HOMEWORK ASSIGNMENTS
245C, SPRING 2024

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2. Homework #2; Due Friday 03 May

Notation. Given a closed ball $B_r(a)$ of center a and radius $r > 0$, we denote by $\hat{B}_r(a)$ the ball of center a and radius $5r$.

Exercise 2.1 $(*)$. *Let $f : \mathbb{R}^d \to [-\infty, +\infty]$ be a $\mathcal{L}^d$–measurable function. Show that if $f \in L^\infty(\mathbb{R}^d)$ then $\|f\|_\infty = \inf_{\alpha>0} \big\{\alpha : \ \lambda_f(\alpha) = 0\big\}$.*

Exercise 2.2. *Let $f \in L^1(\mathbb{R}^d)$ be such that $\|f\|_1 > 0$. Show that there exist $C, R > 0$ such that $M(f)(x) \geq C|x|^{-d}$ for $|x| > R$. Hence, $\|Mf\|_{L^1} = +\infty$ and $\lambda_{M(f)}(\alpha) \geq C'/\alpha$ when α is small, so the estimate in the maximal theorem is essentially sharp.*

Exercise 2.3. *For $f : \mathbb{R}^d \to [-\infty, \infty]$, we define*

$$M_*(f)(x) = \sup_B \bigg\{ \frac{1}{\mathcal{L}^d(B)} \int_B |f(y)|dy : \ x \in B, \ B \subset \mathbb{R}^d \text{ is a non degenerate ball} \bigg\}.$$

Show that $M(f) \leq M_*(f) \leq 2^d M(f)$.

Exercise 2.4. *Let N be sets of finite Lebesgue measure.*

(i) *Show that if $\mathcal{L}^d(N) = 0$ then for every $\delta, \epsilon > 0$, there exists a family $\{B_j\}_{i=1}^\infty$ of disjoint non degenerate closed balls such that* $\operatorname{diam}(B_i) < \delta$,

$$N \subset \cup_{i=1}^\infty B_i \quad \text{and} \quad \sum_{i=1}^\infty \mathcal{L}^d(B_i) < \epsilon.$$

(ii) *Show that if $A \subset \mathbb{R}^d$ is a set of finite Lebesgue measure then for every $\delta, \epsilon > 0$ there exists a family $\{C_j\}_{j=1}^\infty$ of disjoint non-degenerate closed balls such that* $\operatorname{diam}(C_j) <$ *δ and there exists a family $\{B_j\}_{i=1}^\infty$ of disjoint non degenerate closed balls such that* $\operatorname{diam}(B_i) < \delta$ *and*

$$A \subset \big(\cup_{j=1}^\infty C_j\big) \cup \big(\cup_{i=1}^\infty \hat{B}_i\big), \qquad \sum_{j=1}^\infty \mathcal{L}^d(C_j) + 5^d \sum_{i=1}^\infty \mathcal{L}^d(B_i) \leq \mathcal{L}^d(A) + 2\epsilon.$$

(iii) *Conclude that if $A \subset \mathbb{R}^d$ is a set of finite Lebesgue measure then*

$$\mathcal{L}^d(A) = \inf_{\mathcal{F}} \Big\{ \sum_{B \in \mathcal{F}} \mathcal{L}^d(B) \Big\},$$

where the infimum is performed over the set of $\mathcal{F}$, made of countably many non degenerate pairwise disjoint closed balls of radius less than δ, whose union covers A.

Exercise 2.5 $(*)$. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be an l–Lipschitz function. Show that if $A \subset \mathbb{R}^d$ then $\mathcal{L}^d\big(f(A)\big) \leq l^d \mathcal{L}^d(A)$.*

Exercise 2.6 (∗). *Let $O \subset \mathbb{R}^d$ be an open set and let $f \in C^1(O, \mathbb{R}^d)$. Let $a \in O$ and denote by $B_r(a)$ the closed ball of radius r, centered at a and by $D_r(a)$ the interior of $B_r(a)$. Set*

$$L(x) := f(a) + \nabla f(a)(x - a).$$

Show that if $\det(\nabla f)(a) > 0$ then for $r > 0$ small enough

$$f\big(B_r(a)\big) \subset L\Big(B_{r+o(r)}(a)\Big) \quad \text{and} \quad L\Big(B_r(a)\Big) \subset f\Big(B_{r+o(r)}(a)\Big).$$

Hint. By the inverse function theorem, there exists $r_0 > 0$ such that $B_{r_0}(a) \subset O$, $f(D_r(a))$ is open, $f : D_{r_0}(a) \to f(D_{r_0}(a))$ is a bijection with a continuous inverse.

Exercise 2.7 (∗). *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a l–Lipschitz function. We learned that if $A \subset \mathbb{R}^d$ is $\mathcal{L}^d$–measurable then $y \to \mathcal{H}^0\big(f^{-1}(y) \cap A\big)$ is a $\mathcal{L}^d$–measurable function and so, we can define*

$$\bar{\mu}(A) = \int_{\mathbb{R}^d} \mathcal{H}^0\big(f^{-1}(y) \cap A\big) dy.$$

Show that $\bar{\mu}$ can be extended to an outer measure μ which is a Radon measure on $\mathbb{R}^d$. Show that every $\mathcal{L}^d$–measurable set is μ–measurable.

Exercise 2.8 (∗). *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and μ be as in Exercise 2.7.*

(i) *Show that $\mu << \mathcal{L}^d$ (μ is absolutely continuous with respect to $\mathcal{L}^d$).*
(ii) *Show that if we further assume that $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and $a \in \mathbb{R}^d$ is such that $\det(\nabla f(a)) > 0$, then*

$$\lim_{r \to 0} \frac{\mathcal{L}^d\big[f\big(B_r(a)\big)\big]}{r^d} = \lim_{r \to 0} \frac{\mathcal{L}^d\big[L\big(B_r(a)\big)\big]}{r^d}.$$

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