HOMEWORK ASSIGNMENTS
245C, SPRING 2024

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4. Homework #4: Due Friday 31 May

Exercise 4.1. *Let $O \subset \mathbb{R}^d$ and let $u \in C^2(O)$ be a harmonic function in the sense that $\triangle u = 0$ on O. Show that if $r > 0$, $x \in O$ and $B_r(x) \subset O$ then if ν is the surface measure ($(d-1)$-Hausdorff dimensional measure) then*

$$u(x) = \frac{1}{\nu(\partial B_r(x))} \int_{\partial B_r(x)} u d\nu = \frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} u dy$$

Hint. Set

$$\phi(r) = \frac{1}{\nu(\partial B_r(x))} \int_{\partial B_r(x)} u d\nu.$$

Show that

$$\phi'(r) = \frac{1}{\nu(\partial B_1(0))} \int_{\partial B_1(0)} \nabla u(x + rw) \cdot w \nu(dw) = 0.$$

Use the change of variables formula

$$\int_{B_r(x)} u dy = \int_0^r \Big(\int_{\partial B_s(x)} u d\nu \Big) ds$$

Exercise 4.2 (∗)**.** *Let $O \subset \mathbb{R}^d$ and let $u \in C^2(O)$ be a harmonic function. Show that $u \in C^\infty(O)$.*

Hint. Let $(\varrho_\epsilon)_\epsilon$ be the standard mollifiers. Use Exercise 4.1 to show that $\varrho_\epsilon * u = u$.

Exercise 4.3. *Assume that $D \subset \mathbb{C}$ is an open set and $f : D \to \mathbb{C}$ is differentiable on .D Show that $u : (x, y) \to \mathrm{Re}(f(x + iy))$ and $v : (x, y) \to \mathrm{Im}(f(x + iy))$ are differentiable on D and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Conclude that if u and v are of class C^2 then they are harmonic functions.

Exercise 4.4. *If $f \in C^\infty$ show that $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β if and only if $\partial^\alpha (x^\beta f)$ is bounded for all multi-indices α, β.*

Exercise 4.5 (∗)**.** *Suppose that Σ is a σ–algebra and (X, Σ, μ) is a measure space. Suppose that $-\infty < a < b < +\infty$ and $f : X \times [a, b] \to \mathbb{R}$ is such that $f(\cdot, t) \in L^1(\mu)$ for each $t \in [a, b]$. Let*

$$F(t) := \int_X f(x, t) \mu(dx)$$

(i) *Suppose there exists $g \in L^1(\mu)$ such that $|f(x, t)| \le g(x)$ for all $x \in X$ and $t \in [a, b]$. Show that if $\lim_{t \to t_0} f(x, t) = f(x, t_0)$ for every $x \in X$ then*

$$\lim_{t \to t_0} F(t) = F(t_0).$$

(ii) *Suppose that $\partial f/\partial t$ exists and there exists $h \in L^1(X)$ such that $|(\partial f/\partial)(t,x)| \leq h(x)$ for all $x \in X$ and $t \in [a,b]$. Show that F is differentiable and*

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\mu(dx).$$

Exercise 4.6 (∗)**.** *Suppose that $f \in L^1$, $g \in C^k$ and $\partial^\alpha g$ is bounded for $|\alpha| \leq k$. Show that $f * g \in C^k$ and $\partial^\alpha (f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leq k$.*

Exercise 4.7 (∗)**.** *Suppose that $0 < p < q < r \leq \infty$.*

(i) *Show that $L^q \subset L^p + L^r$.*

(ii) *Let $\lambda \in (0,1)$ be defined by $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$. Show that $L^p \cap L^r \subset L^q$ and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$*

Proof. (i) Let $f \in L^q$ and set

$$E := \{|f| > 1\}, \quad g := f\chi_E, \quad h := f\chi_{E^c}.$$

We have $f = g + h$, $|g|^p = |f|^p \chi_E \leq |f|^q \chi_E$ and $|h|^r = |f|^r \chi_{E^c} \leq |f|^q \chi_{E^c}$.

(ii) We only consider the case $r < +\infty$ since the other case is rather straightforward to treat. We have

$$\|f\|_q^q = \int_\Omega |f|^{\lambda q} |f|^{(1-\lambda)q} d\mu \leq \left\| |f|^{\lambda q} \right\|_{\frac{p}{\lambda q}} \left\| |f|^{(1-\lambda)q} \right\|_{\frac{r}{(1-\lambda)q}} = \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q}$$

□

Exercise 4.8. *Let $f(x) = 1/2 - x$ on $[0,1)$ and extend f periodically to $\mathbb{R}$.*

(i) *Show that*

$$\hat{f}(0) = 0, \quad \textit{and} \quad \hat{f}(k) = \frac{1}{2\pi i k}, \quad \textit{if} \quad k \neq 0.$$

(ii) *Show that (use Parseval inequality)*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Exercise 4.9 ((∗) Wirtinger's inequality)**.** *Show that if $f \in C^1([a,b])$ is such that $f(a) = f(b) = 0$ then*

$$\int_a^b f^2(x)dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b (f'(x))^2 dx$$

Hint: It suffices to prove the result when $a = 0$ and $b = 1/2$. Extend f to $[-0.5, 0.5]$ by setting $f(-x) = -f(x)$ and then extend f periodically on $\mathbb{R}$. Check that $f \in C^1(\mathbb{T})$ and apply Parseval inequality.

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