HOMEWORK ASSIGNMENTS
245C, SPRING 2024

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3. Homework #3: Due Friday 17 May

Exercise 3.1. *Let $(X, \mathcal{T})$ be a topological space and let $B \subset X$.*
(i) Show that $\mathrm{int}(B)$ *is the union of the open sets contained in B.*
(ii) Show the $\bar{B}$ is the intersection of the closed sets containing B.

Exercise 3.2. *Let $(X, \mathcal{T})$ be a topological space and let $B \subset X$. Show that $f : X \to \mathbb{R}$ is continuous if and only if $f^{-1}(J) \in \mathcal{T}$ for every open set $J \subset \mathbb{R}$.*

Exercise 3.3. *Let $(X, \mathcal{T})$ be a topological space and let $K \subset X$.*
(i) Show that if K is closed and X is compact, then K is compact.
(ii) Show that if X is a Hausdorff space and K is compact then K is closed.

Hint. (i) Use the fact that if $\{O_i\}_{i\in I}$ is an open cover of K, then $\{K^c\} \cup \{O_i\}_{i\in I}$ is an open cover of X. (ii) Show that for every $x \notin K$ there exist two disjoint open sets U, V such that $x \in U$ and $F \subset V$.

Exercise 3.4 $(*)$**.** *Let X be an LCH space and let μ be a Radon measure on X.*
(i) Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Show that $\mu(N) = 0$. The complement of N, denoted by $\mathrm{spt}(\mu)$ *is called the support of μ.*
(ii) Show that $x \in \mathrm{spt}(\mu)$ if and only if $\int_X f d\mu > 0$ for every $f \in C_c(X, [0, 1])$ such that $f(x) > 0$.

Exercise 3.5 $(*)$**.** *Let X be an LCH space and let μ be a Radon measure on X. Show that μ is inner regular on every σ–finite set.*

Exercise 3.6 $(*)$**.** *Let $X = \mathbb{N}$ with the discrete topology. Show that $C_0(X)^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$.*

Exercise 3.7. *Let $p \in [1, +\infty)$ and suppose that $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that*

$$\lim_{y\to 0} \left\| \frac{\tau_{-y} f - f}{y} - h \right\|_p = 0,$$

we call h the strong L^p–derivative of f. If $f \in L^p(\mathbb{R}^d)$, L^p–partial derivatives of f are defined similarly.

Suppose that p and q are conjugate exponents, $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, and $\partial_j f$, the strong L^p–partial derivatives of f exists. Show that the ordinary derivative $\frac{\partial}{\partial x_j}(f * g)$ exist and equal $(\partial_j f) * g$.

Exercise 3.8 $(*)$**.** *Let $p \in [1, +\infty)$ and suppose that $f \in L^p(\mathbb{R})$. Show that the following are equivalent*

(i) *The strong L^p–derivative of f exists, call it h.*

(ii) *f is absolutely continuous on every bounded interval (perhaps after modification on a null set) and its pointwise derivative f' is in L^p, in which case $h = f'$.*

For "only if" use exercise 3.7 with $g \in C_c(\mathbb{R})$ such that $\int_{\mathbb{R}} g = 1$, $f * g_t \to f$ and $(f * g_t)' \to h$ as $t \to 0$. For "if", write

$$\frac{f(x+y) - f(x)}{y} - f'(x) = \frac{1}{y} \int_0^y \big(f'(x+t) - f'(x)\big) dt.$$

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