HOMEWORK ASSIGNMENTS
245C, SPRING 2024

WILFRID GANGBO

1. Homework #1; Due Wednesday 17 April

In this homework, $(\Omega, \mathcal{M}, \mu)$ is an outer measure space and $f : \Omega \to [-\infty, +\infty]$ is measurable function which assume finite values μ–almost everywhere.

Exercise 1.1 $(*)$**.** *Let $g \in C(\mathbb{R}^d)$ be such that for all $a \in \mathbb{R}^d$, there exists $p \in \mathbb{R}^d$ such that*

$$g(x) \geq g(a) + (x - a, p) \qquad \forall x \in \mathbb{R}^d.$$

Show that g is convex.

Exercise 1.2 $(*)$**.** *Let $g \in C^2(\mathbb{R}^d)$. Show that g is convex if and only if $\nabla^2 g \geq 0$. In this case, show that for all $a \in \mathbb{R}^d$ we have*

$$g(x) \geq g(a) + (x - a, \nabla g(a)) \qquad \forall x \in \mathbb{R}^d.$$

Exercise 1.3. *Let $p \in [1, +\infty)$.*

(i) *Let $g : \Omega \to [-\infty, +\infty]$ be measurable functions which assume finite values μ–almost everywhere. Show that if $c \in \mathbb{R}$ then $[cf]_p = |c|[f]_p$, and $[f + g]_p \leq 2\big([f]_p^p + [g]_p^p\big)^{1/p}$.*

(ii) *Deduce that* weak$(L^p(\Omega))$ *is a vector space.*

(iii) *Show that the "balls" $\{g \in \mathrm{weak}(L^p(\Omega)) : [g - f]_p < r\}_{\{f \in \mathrm{weak}(L^p(\Omega)), r > 0\}}$ generate a topology on* weak$(L^p(\Omega))$ *that makes* weak$(L^p(\Omega))$ *into a topological vector space.*

Exercise 1.4 $(*)$**.** *Let $p \in [1, +\infty)$.*

(i) *Show that if $f \in \mathrm{weak}(L^p(\Omega))$ and $\mu\big(\{x \in \Omega : \ f(x) \neq 0\}\big) < +\infty$ then $f \in L^q(\Omega)$ for any $q \in [1, p)$.*

(ii) *Show that if $f \in \mathrm{weak}(L^p(\Omega)) \cap L^\infty(\Omega)$ then $f \in L^q(\Omega)$ for any $q \in (p, +\infty)$.*

Exercise 1.5 $(*)$**.** *For $A > 0$, let $E(A) = \{|f| > A\}$ and set*

$$h_A = f\chi_{\Omega \setminus E(A)} + A\,\mathrm{sgn}(f)\chi_{E(A)}, \quad g = f - h_A.$$

Show that

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \qquad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \text{if} \quad \alpha < A \\ 0 & \text{if} \quad \alpha \geq A \end{cases}$$

Exercise 1.6 $(*)$**.** *Let $p \in [1, +\infty)$. Show that $f \in L^p(\Omega)$ if and only if $\sum_{k=-\infty}^{+\infty} 2^{pk} \lambda_f(2^k) < +\infty$.*

Exercise 1.7 $(*)$**.** *Let $p \in (0, +\infty)$. Show that if $f \in L^p(\Omega)$ then $\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0$.*

Department of Mathematics, UCLA, Los Angeles, CA 90095
Email address: wgangbo@math.ucla.edu