying

$$\xi = \ker \alpha$$

and

$$\alpha \wedge d\alpha^n \neq 0$$

Question: $\boxed{\text{Which } (2n+1)\text{-mflds admit a contact str?}}$

note: $d\alpha|_\xi$ is a non-degenerate 2-form.

so $\exists$ a complex structure

$$J : \xi \to \xi \quad (\text{i.e. } J^2 = -id_\xi)$$

such that

$$d\alpha(v, Jv) > 0$$

$$\forall\ v \neq 0 \text{ in } \xi.$$

so given a contact structure ξ on M we see

$$TM = \xi \oplus \mathbb{R}$$

complex bundle

so the structure group of TM admits a reduction to $U(n) \times \mathbb{1}$

such a reduction is called an <u>almost</u> <u>contact</u> <u>structure</u>

<u>Examples:</u>

1) any oriented plane field ξ on a 3-manifold M is an almost contact structure (fix metric and J is rotate counterclockwise by $\pi/2$)

2) M^5 closed oriented has an almost contact str

$$\Longleftrightarrow$$

$w_2(M) \in H^2(M; \mathbb{Z}_2)$ has an integral lift

Stiefel-Whitney

Refined Question: Which almost contact structures are homotopic to contact structures?

Dim 3: Martinet '71: any oriented 3-manifold admits a contact structure

Lutz '71: any oriented plane field on a 3-mfd is homotopic to a ct. str.

Th$^{\underline{m}}$ (E, indep. Casel, Pancoli and Presas):

an any closed oriented 5-manifold any almost contact str is homotopic to a contact structure

Precursors: Geiges '92: $\pi_1 = 1$

Geiges-Thomas '01: some π_1's

CPP '12: If H^2 has no 2-torsion (removed that restriction later)

Remark:

1) Our approach to this theorem involves open books

2) The approach is very similar to Giroux's strategy for the analogous theorem in dimension 7 and above.

3) It is likely that there will be a uniform proof of existance in dims ≥ 7 so note cases $3, 5, \geq 7$ are different.

Reason (?): Character of Stein/Weinstein mflds in dim 2, 4 and ≥ 6.

II Open Books & Almost Ct Str's

(Y,π) is an open book for M^5 if

1) Y is 3-manifold in M with

$$N(Y) = Y \times D^2$$

2) $\pi : (M - Y) \to S^1$ is fibration such that:

$$\begin{tikzcd} (N(Y)-Y) \arrow[r] & D^2 - \{(0,0)\} \arrow[r] & S^1 \\ (p,(r,\theta)) \arrow[r, mapsto] \arrow[rr, bend right, "\pi|_{N(Y)-Y}"'] & (r,\theta) \arrow[r, mapsto] & \theta \end{tikzcd}$$

Y is called the binding

$X = \overline{\pi^{-1}(\theta)}$ is called the page.

note: $(M \setminus Y) \setminus X = X \times [0,1]$

so $M \setminus Y = X \times [0,1] \big/ (x,1) \sim (\phi(x), 0)$

for some diffeo. $\phi : X \to X$ (with $\phi|_{\partial X} = id$) ← monodromy

Fix an open book (Y, π) for M^5

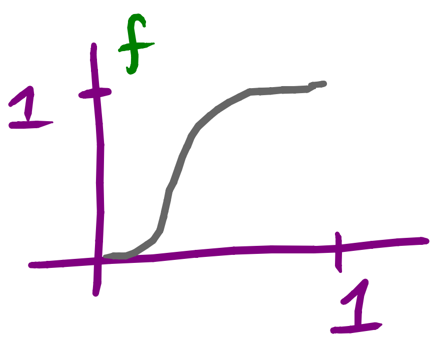
for any plane field ξ on Y

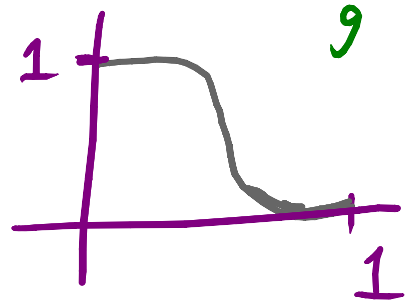
there is a 1-form α : $\xi = \ker \alpha$

on $N(Y) = Y \times D^2$ let

$$\boxed{\beta = f(r)\, d\theta + g(r)\, \alpha}$$

where





extend β to rest of M by $d\pi$

$\ker \beta$ is a hyperplane field on M

so we have a map

$$H : \left\{ \begin{matrix} \text{homotopy classes} \\ \text{plane field on } Y \end{matrix} \right\} \to \left\{ \begin{matrix} \text{homotopy classes} \\ \text{of hyperplane} \\ \text{fields on } M \end{matrix} \right\}$$

depends on (Y, π)

Thm (E):

H is well-defined

if pages of (Y, π) only have handles of index ≤ 2, then H is onto

Remarks:

1) Analogous Thm true in all dims
2) Such open books exist by Quinn '79

Idea of Proof:

only if TM trivial

$$\left\{\begin{array}{l}\text{htpy classes} \\ \text{hyp. plane} \\ \text{on } M^5\end{array}\right\} \longleftrightarrow [M^5, S^4]$$

$$[M^5, S^4] \longleftrightarrow \left\{\begin{array}{l}\text{framed } S^1\text{'s} \\ \text{in } M^5\end{array}\right\}$$

Thom-Pontryagin

S^1 from

So ξ determined by $[\gamma] \in H_1(M)$ and elt of $\mathbb{Z}_2$

$[\gamma]$ " $=$ " " $\frac{1}{2} P.D.(c_1(\xi))$ "

moreover $[\gamma]$ determines $\}$ on 4-skelata of M

and elt of $\mathbb{Z}_2$ determines $\}$ on 5-skelata

Similarly η plane field on Y^3 determined by framed link in Y^3

Given conditions on (Y, π) easy to see

$$H_1(Y) \to H_1(M) \quad \text{surjective}$$

also, clearly

$$\mathbb{Z} \to \mathbb{Z}_2 \quad \text{surjective}$$

Can turn this into surjectivity of H

N.B. since TM not really trivial need to work with "difference classes".

we can improve on above Th$^{\underline{m}}$

Th^m(E): Given an almost ct str (η, J) on M and open book (Y, π) as above, then $\exists$ an overtwisted contact str ξ on Y and an almost complex str J' on $H(\xi)$ such that

- $(\eta, J) \simeq (H(\xi), J')$
- on $N(Y) = Y \times D^2$, ξ is a J' complex sub-bundle of $H(\xi)$
- $H(\xi)$ is tangent to pages of (Y, π) outside $N(Y)$ (and so J' gives an almost complex str on each page).

The 1st and 3rd points are almost obvious.

2nd point is more complicated need to carefully analyze construction of $H(3)$ and consider complex bundles over Y.

III Outline of Proof

let (η, J) be an almost contact structure on M.
and (Y, π) an open book as above.
We know $\exists$ overtwisted ξ on Y and J' on $H(\xi)$ as in last th$^{\underline{m}}$

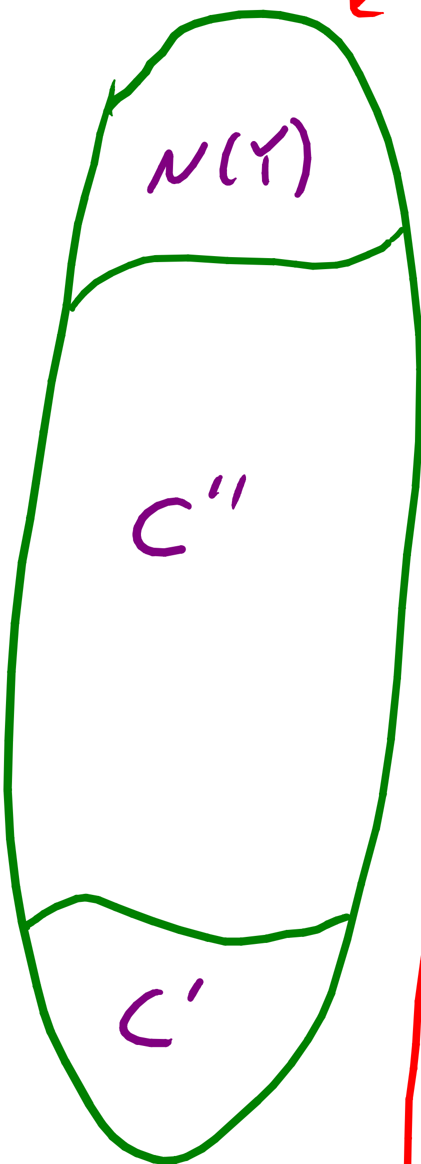
let ϕ be monodromy of (Y, π) and isotope so ϕ fixes a 4-ball X' in the page X

set $X'' = X \setminus X'$ and

$$C' = X' \times [0,1] \big/ (x,1) \sim (\phi(x),1) \cong B^4 \times S^1$$

$$C'' = X'' \times [0,1] \big/ (x,1) \sim (\phi(x),1)$$

So M is



③ now have a loop of overtwisted contact strs on $\partial N(Y) = S^1 \times Y^3$ can use Eliashberg '89 to extend ct str in ② over $N(Y)$.

② now each page of C'' is a cobordism S^3 to Y. Can arrange only 1 and 2 handles. Cieliebak-Eliashberg '12 now give a Weinstein str. on each page. Can use to extend ct str from ① to C''.

① construct contact structure here using E-Pancholi '11 s.t. on $\partial B^4 \times pt = S^3 \times pt$ get overtwisted ct str.

IV Part ① of Proof

Proposition (E-Pancholi '11):

$\exists$ a contact structure ξ on $S^1 \times B^4$ such that on

$$\operatorname{nbhd}(\partial(S^1 \times B^4)) = \underset{\theta}{S^1} \times \underset{t}{[1/2, 1]} \times S^3$$

ξ is given by

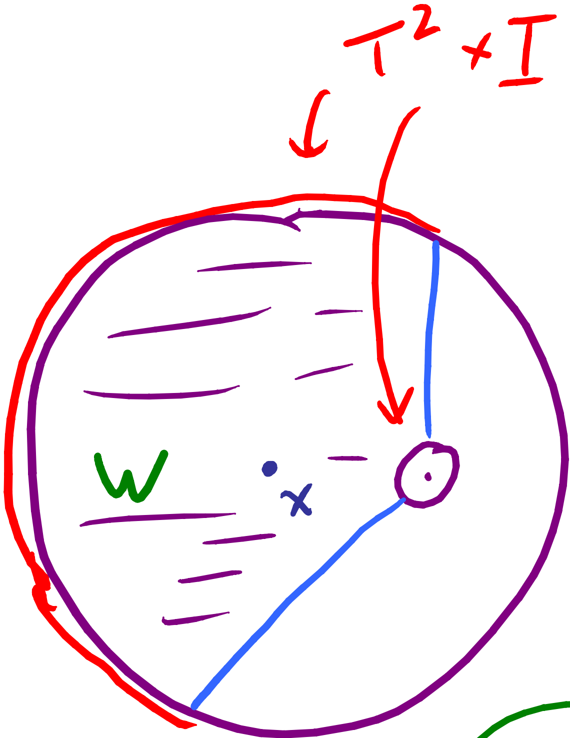
$$\ker(K\,d\theta + t\,\alpha_{ot})$$

where $K > 0$ any constant and α_{ot} is "minimally twisting" overtwisted contact str on S^3.

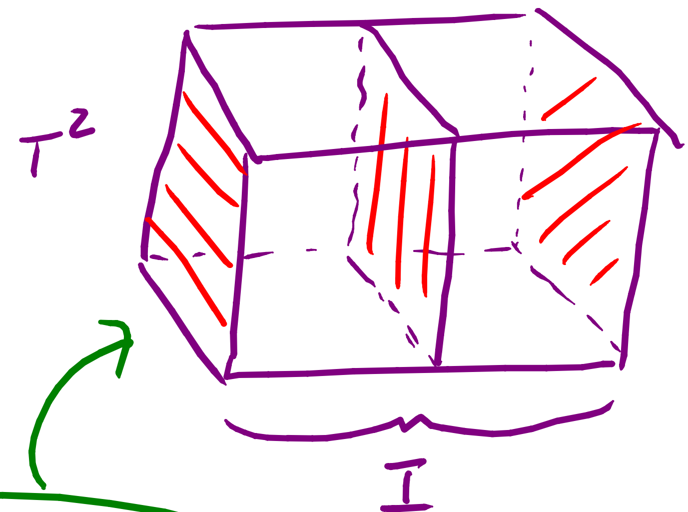
Sketch of Proof:

Use toric symplectic picture on $T^2 \times \mathbb{R}^2$ with $d(\beta = p_1 d\theta_1 + p_2 d\theta_2)$

$(\theta_1, \theta_2\ p_1 p_2)$



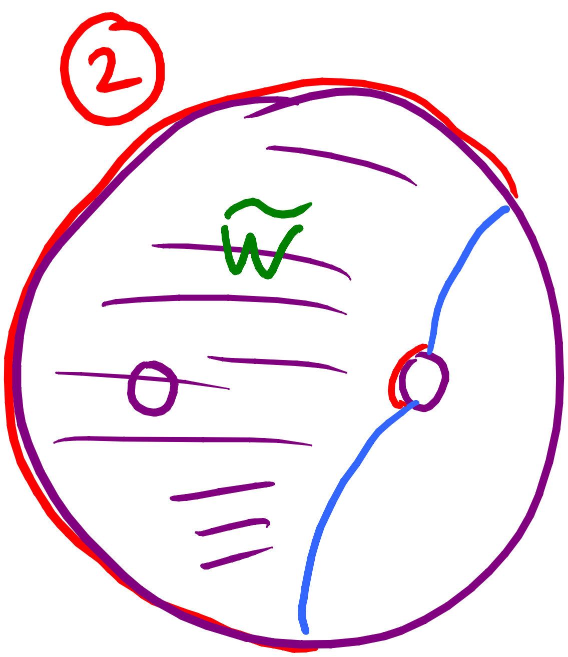
and β induces contact structure on $T^2 \times I$



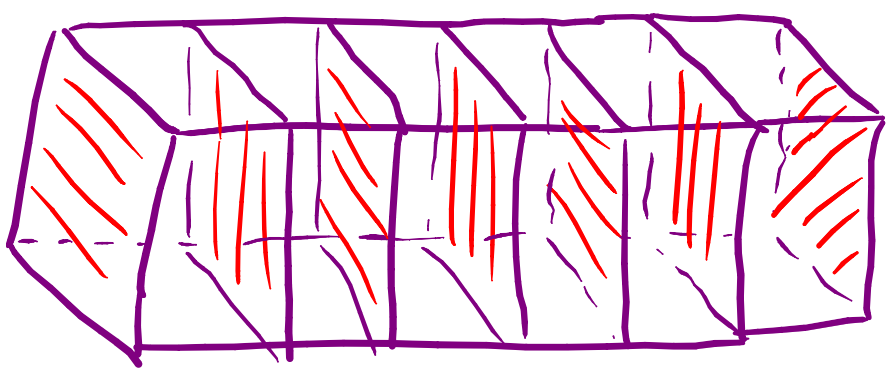
collapse fol^n on ∂ to get tight contact str on S^3

Take 2-fold branch cover over $T^2 \times \{x\}$

on 1 still have ct str above

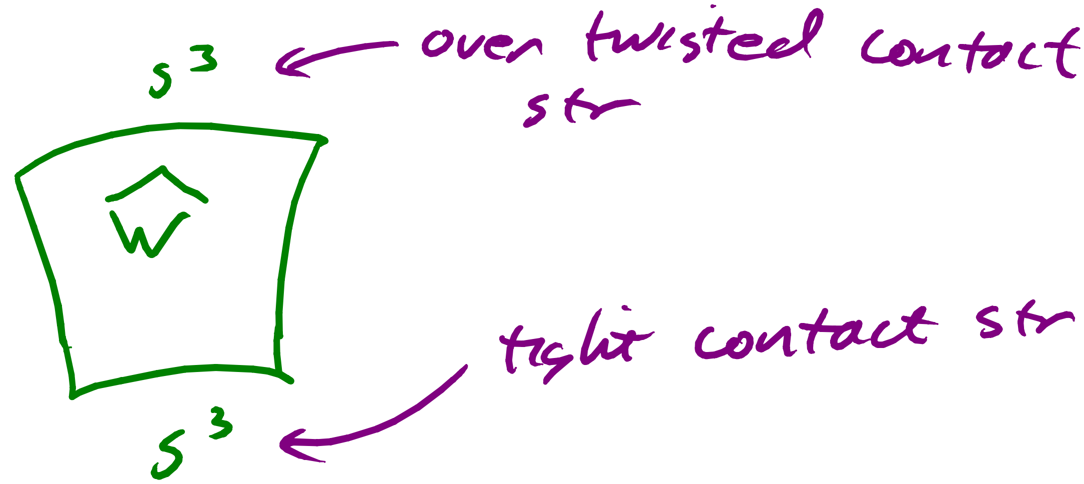


on 2 have



If we collapse all fol^{n}s on all "blue" tori (i.e. tori above blue lines) of $\widetilde{W}$

then we get a cobordism

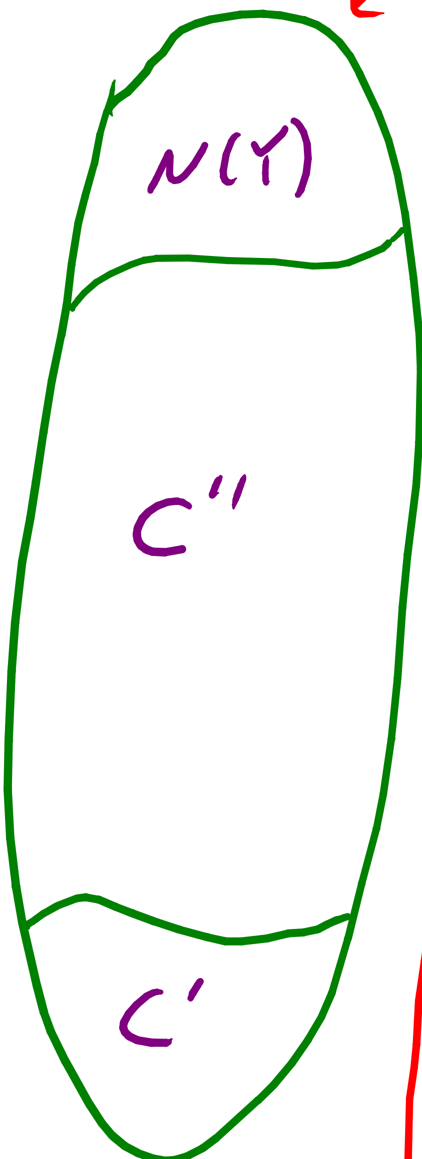


$\widehat{W}$ is an exact symplectic cobordism <u>except</u> it is singular along a torus

<u>Nice Trick:</u> You can make $T^2 \times \{x\} \times S^1$ in $W \times S^1$ a contact submanifold so $\widehat{W} \times S^1$ has a contact structure

Now just glue standard symplectic $B^4 \times S^1$ to bottom of $\widehat{W} \times S^1$ ✓

This finishes 1



3 now have a loop of overtwisted contact strs on $\partial N(Y) = S^1 \times Y^3$ can use Eliashberg '89 to extend ct str in 2 over $N(Y)$.

2 now each page of C'' is a cobordism S^3 to Y. Can arrange only 1 and 2 handles. Cieliebak-Eliashberg '12 now give a Weinstein str. on each page. Can use to extend ct str from 1 to C''.

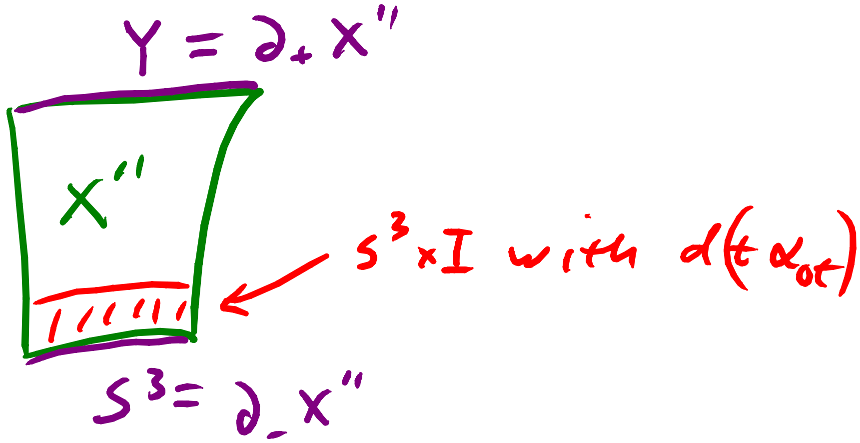
1 construct contact structure here using E-Pancholi '11 s.t. on $\partial B^4 \times pt = S^3 \times pt$ get overtwisted ct str.

Done

V Part (2) of Proof

note: $C'' \setminus X'' = X'' \times [0,1]$

where



let f be a Morse function on X'' with only critical pts of index ≤ 2.

set $f_0 = f$ and $f_1 = f \circ \phi$ and

let f_t be a generic family of functions

think of $f_t : \underbrace{X'' \times \{t\}}_{X''_t} \longrightarrow \mathbb{R}$

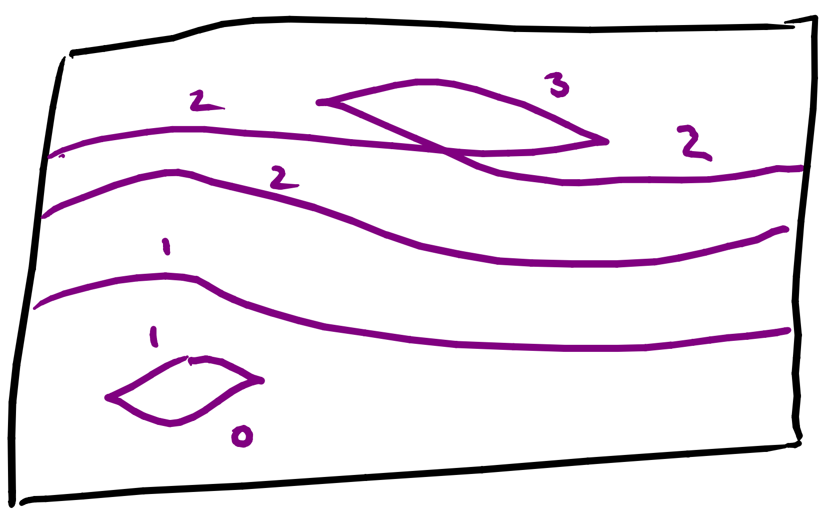
$Th^{\underline{m}}$ (E):

after possibly changing open books for M we can assume f_t has no critical points of index > 2 (but there might be birth/death moments for 1/2 pairs)

Idea of Proof:

Use Fenn-Rourke's '79 extentions of Kirby's '78 proof that Kirby calc works in non-simply conn. mfds.

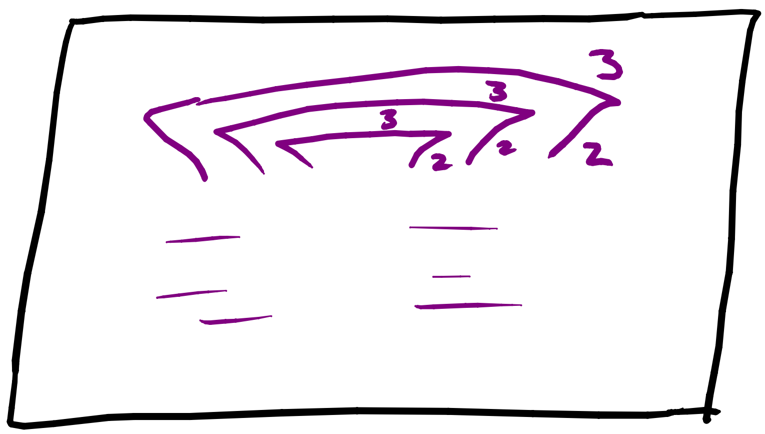
look at Cerf graphic: in $[0,1] \times \mathbb{R}$ plot critical points of f_t



Standard Facts:

- can elliminate birth/death of 0\1 and 3\4 pairs

- can arrange 2/3 pairs like



now take topmost 2/3 pair and push cusps off end

can surger X_t''s to turn top line of crit points into index 2 crit points and replace X_t'' with $X_t'' \#(S^2 \times S^2)$

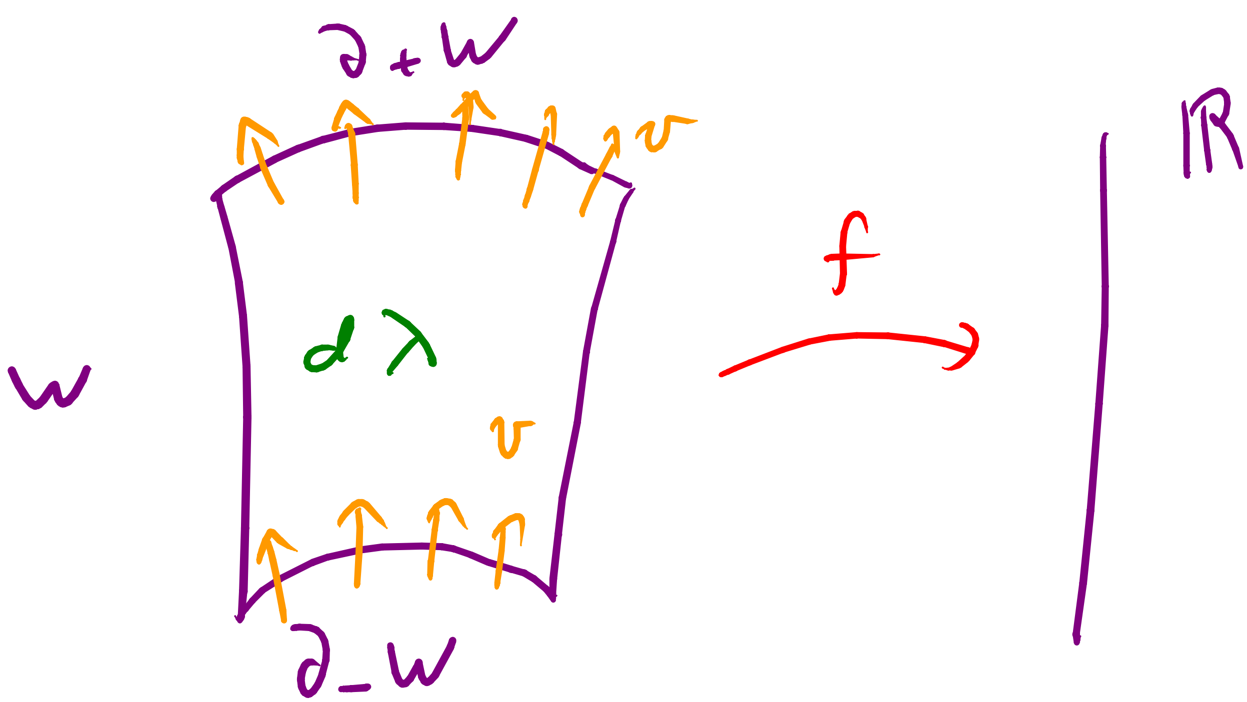
lemma (Saeki '87 + ε):

S^5 has an open book with page
$$[(S^2 \times S^2) \# (S^2 \times S^2)] - D^4$$
and monodromy ϕ_{stab} such that pages have Morse functions with only index 0 and 2 critical pts

- So if we take $X_t'' \# (S^2 \times S^2)$ and connect sum with $S^2 \times S^2$ again then we get a new page X_t'''
- if we use monodromy $\phi \circ \phi_{stab}$ the resulting manifold is $M \# S^5 = M$
- and f_t for these pages have one less family of index 3 crit pts.
- Can continue to elliminate all index 3 critical points!

definition: a Weinstein structure on W is a tuple (λ, v, f) where

- $d\lambda$ is symplectic
- v vector field s.t.
$$\iota_v d\lambda = \lambda$$
- f is a Morse function s.t. v is gradient like for f (i.e. $df(v) \geq \delta(|v|^2 + |df|^2)$)
- $\partial_- W, \partial_+ W$ are level sets of f



Thm (Eliashberg '98, Weinstein '91):

given $t\alpha_{0t}$ near $\partial_- X''_0$, f_0 and J'
$\exists$ Weinstein str on X''_0: (λ_0, v_0, f_0)
s.t. J' compatible with $d\lambda_0$ and
all 2-handles attached to Legendrian
knots with overtwisted disk in
complement

Now use ϕ to get: (λ_1, v_1, f_1) on X''_1

Thm (Cieliebak-Eliashberg '12):

Given $(X''_0, \lambda_0, v_0, f_0)$ & $(X''_1, \lambda_1, v_1, f_1)$
as above and
1) a family f_t, $t\in[0,1]$, with no crit. points of index >2.
2) non-degenerate 2-forms β_t interpolating between $d\lambda_0 = \beta_0$ and $d\lambda_1 = \beta_1$

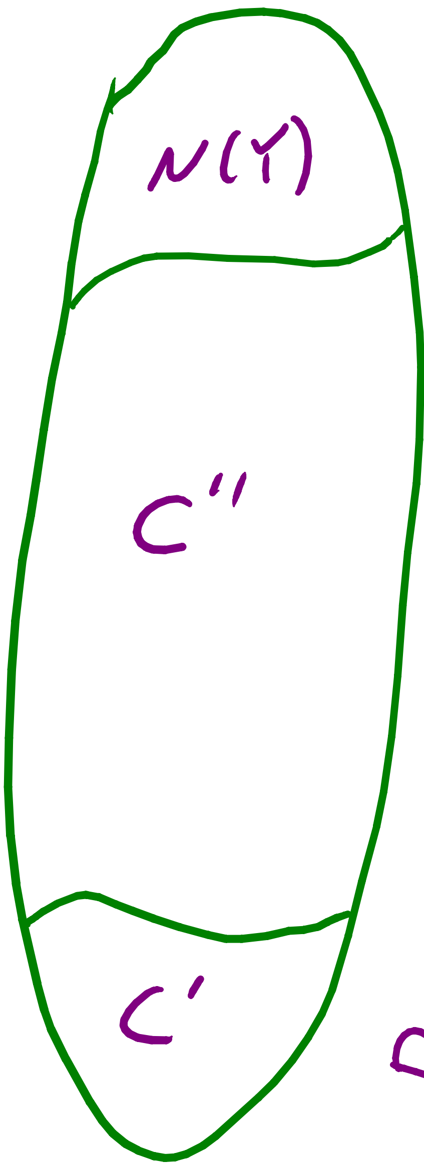
then $\exists$ (λ_t, v_t, f_t) Weinstein strs agreeing with $t\alpha_{0t}$ near $\partial_- X''_t$.

- the complex structure J on $H(\xi)$ gives almost complex structures J_t on X''_t
- these give non-degenerate 2-forms interpolating between $d\lambda_0$ and $d\lambda_1$
- Cieliebak - Eliashberg give λ_t on X''_t.
- the form

$$\alpha = K\,dt + \lambda_t$$

is contact on $X'' \times [0,1]$ for all large K and induces a contact form on C'' that extends the one on C'

This finishes 1 and 2



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2 now each page of C'' is a cobordism S^3 to Y. Can arrange only 1 and 2 handles. Cieliebak-Eliashberg '12 now give a Weinstein str. on each page. Can use to extend ct str from 1 to C''.

Done

1 construct contact structure here using E-Pancholi '11 st. on $\partial B^4 \times pt = S^3 \times pt$ get overtwisted ct str.

Done

VI Part 3 of Proof

note $\partial N(Y) = \partial(C' \cup C'') = S^1 \times Y$

so the $\lambda_t|_{Y_t}$ give a loop of overtwisted contact structures on Y

lemma: Can arrange there is a fixed overtwisted disk D in Y for all $\xi_t = \ker \lambda_t|_{Y_t}$

Th^m (Eliashberg '89):

$$\left\{ \begin{array}{l} \text{overtwisted} \\ \text{ct. str. on } Y \\ \text{with } D \text{ o.t.} \\ \text{disk} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{plane fields} \\ \text{on } Y \text{ that} \\ \text{agree at a} \\ \text{point} \end{array} \right\}$$

is a weak homotopy equivalence

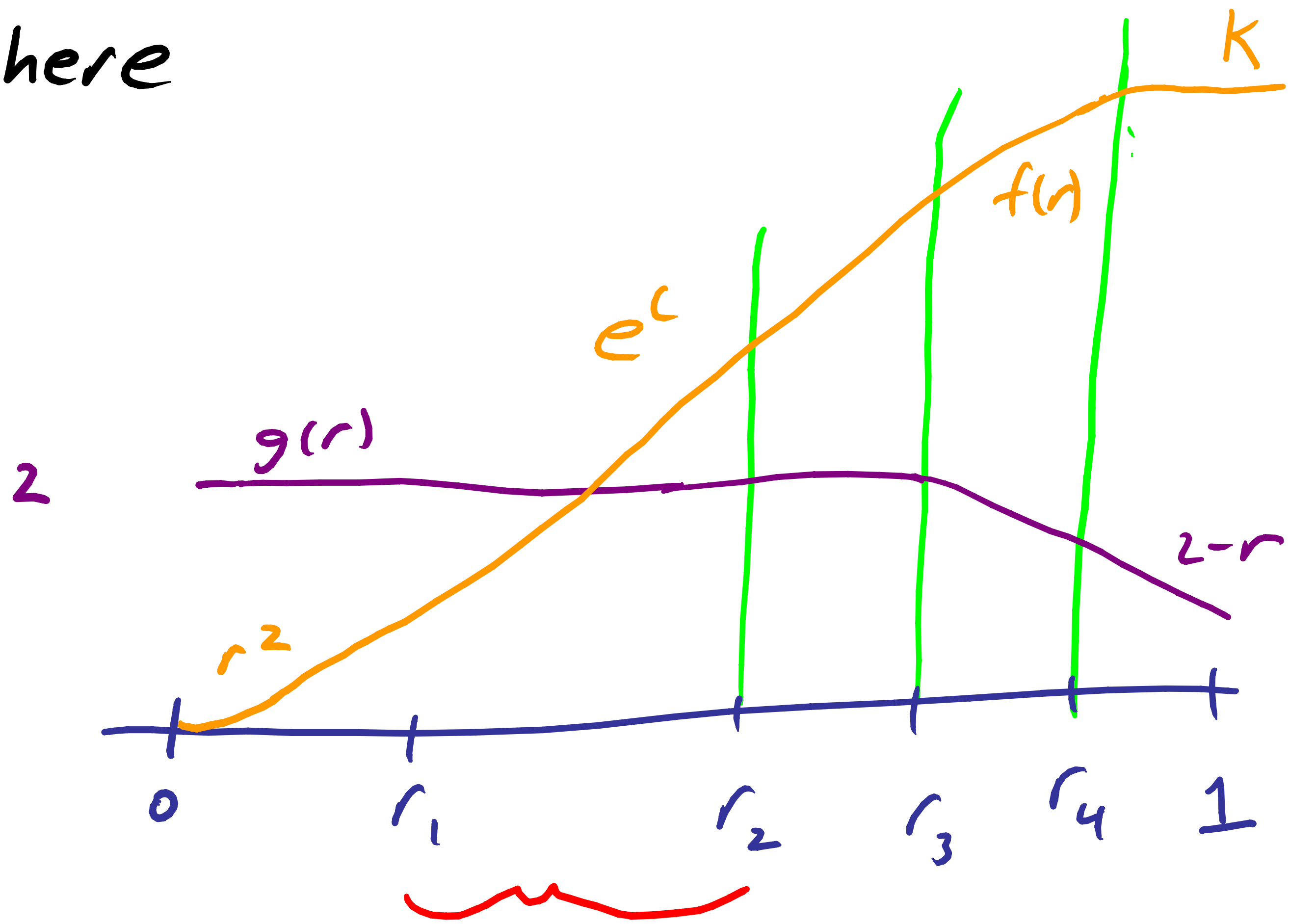
by construction $H(\xi)$ above the $\xi_t = \ker \lambda_t|_{Y_t}$ form a contractible loop

so $\exists\ \lambda_{r,\theta}$ contact forms on Y for each $(r,\theta) \in D^2$

set

$$\beta = f(r)\,d\theta + g(r)\,\alpha^r_\theta$$

where



$\lambda_{r,\theta}$ indep of r outside $[r_1, r_2]$

we have (denote $\lambda_{r\theta}$ by λ)

$$\beta\wedge d\beta\wedge d\beta = g(f'g - g'f)\, dr\wedge d\theta\wedge\lambda\wedge d\lambda$$
$$- g^2 dr\wedge d\theta\wedge\frac{\partial\lambda}{\partial r}\wedge\left(f d\lambda - g\lambda\wedge\frac{\partial\lambda}{\partial\theta}\right)$$

Key observation: by choosing K large can make f' large and first term dominates second

So β is contact and extends contact structure from $C'\cup C''$ to $N(Y)$.

So we are done!

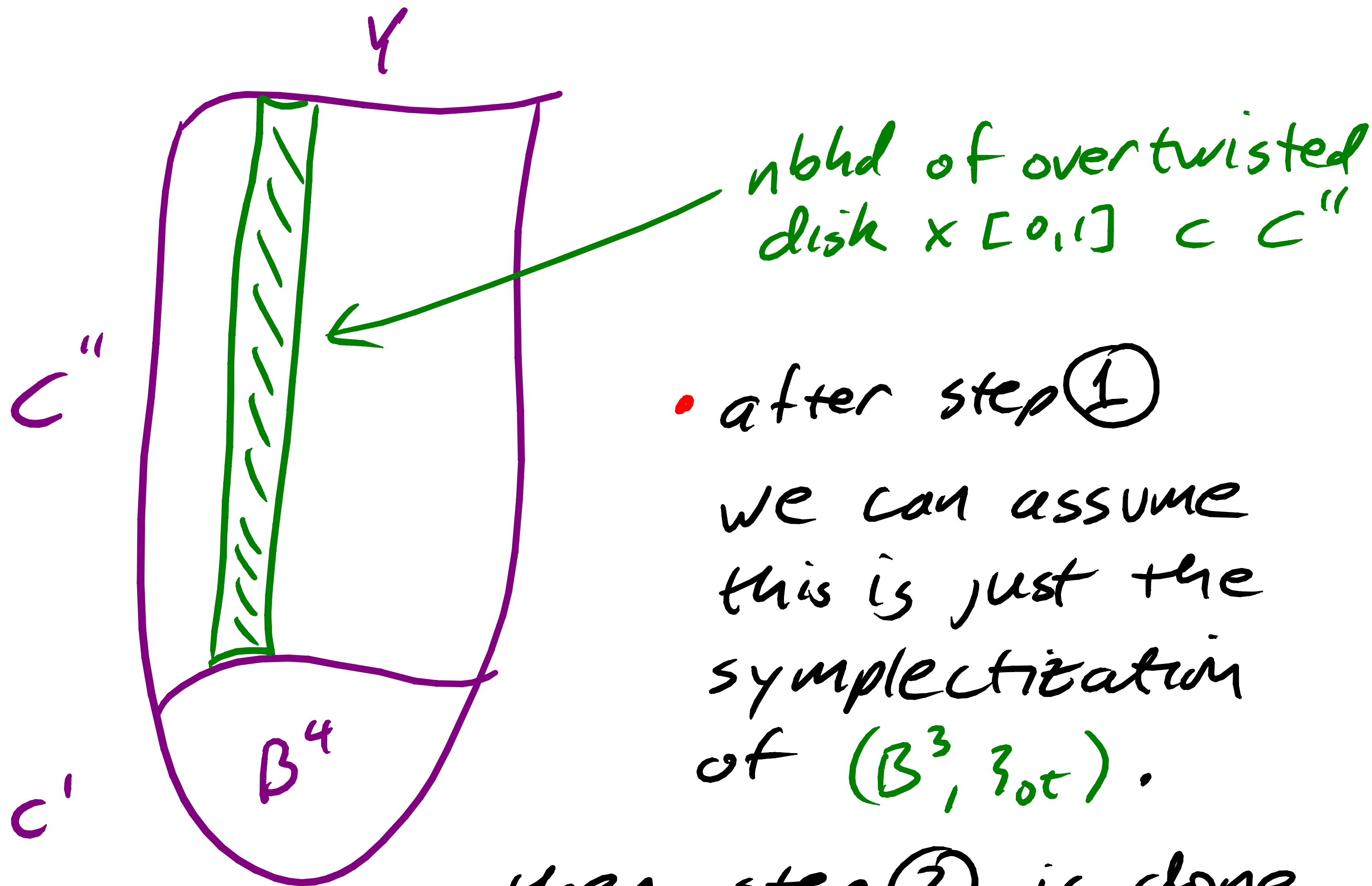
Remarks:

1) One can see that in each step we are just homotoping the almost contact str.

2) The only tricky part now is the lemma:

lemma: $\boxed{\text{Can arrange there is a fixed overtwisted disk } D \text{ in } Y \text{ for all } \xi_t = \ker \lambda_t|_{Y_t}}$

to see this just need relative version of Cieliebak-Eliashberg



- after step 1 we can assume this is just the symplectization of (B^3, ξ_{ot}).
- then step 2 is done relative to this so that lemma is true.
- this is not hard but makes writing the details of the proof a bit messy.

Thanks

for

Your

attention