OPTIMAL TRANSPORT FOR THE SYSTEM OF ISENTROPIC EULER EQUATIONS

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ABSTRACT. We introduce a new variational time discretization for the system of isentropic Euler equations. In each timestep the internal energy is reduced as much as possible, subject to a constraint imposed by a new cost functional that measures the deviation of particles from their characteristic paths.

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1. INTRODUCTION

The system of isentropic Euler equations models the dynamics of compressible fluids under the simplifying assumption that the thermodynamical entropy is constant in space and time. At each instance in time, the state of the fluid is determined by the density ϱ , which characterizes the distribution of mass and is therefore non-negative, and by the momentum field $\mathbf{m} := \varrho \mathbf{u}$, where $\mathbf{u}$ is the Eulerian velocity. As long as $\varrho > 0$, the velocity is uniquely determined by the momentum; if $\varrho = 0$, however, which corresponds to the vacuum, then $\mathbf{u}$ is undefined.

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The isentropic Euler equations form a system of hyperbolic conservation laws

$$\begin{aligned}\partial_t \varrho + \nabla \cdot \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \nabla \cdot (\varrho^{-1} \mathbf{m} \otimes \mathbf{m}) + \nabla P(\varrho) &= 0\end{aligned}\tag{1.1}$$

for the unknown functions

$$(\varrho, \mathbf{m}): [0, \infty) \times \mathbf{R}^d \longrightarrow \mathbb{H},$$

with $\mathbb{H} := ((0, \infty) \times \mathbf{R}^d) \cup \{(0, 0)\}$. The first equation in (1.1) is called the continuity equation and implies that mass is conserved; the second equation, called the momentum equation, models the conservation of momentum. Notice that we need $\mathbf{m} = 0$ whenever $\varrho = 0$, for the term quadratic in $\mathbf{m}$ to be well-defined (in fact equal to zero) in vacuum. The system (1.1) is then equivalent to

$$\begin{aligned}\partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho) &= 0,\end{aligned}$$

and we can even replace the second equation (formally) by

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla U'(\varrho) = 0.\tag{1.2}$$

We call (1.2) the velocity equation. Since we are interested in the Cauchy problem for (1.1), we assume that initially the fluid is determined by suitable data

$$(\varrho, \mathbf{m})(t = 0, \cdot) = (\bar{\varrho}, \bar{\mathbf{m}}).$$

Note that the pressure P , which appears in the momentum equation, depends only on the density ϱ because the thermodynamical entropy is assumed constant.

One can check that for sufficiently smooth solutions, the functions $(\varrho, \mathbf{m})$ satisfy an additional conservation law for the total energy, which is defined as

$$E(r, m) := \frac{|m|^2}{2r} + U(r) \quad \text{for all } (r, m) \in \mathbb{H}.\tag{1.3}$$

The function $U: [0, \infty) \longrightarrow \mathbf{R}$ denotes the internal energy of the fluid. Notice that in thermodynamics one typically considers the *specific* internal energy (energy per unit mass), whereas we prefer to work with the internal energy directly. Then

$$P(r) := U'(r)r - U(r) \quad \text{for all } r \geq 0.\tag{1.4}$$

Any sufficiently smooth solution $(\varrho, \mathbf{m})$ of (1.1) also satisfies

$$\partial_t \left(\frac{|\mathbf{m}|^2}{2\varrho} + U(\varrho) \right) + \nabla \cdot \left(\left(\frac{|\mathbf{m}|^2}{2\varrho} + Q(\varrho) \right) \mathbf{u} \right) = 0,\tag{1.5}$$

where $Q(\varrho) := U'(\varrho)\varrho$. The function $U'(\varrho)$ is called the specific enthalpy.

It is well-known that typically solutions of (1.1) are not smooth: No matter how regular the initial data is, jump discontinuities can occur in finite time. These jumps form along codimension-one submanifolds in space-time and are called shocks. In shocks, total energy is dissipated (e.g. transformed into heat), so (1.5) cannot hold anymore and must therefore be relaxed to an inequality

$$\partial_t \left(\frac{|\mathbf{m}|^2}{2\varrho} + U(\varrho) \right) + \nabla \cdot \left(\left(\frac{|\mathbf{m}|^2}{2\varrho} + Q(\varrho) \right) \mathbf{u} \right) \leq 0 \quad \text{in } \mathcal{D}'([0, \infty) \times \mathbf{R}^d).\tag{1.6}$$

For the same reason, also (1.1) must be considered in distributional sense. Note that the continuity equation in (1.1) and inequality (1.6) suggest the following natural bounds for solutions of the isentropic Euler equations.

(1) The total mass is conserved in time:

$$\frac{d}{dt} \int_{\mathbf{R}^d} \varrho(t, x) dx = 0 \quad \text{for all } t \geq 0.$$

(2) The total energy is nonincreasing in time:

$$\frac{d}{dt} \int_{\mathbf{R}^d} E(\varrho, \mathbf{m})(t, x) dx \leq 0 \quad \text{for a.e. } t \geq 0.$$

For the case of one-dimensional flows, global existence of solutions to (1.1) satisfying only the natural bounds of finite mass and total energy, has been established recently in [23]. This work generalizes earlier results obtained under more stringent boundedness assumptions; see the references in [23] for further information.

In this paper, we consider internal energies for which

$$\text{the map } r \mapsto r^d U(r^{-d}) \text{ is strictly convex and nonincreasing on } (0, \infty). \quad (1.7)$$

We refer the reader to Section 5.1 for an explanation why this assumption is natural. This class includes the important special case of polytropic fluids, where

$$U(\varrho) = \frac{\kappa \varrho^\gamma}{\gamma - 1} \quad \text{and} \quad P(\varrho) = \kappa \varrho^\gamma, \quad (1.8)$$

with adiabatic coefficient $\gamma > 1$ and $\kappa > 0$ some constant. It also includes the case of isothermal fluids, where $U(\varrho) = \kappa \varrho \log \varrho$ and $P(\varrho) = \kappa \varrho$. In fact, if we replace the internal energy in (1.8) by $U(\varrho) = \kappa(\varrho^\gamma - \varrho)/(\gamma - 1)$, which does not change the pressure and modifies the total energy only by a constant (since the total mass is conserved), then the isothermal case follows as the limiting case as $\gamma \rightarrow 1$. For what follows, the details of U will not be important as long as (1.7) holds.

We already mentioned that it is a natural assumption that the total energy of the fluid does not increase over time. In this paper, we propose to consider solutions of (1.1) that satisfy a much stronger condition: We will present a time discretization that tries to implement the idea that the total energy of solutions of (1.1) not only is nonincreasing in time, but in fact *decreases as fast as possible*. We will make more precise what we mean by that in later sections.

Since weak solutions of hyperbolic conservation laws may be nonunique, additional conditions are needed to single out the physically relevant among all possible weak solutions. The existing literature discusses several ways to impose these extra assumptions. One possibility, which is natural for conservation laws that are motivated by physics, is to require so-called *entropy inequalities*. In the context of the isentropic Euler equations (1.1), an entropy is a function $\eta: \mathbb{H} \rightarrow \mathbf{R}$ with the following property: on every open subset of $[0, \infty) \times \mathbf{R}^d$ where $(\varrho, \mathbf{m})$ is smooth, we have the additional conservation law

$$\partial_t \eta(\varrho, \mathbf{m}) + \nabla \cdot q(\varrho, \mathbf{m}) = 0. \quad (1.9)$$

Here q is a suitable function, called the entropy-flux, which is determined by the entropy η and the flux in (1.1). Note that this entropy should not be confused with the thermodynamical entropy. In fact, the most important example of an entropy for (1.1) is the total energy (1.3). It is a convex function of $(\varrho, \mathbf{m})$.

Since solutions of (1.1) are typically not smooth, we cannot expect (1.9) to hold globally. Instead one requires (1.9) as an inequality in distributional sense *for all convex entropies*. In the one-dimensional case, there exists a large family of convex entropies for (1.1), and the corresponding entropy inequalities play a crucial role

in the global existence results available. We refer the reader to [8, 11, 12, 14, 23–25, 27, 28] for more details. In the multidimensional case, however, the only nontrivial convex entropy seems to be the total energy (1.3). As a consequence, the problem of global existence of solutions of (1.1) is largely open. The situation is similar for other systems of multidimensional hyperbolic conservation laws.

In order to overcome this difficulty, Dafermos [9] introduced a different entropy condition, called the *entropy rate admissibility criterion*. To explain this notion, let us consider a one-dimensional conservation law of the form

$$\partial_t v + \partial_x f(v) = 0 \quad \text{in } [0, \infty) \times \mathbf{R} \quad (1.10)$$

that allows for one convex entropy/entropy-flux pair (η, q) . For any weak solution v of (1.10) Dafermos defines the total entropy $H_v: [0, \infty) \rightarrow \mathbf{R}$ to be

$$H_v(t) := \int_{\mathbf{R}} \eta(v(t, x)) dx \quad \text{for a.e. } t \geq 0.$$

Then v is called admissible if H_v has the maximal rate of decrease. That is, there exists no weak solution w of (1.10) with the property that for some $\tau \geq 0$

$$v = w \text{ in } [0, \tau) \times \mathbf{R} \quad \text{and} \quad \frac{d}{dt} H_w(t)|_{t=\tau} < \frac{d}{dt} H_v(t)|_{t=\tau}.$$

Dafermos tested the entropy rate admissibility criterion in the case of a scalar conservation law and in the case of a p -system, and found it consistent with classical entropy conditions in the class of piecewise smooth solutions. We refer the reader to [13, 29, 30] for further results. In the case of a Riemann problem for the system of full Euler equations, Hsiao [19] discovered that the entropy rate admissibility criterion and classical entropy conditions are not equivalent. Dafermos' criterion is difficult to implement since it requires a minimization over the set of weak solutions of the conservation law (1.10), which is not easy to characterize.

In this paper, we propose a new variational principle for the multidimensional system of isentropic Euler equations (1.1). It is inspired by the theory of abstract gradient flows on the space of probability measures, as pioneered by Otto [26], and by a variational approximation scheme for the system of elastodynamics that was studied recently by Demoulini, Stuart, and Tzavaras [10]. In order to put our approach into perspective, let us briefly sketch these results.

It was shown by Otto [26] that the porous medium equation

$$\partial_t \varrho - \Delta P(\varrho) = 0 \quad \text{in } [0, \infty) \times \mathbf{R}^d \quad (1.11)$$

can be considered as an abstract gradient flow in the following sense:

(1) We denote by $\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ the space of all $\mathcal{L}^d$ -measurable, nonnegative functions with unit integral and finite second moments, where $\mathcal{L}^d$ is the Lebesgue measure. The space $\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ is equipped with the *Wasserstein distance*, defined by

$$\mathbf{W}(\varrho_1, \varrho_2)^2 := \inf \left\{ \iint_{\mathbf{R}^d \times \mathbf{R}^d} |x_2 - x_1|^2 \gamma(dx_1, dx_2) : \pi^i \# \gamma = \varrho_i \mathcal{L}^d \right\}. \quad (1.12)$$

This number is the minimal quadratic cost required to transport the measure $\varrho_1 \mathcal{L}^d$ to the measure $\varrho_2 \mathcal{L}^d$. The map $\pi^i: \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ is the projection onto the i th component, and $\#$ denotes the pushforward. The probability measure γ on $\mathbf{R}^d \times \mathbf{R}^d$ is called a transport plan, and one can show that the inf in (1.12) is attained.

(2) We now introduce a differentiable structure on $\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ as follows: For any point $\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$, we define the tangent space $\mathbb{T}_{\varrho}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ as

the closure of $\{\nabla\phi: \phi \in \mathcal{D}(\mathbf{R}^d)\}$ in the $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ -norm.

Then the $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ -inner product induces a metric on $\mathbb{T}_{\varrho}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$. This definition is motivated by the fact that for any sufficiently smooth curve $t \mapsto \varrho(t) \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ with $\varrho(0) = \varrho$, there exists a unique $\mathbf{u} \in \mathbb{T}_{\varrho}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ with the property that

$$\partial_t \varrho(0) + \nabla \cdot (\varrho \mathbf{u}) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^d). \quad (1.13)$$

Formally this structure renders $\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ a Riemannian manifold. We define

$$\mathbb{T}\mathcal{P}_{\text{reg}}(\mathbf{R}^d) := \bigcup \left\{ (\varrho, \mathbf{u}) : \varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d), \mathbf{u} \in \mathbb{T}_{\varrho}\mathcal{P}_{\text{reg}}(\mathbf{R}^d) \right\}.$$

(3) If in (1.13) we put $\mathbf{u} = -\varrho^{-1} \nabla P(\varrho)$, then we get the porous medium equation at one instant in time. This vector field is the “gradient” of the internal energy

$$\mathcal{U}(\varrho) := \int_{\mathbf{R}^d} U(\varrho(x)) dx \quad \text{with} \quad P(\varrho) = U'(\varrho)\varrho - U(\varrho),$$

in the sense that $\mathbf{u}$ is the uniquely determined element of minimal length in the *subdifferential* of $\mathcal{U}(\varrho)$ with respect to the Wasserstein distance. The function $\mathbf{u}$ is indeed a tangent vector to $\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ because $\varrho^{-1} \nabla P(\varrho) = \nabla U'(\varrho)$.

This result has been generalized considerably, and we refer the reader to the monographs [4, 31] and to the references therein for more details. The interpretation of dissipative evolution equations as abstract gradient flows suggests a natural time discretization for (1.11): Given a time step $\tau > 0$ and the value $\varrho^n \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ of the approximate solution at time $t^n := n\tau$, the value at time t^{n+1} is defined as

$$\varrho^{n+1} := \operatorname{argmin} \left\{ \frac{\mathbf{W}(\varrho^n, \varrho)^2}{2\tau} + \mathcal{U}(\varrho) : \varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d) \right\}; \quad (1.14)$$

see [22]. As $\tau \rightarrow 0$, this approximation converges to a solution of (1.11).

Since the multidimensional isentropic Euler equations are not a gradient flow, the above framework cannot be applied. There has been a lot of interest recently to develop an analogous theory for *Hamiltonian* systems (see [3] for a first paper), but fundamental questions are still open. For the system of elastodynamics

$$y_{tt} - S(y_x)_x = 0 \quad \text{in } [0, \infty) \times \mathbf{R}, \quad (1.15)$$

where y denotes the displacement (scalar) and S is the Piola-Kirchhoff stress tensor, Demoulini, Stuart, and Tzavaras [10] proposed a variational time discretization that is quite similar to (1.14). They consider the case where $S = W'$, with W being the stored energy of the system, and where S is strictly increasing and convex. Given a time step $\tau > 0$ and the values $y^n, y^{n-1} \in \mathcal{H}^1(\mathbf{R})$ of the approximate solution at the times t^n and t^{n-1} , the value at time t^{n+1} is defined as

$$y^{n+1} := \operatorname{argmin} \left\{ \int_{\mathbf{R}} \left(\frac{|y - 2y^n + y^{n-1}|^2}{2\tau^2} + W(y_x) \right) dx : y \in \mathcal{H}^1(\mathbf{R}) \right\}. \quad (1.16)$$

Equation (1.15) can be rewritten in the form

$$\left. \begin{aligned} u_t - v_x &= 0 \\ v_t - S(u)_x &= 0 \end{aligned} \right\} \quad \text{in } [0, \infty) \times \mathbf{R}, \quad (1.17)$$

which is a system of hyperbolic conservation laws for the deformation gradient $u := y_x$ and the velocity field $v := y_t$. Demoulini *et al.* show that the approximation (1.16) converges to an entropy solution of (1.17) as $\tau \rightarrow 0$. Note that the term

$$\frac{|y - 2y^n + y^{n-1}|}{\tau} = \tau \left| \frac{y - 2y^n + y^{n-1}}{\tau^2} \right|$$

is the product of a second order difference quotient (approximating the acceleration) times the timestep τ . Up to some constant involving the specific mass, which has been neglected for simplicity, the integral over the first term in (1.16) therefore has the physical dimension of an energy, as does the Wasserstein distance in (1.14).

Here is an outline of the paper:

The state of the fluid can be described in terms of density/momentum $(\varrho, \mathbf{m})$ or in terms of probability measures on the tangent bundle $\mathbb{T}\mathbf{R}^d$, as in kinetic theory. In Section 3 we first explain the connection between these representations. Then we introduce the energy functional and study its properties. In Section 4 we define a new functional, called the Minimal Acceleration Cost, that measures the distance between two admissible states of the fluid. We introduce a minimization problem similar to (1.14), with the Minimal Acceleration Cost as the penalizing term. We prove a crucial energy dissipation estimate for this problem. Finally, in Section 5 we construct a sequence of approximate solutions to (1.1) and give some *heuristic* for its convergence. We start investigating rigorously its convergence towards a measure-valued solution; this study will be continued in a future publication.

2. PRELIMINARIES

In this section, we collect some facts from the theory of optimal transport and from geometric measure theory that will be needed later on in this paper.

2.1. Optimal Transport. We denote by $\mathcal{P}(\mathbf{R}^D)$ the space of probability measures on $\mathbf{R}^D$ with finite second moments, and by $\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ its subspace of measures that are absolutely continuous with respect to the D -dimensional Lebesgue measure $\mathcal{L}^D$. By the Radon-Nikodym theorem, for every $\mu \in \mathcal{P}_{\text{reg}}(\mathbf{R}^D)$ there exists a Lebesgue-measurable function g such that $\mu = g\mathcal{L}^D$. When no confusion is possible, we will identify the measure μ and its density g to simplify notation.

Definition 2.1. For any pair of measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{R}^D)$ we denote by $\Gamma(\mu_1, \mu_2)$ the space of probability measures $\gamma \in \mathcal{P}(\mathbf{R}^D \times \mathbf{R}^D)$ such that

$$\pi^i \# \gamma = \mu_i \quad \text{for } i \in \{1, 2\},$$

and we call such measures transport plans. The map $\pi^i: \mathbf{R}^D \times \mathbf{R}^D \rightarrow \mathbf{R}^D$ is the projection onto the i th component, and $\#$ denotes the push-forward of measures. Then the *Wasserstein distance* between the measures μ_1 and μ_2 is defined as

$$\mathbf{W}(\mu_1, \mu_2)^2 := \inf \left\{ \iint_{\mathbf{R}^D \times \mathbf{R}^D} |\mathbf{x}_2 - \mathbf{x}_1|^2 \gamma(d\mathbf{x}_1, d\mathbf{x}_2) : \gamma \in \Gamma(\mu_1, \mu_2) \right\}. \quad (2.1)$$

There always exists a transport plan $\gamma \in \Gamma(\mu_1, \mu_2)$ for which the infimum in (2.1) is attained; see Section 2 of [17]. Such a transport plan is called optimal. We denote by $\Gamma_{\text{opt}}(\mu_1, \mu_2)$ the set of optimal transport plans between μ_1 and μ_2 .

If $\mu_1 = g_1 \mathcal{L}^D$, then the set $\Gamma_{\text{opt}}(\mu_1, \mu_2)$ of optimal transport plans contains only one measure γ , which is induced by a Borel map $\mathbf{r}_1: \mathbf{R}^D \rightarrow \mathbf{R}^D$ as

$$\iint_{\mathbf{R}^D \times \mathbf{R}^D} \varphi(x_1, x_2) \gamma(dx_1, dx_2) = \int_{\mathbf{R}^D} \varphi(x, \mathbf{r}_1(x)) g_1(x) dx$$

for all $\varphi \in \mathcal{C}_b(\mathbf{R}^D)$. We call $\mathbf{r}_1$ the optimal transport map pushing μ_1 forward to μ_2 . It coincides μ_1 -a.e. with the gradient of a lower semicontinuous convex function, and is therefore monotone: we have $(\mathbf{r}_1(x) - \mathbf{r}_1(\bar{x})) \cdot (x - \bar{x}) \geq 0$ for μ_1 -a.e. $x, \bar{x} \in \mathbf{R}^D$. Moreover, there exists a μ_1 -negligible set $N \subset \mathbf{R}^D$ such that for μ_1 -a.e. $\bar{x} \in \mathbf{R}^D \setminus N$ there is a positive semidefinite symmetric matrix $D\mathbf{r}_1(\bar{x})$ such that

$$\lim_{\substack{x \rightarrow \bar{x} \\ x \in \mathbf{R}^D \setminus N}} \frac{\mathbf{r}_1(x) - \mathbf{r}_1(\bar{x}) - D\mathbf{r}_1(\bar{x}) \cdot (x - \bar{x})}{|x - \bar{x}|} = 0; \quad (2.2)$$

see Theorem 3.2 in [1]. If in addition also μ_2 is absolutely continuous with respect to the Lebesgue measure, then the map $\mathbf{r}_1$ is injective μ_1 -a.e., and its inverse $\mathbf{r}_2 := \mathbf{r}_1^{-1}$ is the uniquely determined optimal transport map pushing μ_2 forward to μ_1 .

The space $\mathcal{P}(\mathbf{R}^D)$, equipped with the Wasserstein distance $\mathbf{W}$, forms a separable complete metric space; see Proposition 7.1.5 in [4]. Given some interval $I := (a, b)$, a curve $\mu_t: I \rightarrow \mathcal{P}(\mathbf{R}^D)$ is called absolutely continuous if there exists a Lebesgue-measurable nonnegative function $m \in \mathcal{L}^2(I)$ such that

$$\mathbf{W}(\mu_{t_1}, \mu_{t_2}) \leq \int_{t_1}^{t_2} m(s) ds \quad \text{for all } a \leq t_1 \leq t_2 \leq b.$$

For any such curve, there exists a Borel vector field $\mathbf{u}: (t, x) \mapsto \mathbf{u}_t(x)$ such that

$$\mathbf{u}_t \in \mathcal{L}^2(\mathbf{R}^D, \mu_t) \quad \text{and} \quad \|\mathbf{u}_t\|_{\mathcal{L}^2(\mathbf{R}^D, \mu_t)} = \lim_{s \rightarrow t} \frac{\mathbf{W}(\mu_s, \mu_t)}{|s - t|}$$

for a.e. $t \in I$, and such that the following continuity equation is satisfied:

$$\partial_t \mu_t + \nabla \cdot (\mathbf{u}_t \mu_t) = 0 \quad \text{in } \mathcal{D}'(I \times \mathbf{R}^D). \quad (2.3)$$

Moreover, for a.e. $t \in I$, the velocity field $\mathbf{u}_t$ belongs to the closure in $\mathcal{L}^2(\mathbf{R}^D, \mu_t)$ of the subspace generated by gradient vector fields of the form $\nabla \phi$, with $\phi \in \mathcal{D}(\mathbf{R}^D)$. We refer the reader to Sections 5.5 & 8.3 in [4] for further information.

Definition 2.2. Let $\mu \in \mathcal{P}(\mathbf{R}^D)$ be given. Then the *tangent space* of $\mathcal{P}(\mathbf{R}^D)$ at the measure μ , which we denote by $\mathbb{T}_\mu \mathcal{P}(\mathbf{R}^D)$, is the closure in $\mathcal{L}^2(\mathbf{R}^D, \mu)$ of the set $\{\nabla \phi: \phi \in \mathcal{D}(\mathbf{R}^D)\}$ of gradient vector fields. We define

$$\mathbb{T} \mathcal{P}(\mathbf{R}^D) := \bigcup \left\{ (\varrho, \mathbf{u}): \varrho \in \mathcal{P}(\mathbf{R}^D), \mathbf{u} \in \mathbb{T}_\varrho \mathcal{P}(\mathbf{R}^D) \right\}.$$

As shown in Section 8.5 of [4], there exists an orthogonal decomposition

$$\mathcal{L}^2(\mathbf{R}^D, \mu) = \mathbb{T}_\mu \mathcal{P}(\mathbf{R}^D) \oplus \mathbb{T}_\mu^\perp \mathcal{P}(\mathbf{R}^D), \quad (2.4)$$

where the orthogonal complement is the space of divergence-free vector fields:

$$\mathbb{T}_\mu^\perp \mathcal{P}(\mathbf{R}^D) := \left\{ \mathbf{w} \in \mathcal{L}^2(\mathbf{R}^D, \mu): \nabla \cdot (\mathbf{w} \mu) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^D) \right\}.$$

Similarly, we define the tangent space/bundle over the space $\mathcal{P}_{\text{reg}}(\mathbf{R}^D)$ of measures that are absolutely continuous with respect to the Lebesgue measure. In that case, we write the decomposition (2.4) with $\mathcal{P}_{\text{reg}}(\mathbf{R}^D)$ in place of $\mathcal{P}(\mathbf{R}^D)$.

Notice that the name tangent space is justified by the continuity equation (2.3) because it allows to identify the derivatives along absolutely continuous curves in the Wasserstein space $(\mathcal{P}(\mathbf{R}^D), \mathbf{W})$ with certain square-integrable vector fields. The tangent space $\mathbb{T}_\mu \mathcal{P}(\mathbf{R}^D)$ inherits the inner product from the ambient $\mathcal{L}^2(\mathbf{R}^D, \mu)$. This turns $\mathcal{P}(\mathbf{R}^D)$ formally into a Riemannian manifold.

2.2. Polar Factorization. For any given density $\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$, consider a vector field $\mathbf{r} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$ that satisfies the following nondegeneracy condition:

$$A \subset \mathbf{R}^d \text{ Borel, } \mathcal{L}^d(A) = 0 \implies \mathcal{L}^d(\mathbf{r}^{-1}(A)) = 0. \quad (2.5)$$

Then there exists functions $(\nabla \zeta, \mathbf{s}) \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$ such that

- (1) The function ζ is lower semicontinuous and convex, with $\nabla \zeta$ defined ϱ -a.e.
- (2) The function $\mathbf{s}$ preserves the measures $\varrho \mathcal{L}^d$ in the sense that

$$\int_{\mathbf{R}^d} \varphi(\mathbf{s}(x)) \varrho(x) dx = \int_{\mathbf{R}^d} \varphi(x) \varrho(x) dx \quad \text{for all } \varphi \in \mathcal{D}(\mathbf{R}^d).$$

Moreover, the function $\mathbf{s}$ is the $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ -projection of $\mathbf{r}$ onto the closed bounded subspace of maps that preserves the measure $\varrho \mathcal{L}^d$.

- (3) We have $\mathbf{r}(x) = \nabla \zeta(\mathbf{s}(x))$ for ϱ -a.e. $x \in \mathbf{R}^d$.

This is Brenier's polar factorization; see [6].

Let now $\tau > 0$ and $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$ be given, and consider the polar factorization of the map $\mathbf{r}^\tau := \text{id} + \tau \mathbf{u}$ in terms of functions $(\nabla \zeta^\tau, \mathbf{s}^\tau)$ as above. We write

$$\nabla \zeta^\tau =: \text{id} + \tau \nabla \phi^\tau \quad \text{and} \quad \mathbf{s}^\tau =: \text{id} + \tau \mathbf{w}^\tau,$$

which implies the factorization

$$\mathbf{u} = \nabla \phi^\tau \circ (\text{id} + \tau \mathbf{w}^\tau) + \mathbf{w}^\tau \quad (2.6)$$

ϱ -a.e. Assumption (2.5) is equivalent to the fact that $(\text{id} + \tau \mathbf{u})\#(\varrho \mathcal{L}^d)$ is absolutely continuous with respect to the Lebesgue measure. Then $\nabla \zeta^\tau$ is the unique optimal transport map pushing $\varrho \mathcal{L}^d$ forward to $\varrho^\tau \mathcal{L}^d := (\text{id} + \tau \mathbf{u})\#(\varrho \mathcal{L}^d)$, and

$$\begin{aligned} \int_{\mathbf{R}^d} |\nabla \phi^\tau|^2 \varrho dx &= \frac{\mathbf{W}(\varrho \mathcal{L}^d, \varrho^\tau \mathcal{L}^d)^2}{\tau^2} \\ &= \inf \left\{ \int_{\mathbf{R}^d} |\tilde{\mathbf{u}}|^2 \varrho dx : (\text{id} + \tau \tilde{\mathbf{u}})\#(\varrho \mathcal{L}^d) = \varrho^\tau \mathcal{L}^d \right\} \leq \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho dx. \end{aligned}$$

From this and identity (2.6), we also obtain the estimate

$$\int_{\mathbf{R}^d} |\mathbf{w}^\tau|^2 \varrho dx \leq 2 \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho dx + 2 \int_{\mathbf{R}^d} |\nabla \phi^\tau \circ (\text{id} + \tau \mathbf{w}^\tau)|^2 \varrho dx \leq 4 \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho dx.$$

Here we used the fact that the map $\text{id} + \tau \mathbf{w}^\tau$ preserves the measure $\varrho \mathcal{L}^d$.

Proposition 2.3. *Let $\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$ be given, and consider a sequence $\tau^k \rightarrow 0$ such that the push-forward measures $(\text{id} + \tau^k \mathbf{u})\#(\varrho \mathcal{L}^d)$ are absolutely continuous with respect to the Lebesgue measure for all k . Let $(\nabla \phi^k, \mathbf{w}^k)$ define the polar factorization of $\mathbf{u}$ as in (2.6), with τ replaced by τ^k . Then*

$$(\nabla \phi^k, \mathbf{w}^k) \rightharpoonup (\mathbf{v}, \mathbf{w}) \quad \text{weakly in } \mathcal{L}^2(\mathbf{R}^d, \varrho), \quad (2.7)$$

where $\mathbf{u} = \mathbf{v} + \mathbf{w}$ is the uniquely determined orthogonal decomposition of $\mathbf{u}$ into a tangent vector field $\mathbf{v} \in \mathbb{T}_\varrho \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and a vector field $\mathbf{w}$ satisfying

$$\nabla \cdot (\mathbf{w} \varrho) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^d).$$

Proof. Notice first that since $(\nabla\phi^k, \mathbf{w}^k)$ are uniformly bounded in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$, by reflexivity and the Banach-Alaoglu theorem we can extract a subsequence (which we still label $(\nabla\phi^k, \mathbf{w}^k)$ for simplicity) such that (2.7) holds. We are going to prove that the limit functions $(\mathbf{v}, \mathbf{w})$ are uniquely determined by the orthogonal decomposition (2.4), and therefore the whole sequence converges, not only a subsequence.

We first consider $\mathbf{v}$. Note that the gradient vector fields $\nabla\phi^k$ are in $\mathbb{T}_{\varrho}\mathcal{P}_{\text{reg}}(R^d)$, which is a closed subspace of $\mathcal{L}^2(\mathbf{R}^d, \varrho)$. The weak limit $\mathbf{v}$ must therefore also be a tangent vector. Since $\text{id} + \tau^k \mathbf{w}^k$ preserves the measure $\varrho\mathcal{L}^d$, we have

$$\int_{\mathbf{R}^d} \varphi \circ (\text{id} + \tau^k \mathbf{u}) \varrho dx = \int_{\mathbf{R}^d} \varphi \circ (\text{id} + \tau^k \nabla\phi^k) \varrho dx \quad (2.8)$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^d)$ and all k . We write

$$\frac{1}{\tau^k} \int_{\mathbf{R}^d} \left(\varphi \circ (\text{id} + \tau^k \mathbf{u}) - \varphi \right) \varrho dx = \int_{\mathbf{R}^d} \psi^k \cdot \mathbf{u} \varrho dx,$$

with function ψ^k defined by

$$\psi^k(x) := \int_0^1 \nabla\varphi(x + \theta\tau^k \mathbf{u}(x)) d\theta \quad \text{for } \varrho\text{-a.e. } x \in \mathbf{R}^d.$$

Note that $\tau^k \mathbf{u} \rightarrow 0$ strongly in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ as $k \rightarrow \infty$, which implies convergence pointwise ϱ -a.e. On the other hand, we have the uniform bound

$$\|\psi^k\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \leq \|\nabla\varphi\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \quad \text{for all } k,$$

so the dominated convergence theorem yields $\psi^k \rightarrow \nabla\varphi$ strongly in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$. Recall that $\varrho\mathcal{L}^d$ is a probability measure. We therefore obtain that

$$\lim_{k \rightarrow \infty} \frac{1}{\tau^k} \int_{\mathbf{R}^d} \left(\varphi \circ (\text{id} + \tau^k \mathbf{u}) - \varphi \right) \varrho dx = \int_{\mathbf{R}^d} \nabla\varphi \cdot \mathbf{u} \varrho dx \quad (2.9)$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^d)$. Similarly, we can write

$$\frac{1}{\tau^k} \int_{\mathbf{R}^d} \left(\varphi \circ (\text{id} + \tau^k \nabla\phi^k) - \varphi \right) \varrho dx = \int_{\mathbf{R}^d} \Psi^k \cdot \nabla\phi^k \varrho dx,$$

with function Ψ^k defined by

$$\Psi^k(x) := \int_0^1 \nabla\varphi(x + \theta\tau^k \nabla\phi^k(x)) d\theta \quad \text{for } \varrho\text{-a.e. } x \in \mathbf{R}^d.$$

We have $\tau^k \nabla\phi^k \rightarrow 0$ strongly in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$, which together with the uniform boundedness of Ψ^k implies that $\Psi^k \rightarrow \nabla\varphi$ in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$. This gives

$$\lim_{k \rightarrow \infty} \frac{1}{\tau^k} \int_{\mathbf{R}^d} \left(\varphi \circ (\text{id} + \tau^k \nabla\phi^k) - \varphi \right) \varrho dx = \int_{\mathbf{R}^d} \nabla\varphi \cdot \mathbf{v} \varrho dx \quad (2.10)$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^d)$, because $\nabla\phi^k \rightarrow \mathbf{v}$ weakly in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$. We then have

$$\int_{\mathbf{R}^d} \nabla\varphi \cdot (\mathbf{u} - \mathbf{v}) \varrho dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathbf{R}^d),$$

as follows from combining equality (2.8) with (2.9) & (2.10). This proves that the weak limit $\mathbf{v}$ coincides with the tangent vector component of the velocity $\mathbf{u}$.

On the other hand, for all $\zeta \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$ we can write

$$\begin{aligned} & \int_{\mathbf{R}^d} \zeta \cdot \left(\nabla \phi^k - \nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) \right) \varrho dx \\ &= \int_{\mathbf{R}^d} (\zeta - \zeta_\delta) \cdot \left(\nabla \phi^k - \nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) \right) \varrho dx \\ &+ \int_{\mathbf{R}^d} \left(\zeta_\delta \circ (\text{id} + \tau^k \mathbf{w}^k) - \zeta_\delta \right) \cdot \left(\nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) \right) \varrho dx, \end{aligned}$$

where $\zeta_\delta \in \mathcal{D}(\mathbf{R}^d)$. The first term on the right-hand side can be estimated as

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} (\zeta - \zeta_\delta) \cdot \left(\nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) - \nabla \phi^k \right) \varrho dx \right| \\ & \leq 2 \left(\int_{\mathbf{R}^d} |\zeta - \zeta_\delta|^2 \varrho dx \right)^{1/2} \left(\int_{\mathbf{R}^d} |\nabla \phi^k|^2 \varrho dx \right)^{1/2}, \end{aligned}$$

which can be made arbitrarily small uniformly in k , by choosing ζ_δ sufficiently close to ζ . Recall that the $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ -norms of both $\nabla \phi^k$ and $\mathbf{w}^k$ are bounded above by some constant times the $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ -norm of $\mathbf{u}$. Similarly, we obtain

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} \left(\zeta_\delta \circ (\text{id} + \tau^k \mathbf{w}^k) - \zeta_\delta \right) \cdot \left(\nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) \right) \varrho dx \right| \\ & \leq \tau^k \|D\zeta_\delta\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \left(\int_{\mathbf{R}^d} |\mathbf{w}^k|^2 \varrho dx \right)^{1/2} \left(\int_{\mathbf{R}^d} |\nabla \phi^k|^2 \varrho dx \right)^{1/2}, \end{aligned}$$

which converges to zero as $\tau^k \rightarrow 0$. We have proved that $\nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) \rightharpoonup \nabla \phi^k$ weakly in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$ as $k \rightarrow \infty$. As a consequence, we find that

$$\mathbf{w}^k = \mathbf{u} - \nabla \phi^k \circ (\text{id} + \tau^k \mathbf{w}^k) \rightharpoonup \mathbf{u} - \mathbf{v}$$

weakly in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$, and thus $\mathbf{w} = \mathbf{u} - \mathbf{v}$. By orthogonality of the decomposition (2.4), the vector field $\mathbf{w}$ then coincides with the divergence-free component of $\mathbf{u}$. $\square$

2.3. Geometric Measure Theory. In this section, we prove a sufficient condition that ensures that the nondegeneracy assumption (2.5) for the polar factorization is satisfied. Before doing this, we need to introduce some terminology.

Definition 2.4. Let $A \subset \mathbf{R}^d$ be a Borel set and $f: A \rightarrow [-\infty, \infty]$ a Borel function. We call $\ell \in [-\infty, \infty]$ the approximate upper limit of f at $\bar{x} \in \mathbf{R}^d$, written

$$\text{ap lim sup}_{x \rightarrow \bar{x}} f(x) = \ell,$$

if ℓ is the infimum of the set of all numbers $t \in \mathbf{R}$ with

$$\lim_{r \rightarrow 0} \frac{|B_r(\bar{x}) \cap \{x \in A: f(x) > t\}|}{|B_r(\bar{x})|} = 0.$$

Here $|\cdot|$ denotes the d -dimensional Lebesgue measure. The approximate lower limit of f at $\bar{x} \in \mathbf{R}^d$ is defined similarly. If $g: A \rightarrow \mathbf{R}^k$ is a Borel map, then we call the vector $\xi \in \mathbf{R}^k$ the approximate limit of g at $\bar{x} \in \mathbf{R}^d$, written

$$\text{ap lim}_{x \rightarrow \bar{x}} g(x) = \xi,$$

if we have

$$\lim_{r \rightarrow 0} \frac{|B_r(\bar{x}) \cap \{x \in A: |g(x) - \xi| \geq \varepsilon\}|}{|B_r(\bar{x})|} = 0 \quad \text{for all } \varepsilon > 0.$$

We call g approximately continuous at $\bar{x} \in \mathbf{R}^d$ if and only if

$$\bar{x} \in \text{dom } g \quad \text{and} \quad \text{ap lim}_{x \rightarrow \bar{x}} g(x) = g(\bar{x}).$$

The map g is called *approximately differentiable* at $\bar{x} \in \mathbf{R}^d$ if and only if there exists a linear map $L: \mathbf{R}^d \rightarrow \mathbf{R}^k$ with the property that

$$\text{ap lim}_{x \rightarrow \bar{x}} \frac{|g(x) - g(\bar{x}) - L(x - \bar{x})|}{|x - \bar{x}|} = 0.$$

Approximate limits ξ and differentials L are uniquely determined if they exist.

We refer the reader to Sections 2.9.12 & 3.1.2 of [16] and 3.1.4 of [18] for details.

Proposition 2.5. *Consider $\mu := \varrho \mathcal{L}^d \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \mu)$ with*

$$\text{ap lim sup}_{x \rightarrow \bar{x}} \frac{|\mathbf{u}(x) - \mathbf{u}(\bar{x})|}{|x - \bar{x}|} < \infty \quad \text{for } \mu\text{-a.e. } \bar{x} \in \mathbf{R}^d. \quad (2.11)$$

For all $\tau > 0$, let μ^τ be the push-forward of μ under the map $\mathbf{r}^\tau := \text{id} + \tau \mathbf{u}$. Then there exists a Lebesgue null set $N \subset \{\tau > 0\}$ with the property that if $\tau > 0$ and $\tau \notin N$, then μ^τ is absolutely continuous with respect to the Lebesgue measure.

Proof. Note first that $\mathbf{r}^\tau \in \mathcal{L}^2(\mathbf{R}^d, \mu)$ for all $\tau > 0$. Therefore we have

$$\int_{\mathbf{R}^d} |y|^2 \mu^\tau(dy) = \int_{\mathbf{R}^d} |\mathbf{r}^\tau(x)|^2 \varrho(x) dx < \infty,$$

which shows that μ^τ has finite second moments and thus $\mu^\tau \in \mathcal{P}(\mathbf{R}^d)$. For proving the absolute continuity of μ^τ we use the following criterion.

Lemma 2.6. *For any measure $\mu \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and any vector field $\mathbf{r} \in \mathcal{C}^1 \cap \text{Lip}(\mathbf{R}^d)$, let $\mu_{\mathbf{r}} \in \mathcal{P}(\mathbf{R}^d)$ be the push-forward of μ under the map $\mathbf{r}$. Then $\mu_{\mathbf{r}}$ is absolutely continuous with respect to the Lebesgue measure if and only if*

$$\det D\mathbf{r}(x) \neq 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbf{R}^d.$$

Proof. The result was proved in Lemma 5.5.3 of [4] for the case that $\mathbf{r}$ is injective. The general case can be established as follows: We first introduce the closed set

$$D := \left\{ x \in \mathbf{R}^d: \det D\mathbf{r}(x) = 0 \right\}.$$

By Sard's theorem, the image $\mathbf{r}(D)$ has zero Lebesgue measure. But

$$\mu_{\mathbf{r}}(\mathbf{r}(D)) = \mu(\mathbf{r}^{-1}(\mathbf{r}(D))) \geq \mu(D)$$

because $\mathbf{r}^{-1}(\mathbf{r}(D)) \supset D$. We conclude that if $\mu(D) > 0$, then $\mu_{\mathbf{r}}(\mathbf{r}(D)) > 0$, thus $\mu_{\mathbf{r}}$ fails to be absolutely continuous with respect to the Lebesgue measure.

Conversely, suppose that $\mu(D) = 0$. Let $N \subset \mathbf{R}^d$ be a Borel set with $\mathcal{L}^d(N) = 0$. We are going to prove that then $\mu_{\mathbf{r}}(N) = 0$ as well. Let $E := (\text{spt } \mu) \setminus D$ such that $\det D\mathbf{r}(x) \neq 0$ for $\mathcal{L}^d$ -a.e. $x \in E$. By assumption, we can write

$$\mu_{\mathbf{r}}(N) = \mu(\mathbf{r}^{-1}(N)) = \mu(\mathbf{r}^{-1}(N) \cap E). \quad (2.12)$$

On the other hand, according to Lemma 2.74 in [2], $\mathcal{L}^d$ -almost all of E can be covered by a sequence of pairwise disjoint compact sets $\{E_i\}$ with the property that the map $\mathbf{r}_i := \mathbf{r}|_{E_i}$ is one-to-one with Lipschitz inverse, and that $\mathbf{r}$ is differentiable at every point of E_i . Then we can estimate

$$\begin{aligned} \mathcal{L}^d(\mathbf{r}^{-1}(N) \cap E_i) &= \mathcal{L}^d(\mathbf{r}_i^{-1} \circ \mathbf{r}_i(\mathbf{r}^{-1}(N) \cap E_i)) \\ &\leq (\|\mathbf{r}_i^{-1}\|_{\text{Lip}(\mathbf{R}^d)})^d \mathcal{L}^d(\mathbf{r}_i(\mathbf{r}^{-1}(N) \cap E_i)) = 0 \end{aligned}$$

since $\mathbf{r}_i(\mathbf{r}^{-1}(N) \cap E_i) \subset N$. Summing up over all E_i , we get $\mathcal{L}^d(\mathbf{r}^{-1}(N) \cap E) = 0$. But μ is assumed to be absolutely continuous with respect to the Lebesgue measure. Therefore we have $\mu(\mathbf{r}^{-1}(N) \cap E) = 0$, and thus $\mu_{\mathbf{r}}(N) = 0$, by (2.12). $\square$

We can now finish the proof of Proposition 2.5.

Step 1. We assume first that the vector field $\mathbf{u}$ also satisfies $\mathbf{u} \in \mathcal{C}^1 \cap \text{Lip}(\mathbf{R}^d)$. Then $\mathbf{r}^\tau \in \mathcal{C}^1 \cap \text{Lip}(\mathbf{R}^d)$ for all $\tau > 0$, so we can apply Lemma 2.6. We must control the set of all $x \in \mathbf{R}^d$ for which $\det D\mathbf{r}^\tau(x) = 0$ or, equivalently, for which at least one eigenvector of $D\mathbf{r}^\tau(x)$ equals $-1/\tau$. For any $\tau > 0$ we define

$$N(x, \tau) := \max \left\{ \begin{array}{l} n \in \mathbf{N}: \quad \text{there exist } n \text{ eigenvalues } \lambda_1, \dots, \lambda_n \text{ of } D\mathbf{r}^\tau(x) \\ \quad \text{with corresponding eigenvectors } u_1, \dots, u_n \text{ such} \\ \quad \text{that } \text{Re } \lambda_j < -1/\tau \text{ for all } j, \text{ and such that the} \\ \quad \text{family } \{u_j\} \text{ is linearly independent} \end{array} \right\} \quad (2.13)$$

for all $x \in \mathbf{R}^d$. Since $D\mathbf{r}^\tau \in \mathcal{C}(\mathbf{R}^d)$ and since the eigenvalues of a matrix depend continuously on the matrix coefficients, the map $x \mapsto N(x, \tau)$ is Borel for all $\tau > 0$. Moreover, for fixed $x \in \mathbf{R}^d$ the map $\tau \mapsto N(x, \tau)$ is monotonically nondecreasing, with range contained in $[0, d]$. We now define the family of integrals

$$I(\tau) := \int_{\mathbf{R}^d} N(x, \tau) \varrho(x) dx \quad \text{for all } \tau > 0.$$

Then $\tau \mapsto I(\tau)$ is monotonically nondecreasing, with range contained in $[0, d]$. In particular, the map I is of bounded variation, so the set of all $\tau > 0$ with

$$\lim_{t \rightarrow \tau+} I(t) \neq \lim_{t \rightarrow \tau-} I(t) \quad (2.14)$$

is at most countable. Note that (2.14) can happen if and only if

$$\lim_{t \rightarrow \tau+} N(x, t) \neq \lim_{t \rightarrow \tau-} N(x, t)$$

for all x in a set of positive μ -measure. By definition (2.13), this means that $D\mathbf{r}^\tau(x)$ has at least one eigenvalue with real part equal to $-1/\tau$. For all $\tau \in \mathbf{R}$ for which (2.14) does not hold, we conclude that for μ -a.e. $x \in \mathbf{R}^d$ the eigenvalues of $D\mathbf{r}^\tau(x)$ have real parts different from $-1/\tau$, which entails that $\det D\mathbf{r}^\tau(x) \neq 0$. This proves the proposition in the case of a smooth vector field.

Step 2 Consider now $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \mu)$ satisfying (2.11). Let $S := \text{spt } \mu$ and

$$A := \left\{ \bar{x} \in S: \text{ap } \limsup_{x \rightarrow \bar{x}} \frac{|\mathbf{u}(x) - \mathbf{u}(\bar{x})|}{|x - \bar{x}|} < \infty \right\}.$$

By assumption, we have $\mu(\mathbf{R}^d \setminus A) = 0$ and thus $\mathcal{L}^d(S \setminus A) = 0$. We apply Theorem 3 in Section 3.1.4 of [18] to obtain a nondecreasing sequence of closed sets $F_n \subset S$, and a sequence of maps $\mathbf{u}_n \in \mathcal{C}^1 \cap \text{Lip}(\mathbf{R}^d)$ such that

$$\mathbf{u}_n(x) = \mathbf{u}(x) \quad \text{for all } x \in F_n \text{ and } n \in \mathbf{N},$$

and $\mathcal{L}^d(S \setminus \bigcup_{n \in \mathbf{N}} F_n) = 0$. This implies $\mu(\mathbf{R}^d \setminus \bigcup_{n \in \mathbf{N}} F_n) = 0$ since $\mu \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$. For all $\tau > 0$ and $n \in \mathbf{N}$ we define the map $\mathbf{r}_n^\tau := \text{id} + \tau \mathbf{u}_n$, which in F_n coincides with $\mathbf{r}^\tau$. Let μ_n^τ be the push-forward of the restriction $\mu|_{F_n}$ under $\mathbf{r}_n^\tau$. By Step 1, there exists a countable set $N_n \subset \{\tau > 0\}$ such that for all $\tau > 0$ with $\tau \notin N_n$, the measure μ_n^τ is absolutely continuous with respect to the Lebesgue measure. For such τ we define a Borel density ϱ_n^τ by $\mu_n^\tau =: \varrho_n^\tau \mathcal{L}^d$.

Consider now $N := \bigcup_{n \in \mathbf{N}} N_n$, which is countable and thus a Lebesgue null set. For any $\tau > 0$ with $\tau \notin N$, the sequence of densities ϱ_n^τ is monotonically nondecreasing almost everywhere, as follows from the area formula (see Theorem 3.2.3 in [16]) and from the fact that the sequence of closed sets F_k is nondecreasing. Therefore the function $\varrho^\tau: \mathbf{R}^d \rightarrow [0, \infty]$ defined by

$$\varrho^\tau(x) := \lim_{n \rightarrow \infty} \varrho_n^\tau(x) \quad \text{for a.e. } x \in \mathbf{R}^d$$

is Borel. By monotone convergence we have that for all $\varphi \in \mathcal{C}_b(\mathbf{R}^d)$ nonnegative

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi(y) \varrho^\tau(y) dy &= \lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \varphi(y) \varrho_n^\tau(y) dy \\ &= \lim_{n \rightarrow \infty} \int_{F_n} \varphi(\mathbf{r}^\tau(x)) \varrho(x) dx \\ &= \int_{\bigcup_{n \in \mathbf{N}} F_n} \varphi(\mathbf{r}^\tau(x)) \varrho(x) dx = \int_{\mathbf{R}^d} \varphi(\mathbf{r}^\tau(x)) \varrho(x) dx. \end{aligned}$$

This shows that for all $\tau > 0$ with $\tau \notin N$, the push-forward measure $\mu^\tau = \varrho^\tau \mathcal{L}^d$. $\square$

3. DESCRIPTION OF FLUIDS

We denote by $\mathbf{TR}^d$ the tangent bundle over $\mathbf{R}^d$, and we denote elements in the tangent bundle by bold symbols, such as $\mathbf{x} = (x, \xi)$. Here $x \in \mathbf{R}^d$ is called position and $\xi \in \mathbb{T}_x \mathbf{R}^d \equiv \mathbf{R}^d$ is called velocity. We will assume that the tangent bundle is equipped with the Euclidean inner product, so that $\mathbf{TR}^d$ is isomorphic to $\mathbf{R}^{2d}$.

As explained in the Introduction, the state of an isentropic compressible fluid is completely determined by the density ϱ , which characterizes the distribution of mass, and the velocity field $\mathbf{u}$. This is a special case of a more flexible description of fluids in terms of probability measures on the tangent bundle, which we also call the state space. In fact, assume that $\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$. Then we can define a measure $\mu \in \mathcal{P}(\mathbf{TR}^d)$ such that for all $\varphi \in \mathcal{C}_b(\mathbf{TR}^d)$ we have

$$\int_{\mathbf{TR}^d} \varphi(\mathbf{x}) \mu(d\mathbf{x}) := \int_{\mathbf{R}^d} \varphi(x, \mathbf{u}(x)) \varrho(x) dx. \quad (3.1)$$

As in kinetic theory, the measure μ describes the mass carried by particles that are located at positions x and have velocities ξ . The description of fluids in terms of probability measures on the tangent space is advantageous mathematically because the space $\mathcal{P}(\mathbf{TR}^d)$ is a separable complete metric space; see Section 2.1.

Definition 3.1 (Energy). For any $\mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d)$ the *kinetic energy* is defined as

$$\mathcal{K}(\mu) := \int_{\mathbb{T}\mathbf{R}^d} \frac{1}{2} |\xi|^2 \mu(d\mathbf{x}).$$

Let $U: [0, \infty) \rightarrow \mathbf{R}$ be a proper, lower semicontinuous, convex function satisfying (1.7) and $U(0) = 0$. Let us also assume for simplicity that U is nonnegative (which includes the case (1.8)). For any $\mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d)$ the *internal energy* is defined as

$$\mathcal{U}(\mu) := \begin{cases} \int_{\mathbf{R}^d} U(\varrho(x)) dx & \text{if } \pi\#\mu = \varrho\mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\pi: \mathbb{T}\mathbf{R}^d \rightarrow \mathbf{R}^d$ denotes the projection onto the spatial component. We define the *total energy* of $\mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d)$ as the sum $\mathcal{E}(\mu) := \mathcal{K}(\mu) + \mathcal{U}(\mu)$, and the domain $\text{dom } \mathcal{E}$ of the functional $\mathcal{E}$ as the set of all measures μ with $\mathcal{E}(\mu) < \infty$.

One can show that the energy functionals of Definition 3.1 are lower semicontinuous with respect to the narrow convergence of measures in $\mathcal{P}(\mathbb{T}\mathbf{R}^d)$, and displacement convex. We refer the reader to Section 9 of [4] for more details.

Notice that the kinetic energy is finite for all $\mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d)$, by definition. Boundedness of the internal energy, however, requires that the spatial marginal of μ is absolutely continuous with respect to the Lebesgue measure, and thus induced by a density ϱ . We will write $\mathcal{E}(\varrho, \mathbf{u})$ for the total energy of the measure $\mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d)$ induced by a density/velocity pair $(\varrho, \mathbf{u})$ as in relation (3.1). In a similar way, we define $\mathcal{K}(\varrho, \mathbf{u})$ and $\mathcal{U}(\varrho)$. Note that the internal energy only depends on ϱ .

We now prove a crucial convexity estimate for the internal energy.

Proposition 3.2. *Consider measures $\varrho_i\mathcal{L}^d \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and optimal transport maps $\mathbf{r}_i$ that push $\varrho_i\mathcal{L}^d$ forward to some measure $\hat{\varrho}\mathcal{L}^d \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ for $i \in \{0, 1\}$. Suppose that the internal energies $\mathcal{U}(\varrho_i)$ are both finite, and that*

$$\nabla U'(\varrho_0) \in \mathcal{L}^2(\mathbf{R}^d, \varrho_0),$$

where U is the specific internal energy of Definition 3.1. Then

$$\mathcal{U}(\varrho_0) + \int_{\mathbf{R}^d} \nabla U'(\varrho_0) \cdot (\mathbf{r}_1^{-1} \circ \mathbf{r}_0 - \text{id}) \varrho_0 dx \leq \mathcal{U}(\varrho_1). \quad (3.2)$$

Proof. We divide the proof into two steps.

Step 1. Since the measures $\varrho_i\mathcal{L}^d$ and $\hat{\varrho}\mathcal{L}^d$ are absolutely continuous with respect to the Lebesgue measure, the optimal transport maps $\mathbf{r}_i$ are uniquely determined and essentially injective. Therefore the inverse maps $\mathbf{t}_i := \mathbf{r}_i^{-1}$ exist and are optimal transport maps pushing $\hat{\varrho}\mathcal{L}^d$ forward to $\varrho_i\mathcal{L}^d$ for $i \in \{0, 1\}$; see Section 2.1 above. For any number $s \in [0, 1]$ we now define the Borel map

$$\mathbf{t}_s(y) := (1-s)\mathbf{t}_0(y) + s\mathbf{t}_1(y) \quad \text{for } \hat{\varrho}\text{-a.e. } y \in \mathbf{R}^d.$$

In the terminology of Definition 9.2.2 in [4] the map

$$s \mapsto \mathbf{t}_s\#(\hat{\varrho}\mathcal{L}^d) \quad \text{for } s \in [0, 1]$$

is called a *generalized geodesic* connecting the two measures $\varrho_i\mathcal{L}^d$. It is shown there that for all s , the measure $\mathbf{t}_s\#(\hat{\varrho}\mathcal{L}^d) =: \varrho_s\mathcal{L}^d$ is absolutely continuous with respect to the Lebesgue measure, and that the map $s \mapsto \mathcal{U}(\varrho_s)$ is convex: We have

$$\mathcal{U}(\varrho_s) \leq (1-s)\mathcal{U}(\varrho_0) + s\mathcal{U}(\varrho_1) \quad \text{for all } s \in [0, 1]; \quad (3.3)$$

see Proposition 9.3.9 of [4]. Now note that

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi(\mathbf{t}_s(y)) \hat{\varrho}(y) dy &= \int_{\mathbf{R}^d} \varphi(\mathbf{t}_s(\mathbf{r}_0(z))) \varrho_0(x) dz \\ &= \int_{\mathbf{R}^d} \varphi(z + s(\mathbf{r}_1^{-1} \circ \mathbf{r}_0(z) - z)) \varrho_0(z) dz \end{aligned}$$

for all $\varphi \in \mathcal{C}_b(\mathbf{R}^d)$ and $s \in [0, 1]$. This shows that the interpolation above is actually induced by the map $\mathbf{r}_1^{-1} \circ \mathbf{r}_0$, which pushes $\varrho_0 \mathcal{L}^d$ forward to $\varrho_1 \mathcal{L}^d$.

Step 2. Rearranging terms in (3.3) we now obtain

$$\frac{\mathcal{U}(\varrho_s) - \mathcal{U}(\varrho_0)}{s} \leq \mathcal{U}(\varrho_1) - \mathcal{U}(\varrho_0) \quad \text{for all } s \in [0, 1].$$

The left-hand side can be estimated from below by

$$\int_{\mathbf{R}^d} \nabla U'(\varrho_0(x)) \cdot \mathbf{u}_s(x) \varrho_0(x) dx \leq \frac{\mathcal{U}(\varrho_s) - \mathcal{U}(\varrho_0)}{s}, \quad (3.4)$$

where $\text{id} + s\mathbf{u}_s$ is the uniquely determined optimal transport map that pushes $\varrho_0 \mathcal{L}^d$ forward to $\varrho_s \mathcal{L}^d$; see Theorem 10.4.6 in [4]. By Proposition 2.3, we have that

$$\mathbf{u}_s \rightharpoonup \mathbf{v} \quad \text{weakly in } \mathcal{L}^2(\mathbf{R}^d, \varrho_0)$$

as $s \rightarrow 0$, where the map $\mathbf{v}$ is the tangent component in the orthogonal decomposition (2.4) of the vector field $\mathbf{u}_0 := \mathbf{r}_1^{-1} \circ \mathbf{r}_0 - \text{id}$. Since $\nabla U'(\varrho_0) \in \mathcal{L}^2(\mathbf{R}^d, \varrho_0)$ we can pass to the limit on the left-hand side of (3.4) and obtain

$$\int_{\mathbf{R}^d} \nabla U'(\varrho_0(x)) \cdot \mathbf{v}(x) \varrho_0(x) dx \leq \mathcal{U}(\varrho_1) - \mathcal{U}(\varrho_0). \quad (3.5)$$

On the other hand, recall that the vector field $\mathbf{w} := \mathbf{u}_0 - \mathbf{v}$ satisfies $\nabla \cdot (\mathbf{w} \varrho_0) = 0$ in the distributional sense. Approximating $U'(\varrho_0)$ by smooth functions, we can thus substitute in (3.5) the full velocity $\mathbf{u}_0$ for $\mathbf{v}$ because the divergence-free part $\mathbf{w}$ is eliminated. This yields the inequality (3.2) and finishes the proof. $\square$

4. STEEPEST DESCENT

As explained in the Introduction, our goal is to implement Dafermos' idea that an admissible solution to the isentropic Euler equations (1.1) should dissipate its total energy as fast as possible. In this section, we will explain the specifics of this minimization. We start by introducing a new cost functional.

4.1. Minimal Acceleration Cost. Our construction is motivated by the following heuristic: Consider a particle located at position $x_1 \in \mathbf{R}^d$ with velocity $\xi_1 \in \mathbf{R}^d$. Assume that during a time interval of length $\tau > 0$, the particle is allowed to move to a new position $x_2 \in \mathbf{R}^d$ and to change its velocity to a new value $\xi_2 \in \mathbf{R}^d$. If we require that the particle follows a path $c: [0, \tau] \rightarrow \mathbf{R}^d$ such that

$$(c, \dot{c})(0) = (x_1, \xi_1) \quad \text{and} \quad (c, \dot{c})(\tau) = (x_2, \xi_2),$$

and such that the average acceleration along the curve, defined as $\frac{1}{\tau} \int_0^\tau |\ddot{c}(t)|^2 dt$, is minimized, then the curve is uniquely determined. It is given by a cubic polynomial, and the minimal average acceleration can be computed explicitly as

$$\frac{1}{\tau} \int_0^\tau |\ddot{c}(s)|^2 ds = 12 \left| \frac{1}{\tau} \left(\frac{x_2 - x_1}{\tau} - \frac{\xi_2 + \xi_1}{2} \right) \right|^2 + \left| \frac{\xi_2 - \xi_1}{\tau} \right|^2,$$

which is a function of the initial and final states (x_1, ξ_1) and (x_2, ξ_2) .

We use this computation to introduce the following functional.

Definition 4.1. For any $\tau > 0$ let the map $A_\tau : \mathbb{TR}^d \times \mathbb{TR}^d \longrightarrow \mathbf{R}$ be given by

$$A_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 := 3 \left| \frac{x_2 - x_1}{\tau} - \frac{\xi_2 + \xi_1}{2} \right|^2 + \frac{1}{4} |\xi_2 - \xi_1|^2 \quad (4.1)$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{TR}^d$. Then the *Minimal Acceleration Cost* is the functional

$$\mathbf{A}_\tau(\mu_1, \mu_2)^2 := \inf \left\{ \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} A_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 \gamma(d\mathbf{x}_1, d\mathbf{x}_2) : \gamma \in \Gamma(\mu_1, \mu_2) \right\}, \quad (4.2)$$

defined for all measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{TR}^d)$ (see Definition 2.1 for notation).

Note the analogy between the definition of the minimal acceleration cost and the Wasserstein distance: For the Wasserstein case, a particle is allowed to move from initial position to final position, by following a path that minimizes the velocity integral $\int_0^1 |\dot{c}(t)|^2 dt$. Since minimizing paths are geodesics, the resulting cost functional is just the distance squared. For the Minimal Acceleration Cost, we minimize the second derivative along the curve and obtain the cost function (4.1).

Note also that $\mathbf{A}_\tau$ is not a distance: Firstly, it is not symmetric in the arguments μ_1 and μ_2 , which follows from the asymmetry of the cost function (4.1). Secondly, it does not vanish if $\mu_1 = \mu_2$. Instead, we have the following relation:

$$\mathbf{A}_\tau(\mu_1, \mu_2) = 0 \iff \mu_2 = F_\tau \# \mu_1,$$

where $F_\tau : \mathbb{TR}^d \longrightarrow \mathbb{TR}^d$ is the *free transport map* defined by

$$F_\tau(\mathbf{x}) := (x + \tau\xi, \xi) \quad \text{for all } \mathbf{x} \in \mathbb{TR}^d. \quad (4.3)$$

This is in agreement with our heuristic: If each particle just follows the straight path determined by its initial velocity, then the acceleration vanishes. Unlike the Wasserstein distance, the Minimal Acceleration Cost depends explicitly on τ .

It will be convenient to rewrite $\mathbf{A}_\tau$ in a slightly different form.

Definition 4.2. For any $\tau > 0$ let the map $W_\tau : \mathbb{TR}^d \times \mathbb{TR}^d \longrightarrow \mathbf{R}$ be given by

$$W_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 := 3 \left| \frac{x_2 - x_1}{\tau} - \frac{\xi_2 - \xi_1}{2} \right|^2 + \frac{1}{4} |\xi_2 - \xi_1|^2. \quad (4.4)$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{TR}^d$. Then the *Minimal Acceleration Distance* is the functional

$$\mathbf{W}_\tau(\mu_1, \mu_2)^2 := \inf \left\{ \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} W_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 \gamma(d\mathbf{x}_1, d\mathbf{x}_2) : \gamma \in \Gamma(\mu_1, \mu_2) \right\}, \quad (4.5)$$

defined for all measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{TR}^d)$ (see Definition 2.1 for notation).

The cost functions A_τ and W_τ are related by the identity

$$A_\tau(\mathbf{x}_1, \mathbf{x}_2) = W_\tau(F_\tau(\mathbf{x}_1), \mathbf{x}_2) \quad \text{for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{TR}^d,$$

where F_τ is the free transport map defined in (4.3). Since F_τ is an automorphism of the vector space $\mathbb{TR}^d$, the push-forward under F_τ maps $\mathcal{P}(\mathbb{TR}^d)$ injectively onto itself. We therefore obtain the following relation

$$\mathbf{A}_\tau(\mu_1, \mu_2) = \mathbf{W}_\tau(F_\tau \# \mu_1, \mu_2) \quad \text{for all } \mu_1, \mu_2 \in \mathcal{P}(\mathbb{TR}^d). \quad (4.6)$$

We will prove below that the functional $\mathbf{W}_\tau$ defines a distance on $\mathcal{P}(\mathbb{TR}^d)$.

Remark 4.3. Cost functionals similar to $\mathbf{A}_\tau$ and $\mathbf{W}_\tau$ have been considered before by Huang & Jordan in their work on the Vlasov-Poisson-Fokker-Planck system and the Kramers equation; see [20, 21]. They use the cost function

$$C_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 := 2 \left| \frac{x_2 - x_1}{\tau} - \frac{\xi_2 + \xi_1}{2} \right|^2 + \frac{1}{2} |\xi_2 - \xi_1|^2.$$

As the heuristic outlined above suggests, the numerical constants used in (4.1) and (4.4) are more “natural.” We will see in Section 4.4 that this choice gives the right energy dissipation. There is also a connection to ultraparabolic equations, already pointed out in [20]: The fundamental solution Γ_τ of the equation

$$\partial_t u = -\xi \cdot \nabla_x u + \Delta_\xi u \quad \text{in } [0, \infty) \times \mathbb{TR}^d \quad (4.7)$$

is given in terms of the cost function (4.1) as

$$\Gamma_t(\mathbf{x}, \bar{\mathbf{x}}) = \frac{\alpha_d}{t^{2d}} \exp \left(- \frac{A_t(\mathbf{x}, \bar{\mathbf{x}})^2}{t} \right),$$

where $\alpha_d > 0$ is some constant depending only on the space dimension. That is, if for suitable initial data $\bar{u}: \mathbb{TR}^d \rightarrow \mathbf{R}$ we define the function

$$u(t, \mathbf{x}) := \int_{\mathbb{TR}^d} \Gamma_t(\mathbf{x}, \bar{\mathbf{x}}) \bar{u}(\bar{\mathbf{x}}) d\bar{\mathbf{x}} \quad \text{for all } (t, \mathbf{x}) \in (0, \infty) \times \mathbb{TR}^d,$$

then u is a solution of the Cauchy problem for (4.7); see [32]. We refer the reader to [5, 7, 15] for further information on the Vlasov-Poisson-Fokker-Planck system.

Proposition 4.4. *For any $\tau > 0$ consider the automorphism G_τ of $\mathbb{TR}^d$ given by*

$$G_\tau(\mathbf{x}) := \left(\sqrt{3} \left(\frac{x}{\tau} - \frac{\xi}{2} \right), \frac{\xi}{2} \right) \quad \text{for all } \mathbf{x} \in \mathbb{TR}^d.$$

Then $\mathbf{W}_\tau$ can be expressed in terms of the Wasserstein distance as

$$\mathbf{W}_\tau(\mu_1, \mu_2) = \mathbf{W}(G_\tau \# \mu_1, G_\tau \# \mu_2) \quad \text{for all } \mu_1, \mu_2 \in \mathcal{P}(\mathbb{TR}^d). \quad (4.8)$$

In particular, the functional $\mathbf{W}_\tau$ defines a distance on $\mathcal{P}(\mathbb{TR}^d)$, and for all pairs of measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{TR}^d)$ there exists a transport plan $\gamma \in \Gamma(\mu_1, \mu_2)$ with

$$\mathbf{W}_\tau(\mu_1, \mu_2)^2 = \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} W_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 \gamma(d\mathbf{x}_1, d\mathbf{x}_2). \quad (4.9)$$

Similarly, the infimum in (4.2) is attained and thus $\mathbf{A}_\tau(\mu_1, \mu_2)$ is a minimum.

Proof. Notice that the push-forward under G_τ maps $\mathcal{P}(\mathbb{TR}^d)$ injectively onto itself. Therefore the push-forward under the linear bijection H_τ defined by

$$H_\tau(\mathbf{x}_1, \mathbf{x}_2) := \left(G_\tau^{-1}(\mathbf{x}_1), G_\tau^{-1}(\mathbf{x}_2) \right) \quad \text{for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{TR}^d,$$

maps the set of transport plans $\Gamma(G_\tau \# \mu_1, G_\tau \# \mu_2)$ injectively onto $\Gamma(\mu_1, \mu_2)$. Since

$$\begin{aligned} & \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} W_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 (H_\tau \# \gamma)(d\mathbf{x}_1, d\mathbf{x}_2) \\ &= \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} |G_\tau(\mathbf{x}_1) - G_\tau(\mathbf{x}_2)|^2 (H_\tau \# \gamma)(d\mathbf{x}_1, d\mathbf{x}_2) \\ &= \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} |\mathbf{z}_1 - \mathbf{z}_2|^2 ((H_\tau^{-1} \circ H_\tau) \# \gamma)(d\mathbf{z}_1, d\mathbf{z}_2) \\ &= \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} |\mathbf{z}_1 - \mathbf{z}_2|^2 \gamma(d\mathbf{z}_1, d\mathbf{z}_2) \end{aligned}$$

for all $\gamma \in \Gamma(G_\tau \# \mu_1, G_\tau \# \mu_2)$, we conclude that $\Gamma_{\text{opt}}(G_\tau \# \mu_1, G_\tau \# \mu_2)$ and the set of transport plans in $\Gamma(\mu_1, \mu_2)$ that minimize (4.5), are in one-to-one correspondence. This implies the identity (4.8) and the existence of $\gamma \in \Gamma(\mu_1, \mu_2)$ satisfying (4.9). Since $\mathbf{W}$ is a metric on $\mathcal{P}(\mathbb{TR}^d)$, and since the push-forward under G_τ is a bijection of $\mathcal{P}(\mathbb{TR}^d)$ to itself, the functional $\mathbf{W}_\tau$ is a metric as well.

For the second part of the proposition we can argue in a similar way. $\square$

With $\mu_1 \in \mathcal{P}(\mathbb{TR}^d)$ a given measure, we will now study the problem of minimizing the functional $\mathbf{W}_\tau(\mu_1, \mu)$ over all $\mu \in \mathcal{P}(\mathbb{TR}^d)$ with prescribed marginal. It turns out that then the minimal acceleration distance reduces to the Wasserstein distance between the marginal measures, up to some constant.

Proposition 4.5. *Let $\mu_1 \in \mathcal{P}(\mathbb{TR}^d)$ be given, and let $\varrho_1 := \pi \# \mu_1$ be its marginal. For any timestep $\tau > 0$ and any $\varrho_2 \in \mathcal{P}(\mathbb{R}^d)$ we then have the equality*

$$\inf \left\{ \mathbf{W}_\tau(\mu_1, \mu)^2 : \mu \in \mathcal{P}(\mathbb{TR}^d), \pi \# \mu = \varrho_2 \right\} = \frac{3}{4\tau^2} \mathbf{W}(\varrho_1, \varrho_2)^2. \quad (4.10)$$

Moreover, the optimal transport plans $\gamma \in \mathcal{P}(\mathbb{TR}^d \times \mathbb{TR}^d)$, for which the infimum on the left-hand side of (4.10) is attained, are completely determined by

$$\begin{aligned} & \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} \varphi(\mathbf{x}, \mathbf{z}) \gamma(d\mathbf{x}, d\mathbf{z}) \\ &:= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\int_{\mathbb{T}_x \mathbb{R}^d} \varphi(\mathbf{x}, z, \beta(\mathbf{x}, z)) \sigma_1(d\xi|x) \right) \hat{\gamma}(dx, dz) \end{aligned} \quad (4.11)$$

for all $\varphi \in \mathcal{C}_b(\mathbb{TR}^d \times \mathbb{TR}^d)$, where $\hat{\gamma} \in \Gamma_{\text{opt}}(\varrho_1, \varrho_2)$ is any optimal transport plan pushing ϱ_1 into ϱ_2 , where $\mu_1 = \sigma_1 \varrho_1$ is the disintegration of μ_1 defined by

$$\int_{\mathbb{TR}^d} \varphi(\mathbf{x}) \mu_1(d\mathbf{x}) =: \int_{\mathbb{R}^d} \left(\int_{\mathbb{T}_x \mathbb{R}^d} \varphi(x, \xi) \sigma_1(d\xi|x) \right) \varrho_1(dx)$$

for all $\varphi \in \mathcal{C}_b(\mathbb{TR}^d)$, and where $\beta(\mathbf{x}, z) := \xi + \frac{3}{2\tau}(z - x)$ for all $(\mathbf{x}, z) \in \mathbb{TR}^d \times \mathbb{R}^d$. If the optimal transport plan $\hat{\gamma}$ is unique, then so is the minimizer γ .

Proof. Let us first show that the inf in (4.10) is attained for some $\mu \in \mathcal{P}(\mathbb{TR}^d)$. Let $\hat{\gamma} \in \Gamma_{\text{opt}}(\varrho_1, \varrho_2)$ be any optimal transport plan, and let $\gamma \in \mathcal{P}(\mathbb{TR}^d \times \mathbb{TR}^d)$ be defined by (4.11) for all test functions $\varphi \in \mathcal{C}_b(\mathbb{TR}^d \times \mathbb{TR}^d)$. Notice that $\pi \# \gamma = \mu_1$,

and that $\pi^2 \# \gamma \in \mathcal{P}(\mathbb{TR}^d)$ with $(\pi \circ \pi^2) \# \gamma = \varrho_2$. Then we have

$$\begin{aligned} \mathbf{W}_\tau(\mu_1, \pi^2 \# \gamma)^2 &\leq \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} W_\tau(\mathbf{x}, \mathbf{z})^2 \gamma(d\mathbf{x}, d\mathbf{z}) \\ &= \frac{3}{4\tau^2} \iint_{\mathbf{R}^d \times \mathbf{R}^d} |z - x|^2 \hat{\gamma}(dx, dz) = \frac{3}{4\tau^2} \mathbf{W}(\varrho_1, \varrho_2)^2, \end{aligned} \quad (4.12)$$

which shows that the infimum on the left-hand side of (4.10) is bounded above. It is trivially bounded below since the functional $\mathbf{W}_\tau$ is nonnegative.

Consider now a minimizing sequence $\{\mu^n\}$ of measures $\mu^n \in \mathcal{P}(\mathbb{TR}^d)$ with

$$\pi \# \mu^n = \varrho_2 \quad \text{for all } n \in \mathbf{N}.$$

Choose transport plans $\gamma^n \in \Gamma(\mu_1, \mu^n)$ such that

$$\mathbf{W}_\tau(\mu_1, \mu^n)^2 = \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} W_\tau(\mathbf{x}, \mathbf{z})^2 \gamma^n(d\mathbf{x}, d\mathbf{z}) \quad \text{for all } n \in \mathbf{N}. \quad (4.13)$$

The measures γ^n exist because of Proposition 4.4, and we may assume that (4.13) is bounded above by some constant independent of n . Since we can estimate

$$\begin{aligned} \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} |\zeta|^2 \gamma^n(d\mathbf{x}, d\mathbf{z}) &\leq 2 \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} \left(|\xi|^2 + 4W_\tau(\mathbf{x}, \mathbf{z})^2 \right) \gamma^n(d\mathbf{x}, d\mathbf{z}) \\ &= 2 \int_{\mathbb{TR}^d} |\xi|^2 \mu_1(d\mathbf{x}) + 8\mathbf{W}_\tau(\mu_1, \mu^n)^2, \end{aligned}$$

and since all other second moments do not depend on n , we find that the sequence $\{\gamma^n\}$ is tight, and thus precompact in the narrow topology; see Lemma 5.2.2 in [4]. Extracting a subsequence if necessary (which we still label $\{\gamma^n\}$ for simplicity), we obtain that $\gamma^n \rightarrow \gamma$ narrowly for some measure $\gamma \in \mathcal{P}(\mathbb{TR}^d \times \mathbb{TR}^d)$ with

$$\pi^1 \# \gamma = \mu_1 \quad \text{and} \quad (\pi \circ \pi^2) \# \gamma = \varrho_2.$$

Then $\mu := \pi^2 \# \gamma$ is a minimizer in (4.10), and we have that

$$\inf \left\{ \mathbf{W}_\tau(\mu_1, \mu)^2 : \mu \in \mathcal{P}(\mathbb{TR}^d), \pi \# \mu = \varrho_2 \right\} = \iint_{\mathbb{TR}^d \times \mathbb{TR}^d} W_\tau(\mathbf{x}, \mathbf{z})^2 \gamma(d\mathbf{x}, d\mathbf{z}). \quad (4.14)$$

Let us further investigate the structure of the optimal transport plan γ . Consider the $(\mathbf{x}, z)$ -marginal of γ , denoted by $\gamma' \in \mathcal{P}(\mathbb{TR}^d \times \mathbf{R}^d)$, and let $\gamma = \sigma' \gamma'$ be the disintegration of γ with respect to γ' , which is defined by

$$\begin{aligned} &\iint_{\mathbb{TR}^d \times \mathbb{TR}^d} \varphi(\mathbf{x}, \mathbf{z}) \gamma(d\mathbf{x}, d\mathbf{z}) \\ &= \iint_{\mathbb{TR}^d \times \mathbf{R}^d} \left(\int_{\mathbb{T}_z \mathbf{R}^d} \varphi(\mathbf{x}, z, \zeta) \sigma'(d\zeta | \mathbf{x}, z) \right) \gamma'(d\mathbf{x}, dz) \end{aligned}$$

for all $\varphi \in \mathcal{C}_b(\mathbb{TR}^d \times \mathbb{TR}^d)$. Since the minimization problem (4.10) does not impose a constraint on the ζ -marginal of $\pi^2 \# \gamma$, for any γ satisfying (4.14) we have

$$\int_{\mathbb{T}_z \mathbf{R}^d} W_\tau(\mathbf{x}, z, \zeta)^2 \sigma'(d\zeta | \mathbf{x}, z) = \inf \left\{ \int_{\mathbb{T}_z \mathbf{R}^d} W_\tau(\mathbf{x}, z, \zeta)^2 \sigma(d\zeta) : \sigma \in \mathcal{P}(\mathbb{T}_z \mathbf{R}^d) \right\}$$

for γ' -a.e. $(\mathbf{x}, z) \in \mathbb{TR}^d \times \mathbf{R}^d$. By strict convexity of the map $\zeta \mapsto W_\tau(\mathbf{x}, z, \zeta)^2$, the unique minimizer of this problem is a Dirac measure located at the velocity $\beta(\mathbf{x}, z)$

defined above, which implies the simplification that

$$\int_{\mathbb{T}_z \mathbf{R}^d} W_\tau(\mathbf{x}, z, \zeta)^2 \sigma'(d\zeta|\mathbf{x}, z) = W_\tau(\mathbf{x}, z, \beta(\mathbf{x}, z))^2 = \frac{3}{4\tau^2} |z - x|^2 \quad (4.15)$$

for γ' -a.e. $(\mathbf{x}, z) \in \mathbb{T}\mathbf{R}^d \times \mathbf{R}^d$. Using (4.14), we then obtain the estimate

$$\begin{aligned} \inf \left\{ \mathbf{W}_\tau(\mu_1, \mu)^2 : \mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d), \pi\#\mu = \varrho_2 \right\} &= \frac{3}{4\tau^2} \iint_{\mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d} |z - x|^2 \gamma(d\mathbf{x}, d\mathbf{z}) \\ &\geq \frac{3}{4\tau^2} \mathbf{W}(\varrho_1, \varrho_2)^2. \end{aligned} \quad (4.16)$$

Note that the function (4.15) does not depend on ξ . Combining (4.12) and (4.16), we obtain that the (x, z) -marginal of γ is an optimal transport map in $\Gamma_{\text{opt}}(\varrho_1, \varrho_2)$. If this set contains only one element, then γ is uniquely determined. $\square$

4.2. Velocity Projection. In this section and the following one, we introduce a two-stage minimization problem that will be the building-block for the time discretization for (1.1) we will discuss in Section 5: In the first step, we minimize the *internal* energy subject to a constraint imposed by the Minimal Acceleration Cost. The resulting minimizer typically involves a velocity that is not a gradient vector field. In order to restore the tangency, we project the minimizer onto the tangent bundle. In this second step, the *kinetic* energy is reduced in an optimal way.

Let us first consider the velocity projection.

Definition 4.6 (Velocity Projection). Let functions

$$\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d) \quad \text{and} \quad \mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$$

be given, and assume that $\mathbf{u}$ is approximately differentiable ϱ -a.e. in the sense of (2.11). As shown in Proposition 2.5, for any $\delta > 0$ we can pick $\tau \in [\delta/2, \delta]$ such that the push-forward measure $(\text{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d) \in \mathcal{P}(\mathbf{R}^d)$ is absolutely continuous with respect to the Lebesgue measure. Let

$$\mathbf{u} =: \nabla\phi^\tau \circ (\text{id} + \tau\mathbf{w}^\tau) + \mathbf{w}^\tau$$

be the uniquely determined polar factorization of $\mathbf{u}$, where $\text{id} + \tau\nabla\phi^\tau$ is the gradient of a convex function, and $\text{id} + \tau\mathbf{w}^\tau$ is a $\varrho\mathcal{L}^d$ -preserving map from $\mathbf{R}^d$ to itself (see Section 2.1 for more details). Then we define the velocity projection

$$\mathbf{P}[\varrho, \mathbf{u}|\tau] := (\varrho, \nabla\phi^\tau).$$

We denote by $\mathbb{T}\mathcal{P}_{\text{reg}}(\mathbf{R}^d, \tau)$ the set of all $(\varrho, \mathbf{u}) \in \mathbb{T}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ such that $\text{id} + \tau\mathbf{u}$ is an optimal transport map pushing $\varrho\mathcal{L}^d$ forward to $(\text{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d)$.

Note that the map $\text{id} + \tau\nabla\phi^\tau$ is the *optimal transport map* pushing $\varrho\mathcal{L}^d$ forward to the measure $\hat{\varrho}^\tau\mathcal{L}^d := (\text{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d)$. It is injective ϱ -a.e., and we have

$$\begin{aligned} \int_{\mathbf{R}^d} |\nabla\phi^\tau|^2 \varrho dx &= \frac{\mathbf{W}(\varrho, \hat{\varrho}^\tau)^2}{\tau^2} \\ &= \inf \left\{ \int_{\mathbf{R}^d} |\mathbf{u}_*|^2 \varrho dx : (\text{id} + \tau\mathbf{u}_*)\#(\varrho\mathcal{L}^d) = \hat{\varrho}^\tau\mathcal{L}^d \right\} \leq \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho dx, \end{aligned} \quad (4.17)$$

which shows that the velocity projection reduces the kinetic energy as much as possible, given the constraint that the new velocity field induces a transport map pushing $\varrho\mathcal{L}^d$ to $\hat{\varrho}^\tau\mathcal{L}^d$. If we define $\hat{\mathbf{u}}^\tau \in \mathcal{L}^2(\mathbf{R}^d, \hat{\varrho}^\tau)$ by $\hat{\mathbf{u}}^\tau := \nabla\phi^\tau \circ (\text{id} + \tau\nabla\phi^\tau)^{-1}$,

then the map $\text{id} - \tau \hat{\mathbf{u}}^\tau$ is the optimal transport map pushing $\hat{\varrho}^\tau \mathcal{L}^d$ forward to $\varrho \mathcal{L}^d$. This follows from Remark 6.2.11 in [4] and the fact that

$$\text{id} - \tau \hat{\mathbf{u}}^\tau = \left((\text{id} + \tau \nabla \phi^\tau) - \tau \nabla \phi^\tau \right) \circ (\text{id} + \tau \nabla \phi^\tau)^{-1} = (\text{id} + \tau \nabla \phi^\tau)^{-1}.$$

In particular, we have $(\hat{\varrho}^\tau, -\hat{\mathbf{u}}^\tau) \in \mathbb{T} \mathcal{P}_{\text{reg}}(\mathbf{R}^d, \tau)$, and so $\hat{\mathbf{u}}^\tau$ is a gradient field.

4.3. Energy Minimization. We now consider the internal energy.

Proposition 4.7. *Let $\tau > 0$ be given and consider functions $(\varrho, \mathbf{u}) \in \mathbb{T} \mathcal{P}_{\text{reg}}(\mathbf{R}^d, \tau)$ with finite total energy. Let $\mu := (\text{id} \times \mathbf{u}) \# (\varrho \mathcal{L}^d)$ (that is, let μ be defined by*

$$\int_{\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x}) \mu(d\mathbf{x}) := \int_{\mathbf{R}^d} \varphi(x, \mathbf{u}(x)) \varrho(x) dx \quad (4.18)$$

for all $\varphi \in \mathcal{C}_b(\mathbb{T}\mathbf{R}^d)$). Then there exists a uniquely determined minimizer

$$\mu^\tau \in \operatorname{argmin} \left\{ \mathbf{A}_\tau(\mu, \mu_*)^2 + \mathcal{U}(\mu_*): \mu_* \in \mathcal{P}(\mathbb{T}\mathbf{R}^d) \right\}. \quad (4.19)$$

It is of the form $\mu^\tau = (\text{id} \times \mathbf{u}^\tau) \# (\varrho^\tau \mathcal{L}^d)$, where $\varrho^\tau \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and $\mathbf{u}^\tau \in \mathcal{L}^2(\mathbf{R}^d, \varrho^\tau)$ have finite total energy and satisfy the following identities ϱ^τ -a.e.:

$$\varrho^\tau \mathcal{L}^d = \left((\text{id} + \tau \mathbf{u})^{-1} \circ \left(\text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau) \right) \right)^{-1} \# (\varrho \mathcal{L}^d), \quad (4.20)$$

$$\mathbf{u}^\tau = \mathbf{u} \circ (\text{id} + \tau \mathbf{u})^{-1} \circ \left(\text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau) \right) - \tau \nabla U'(\varrho^\tau). \quad (4.21)$$

The vector field $\text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau)$ is the gradient of a lower semicontinuous convex function, and the velocity $\mathbf{u}^\tau$ is approximately differentiable ϱ^τ -a.e. as in (2.11).

Remark 4.8. It is remarkable that the minimization problem (4.19) preserves the structure (4.18). That is, if we start with a measure $\mu \in \mathcal{P}(\mathbb{T}\mathbf{R}^d)$ that is induced by a density/velocity pair $(\varrho, \mathbf{u})$, then the minimizer of (4.19) is of the same form, even though we are minimizing over all of $\mathcal{P}(\mathbb{T}\mathbf{R}^d)$.

Proof. Let us first prove that the minimization problem (4.19) has a solution. Since the internal energy $\mathcal{U}(\mu)$ does not take into account the velocity distribution of μ , we can apply Proposition 4.5 to reduce the optimization (4.19) to a minimization problem for densities: Defining $\hat{\varrho}^\tau \mathcal{L}^d := (\text{id} + \tau \mathbf{u}) \# (\varrho \mathcal{L}^d)$, we have that

$$\begin{aligned} & \inf \left\{ \mathbf{A}_\tau(\mu, \mu_*)^2 + \mathcal{U}(\mu_*): \mu_* \in \mathcal{P}(\mathbb{T}\mathbf{R}^d) \right\} \\ &= \inf \left\{ \frac{3}{4\tau^2} \mathbf{W}(\hat{\varrho}^\tau, \varrho_*)^2 + \mathcal{U}(\varrho_*): \varrho_* \in \mathcal{P}(\mathbf{R}^d) \right\}. \end{aligned} \quad (4.22)$$

The latter infimum is bounded above because we may choose $\varrho_* = \varrho \mathcal{L}^d$ and obtain

$$\frac{3}{4\tau^2} \mathbf{W}(\hat{\varrho}^\tau, \varrho)^2 + \int_{\mathbf{R}^d} U(\varrho) dx \leq \frac{3}{4} \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho dx + \int_{\mathbf{R}^d} U(\varrho) dx,$$

which is finite since $(\varrho, \mathbf{u})$ has finite total energy. In particular, in (4.22) it suffices to minimize only over absolutely continuous densities. It is well-known that there exists a uniquely determined minimizer of (4.22), which we denote by $\varrho^\tau \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$; see [4, 26]. In fact, the existence of a minimizer follows from lower semicontinuity of the functionals, while uniqueness is a consequence of displacement convexity; see

Section 9.3.9 in [4]. Therefore there exists a unique, essentially injective, optimal transport map $\mathbf{r}^\tau$ pushing $\varrho^\tau \mathcal{L}^d$ forward to $\hat{\varrho}^\tau \mathcal{L}^d$. It is given by

$$\mathbf{r}^\tau = \text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau) \quad (4.23)$$

ϱ^τ -a.e., and $\mathbf{r}^\tau$ is the gradient of a convex function. Then (4.20) follows.

Note that since $\mathbf{u}$ is an optimal transport velocity and $\text{id} + \tau \mathbf{u}$ is essentially injective, the push-forward measure $\hat{\mu}^\tau := F_\tau \# \mu$ satisfies the identity

$$\int_{\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x}) \hat{\mu}^\tau(d\mathbf{x}) := \int_{\mathbf{R}^d} \varphi(x, \hat{\mathbf{u}}^\tau(x)) \hat{\varrho}^\tau(x) dx$$

for all $\varphi \in \mathcal{C}_b(\mathbb{T}\mathbf{R}^d)$, with velocity $\hat{\mathbf{u}}^\tau \in \mathcal{L}^2(\mathbf{R}^d, \hat{\varrho}^\tau)$ defined $\hat{\varrho}^\tau$ -a.e. by

$$\hat{\mathbf{u}}^\tau := \mathbf{u} \circ (\text{id} + \tau \mathbf{u})^{-1}. \quad (4.24)$$

By (4.6) and Proposition 4.5, the velocity distribution of μ^τ is uniquely determined by the optimal transport map $\mathbf{r}^\tau$ and the velocity $\hat{\mathbf{u}}^\tau$, and we obtain

$$\mathbf{u}^\tau = \beta(\mathbf{r}^\tau, \hat{\mathbf{u}}^\tau \circ \mathbf{r}^\tau, \text{id}) = \hat{\mathbf{u}}^\tau \circ \mathbf{r}^\tau + \frac{3}{2\tau} (\text{id} - \mathbf{r}^\tau)$$

ϱ^τ -a.e. Then (4.21) follows from (4.23) and (4.24).

Finally, since $\mathbf{r}^\tau$ is the gradient of a convex function, the map $\nabla U'(\varrho^\tau)$ is differentiable ϱ^τ -a.e., hence satisfies (2.11) with measure $\mu := \varrho^\tau \mathcal{L}^d$. A similar statement holds for the velocity $\hat{\mathbf{u}}^\tau$. Moreover, the preimage under $\mathbf{r}^\tau$ of any $(\hat{\varrho}^\tau \mathcal{L}^d)$ -null set is negligible with respect to the measure $\varrho^\tau \mathcal{L}^d$. Therefore the composition $\hat{\mathbf{u}}^\tau \circ \mathbf{r}^\tau$ is approximately differentiable $(\varrho^\tau \mathcal{L}^d)$ -a.e. as well. This concludes the proof. $\square$

We can now define the following minimization step.

Definition 4.9 (Energy Minimization). Let $\tau > 0$ be given and consider

$$(\varrho, \mathbf{u}) \in \mathbb{T} \mathcal{P}_{\text{reg}}(\mathbf{R}^d, \tau).$$

Let $(\varrho^\tau, \mathbf{u}^\tau)$ be the pair of functions that determine the minimizer in problem (4.19) of Proposition 4.7. Then we define the energy minimization

$$\mathbf{M}[\varrho, \mathbf{u}|\tau] := (\varrho^\tau, \mathbf{u}^\tau) \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d) \times \mathcal{L}^2(\mathbf{R}^d, \varrho^\tau).$$

Note that the velocity $\mathbf{u}^\tau$ is typically not a gradient vector field.

4.4. Energy Inequality. We now prove a crucial stability estimate for the two-stage minimization introduced above: The total energy is nonincreasing.

Proposition 4.10 (Energy Inequality). Assume that $\tau > 0$ and

$$\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d), \quad \mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho) \quad (4.25)$$

are given with $\mathcal{E}(\varrho, \mathbf{u}) < \infty$. Suppose that $\tau > 0$ is chosen so that $(\text{id} + \tau \mathbf{u}) \# (\varrho \mathcal{L}^d)$ is absolutely continuous with respect to the Lebesgue measure.

(1) If $(\varrho^\tau, \mathbf{u}^\tau) := \mathbf{P}[\varrho, \mathbf{u}|\tau]$, with $\mathbf{P}$ as in Definition 4.6, then

$$\mathcal{E}(\varrho^\tau, \mathbf{u}^\tau) \leq \mathcal{E}(\varrho, \mathbf{u}).$$

(2) Suppose that in addition to (4.25), we have

$$(\varrho, \mathbf{u}) \in \mathbb{T} \mathcal{P}_{\text{reg}}(\mathbf{R}^d, \tau).$$

If $(\varrho^\tau, \mathbf{u}^\tau) := \mathbf{M}[\varrho, \mathbf{u}|\tau]$, with $\mathbf{M}$ as in Definition 4.9, then

$$\mathcal{E}(\varrho^\tau, \mathbf{u}^\tau) + \frac{\tau^2}{6} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx \leq \mathcal{E}(\varrho, \mathbf{u}).$$

Proof. Statement (1) follows immediately because the velocity projection leaves the density and therefore the internal energy unchanged, and replaces the velocity by an optimal transport velocity with minimal kinetic energy; see (4.17).

To prove Statement (2), let $\hat{\varrho}^\tau \mathcal{L}^d := (\text{id} + \tau \mathbf{u}) \# (\varrho \mathcal{L}^d)$ and $\hat{\mathbf{u}}^\tau := \mathbf{u} \circ (\text{id} + \tau \mathbf{u})^{-1}$. By assumption on $\mathbf{u}$ and because of Proposition 4.7, we have that

$$\mathbf{r}^\tau := \text{id} + \tau \mathbf{u} \quad \text{and} \quad \bar{\mathbf{r}}^\tau := \text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau)$$

are optimal transport maps pushing $\varrho \mathcal{L}^d$ and $\varrho^\tau \mathcal{L}^d$ forward to $\hat{\varrho}^\tau \mathcal{L}^d$. Given that both $\mathcal{U}(\varrho)$ and $\mathcal{U}(\varrho^\tau)$ are finite, and since $\nabla U'(\varrho^\tau) \in \mathcal{L}^2(\mathbf{R}^d, \varrho^\tau)$, we obtain

$$\mathcal{U}(\varrho^\tau) + \frac{2\tau^2}{3} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx - \tau \int_{\mathbf{R}^d} \nabla U'(\varrho^\tau) \cdot \tilde{\mathbf{u}}^\tau \varrho^\tau dx \leq \mathcal{U}(\varrho), \quad (4.26)$$

where $\tilde{\mathbf{u}}^\tau := \hat{\mathbf{u}}^\tau \circ \bar{\mathbf{r}}^\tau$ (recall that $(\text{id} + \tau \mathbf{u})^{-1} = \text{id} - \tau \hat{\mathbf{u}}^\tau$). We used Proposition 3.2. Then we have $\mathbf{u}^\tau = \tilde{\mathbf{u}}^\tau - \tau \nabla U'(\varrho^\tau)$, which implies the identity

$$\mathcal{K}(\varrho^\tau, \mathbf{u}^\tau) - \frac{\tau^2}{2} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx + \tau \int_{\mathbf{R}^d} \nabla U'(\varrho^\tau) \cdot \tilde{\mathbf{u}}^\tau \varrho^\tau dx = \mathcal{K}(\varrho^\tau, \tilde{\mathbf{u}}^\tau). \quad (4.27)$$

Adding (4.26) and (4.27) and noticing that $\mathcal{K}(\varrho^\tau, \tilde{\mathbf{u}}^\tau) = \mathcal{K}(\varrho, \mathbf{u})$, we conclude. $\square$

Remark 4.11. There is an alternative way to prove Proposition 4.10 that highlights the role of the convexity of the internal energy and the Wasserstein distance along generalized geodesics, and of the curvature properties of the Wasserstein space.

Let us first consider Statement (2) of Proposition 4.10. Under the assumptions made there, let $\hat{\varrho}^\tau \mathcal{L}^d := (\text{id} + \tau \mathbf{u}) \# (\varrho \mathcal{L}^d)$. Notice that then

$$\mathcal{E}(\varrho, \mathbf{u}) = \frac{\mathbf{W}(\varrho, \hat{\varrho}^\tau)^2}{2\tau^2} + \mathcal{U}(\varrho). \quad (4.28)$$

We now define the functional

$$\Phi(\varrho_*) := \frac{\mathbf{W}(\varrho_*, \hat{\varrho}^\tau)^2}{2\tau^2} + \mathcal{U}(\varrho_*) \quad \text{for all } \varrho_* \in \mathcal{P}(\mathbf{R}^d).$$

We already know that the vector fields

$$\mathbf{r}^\tau := \text{id} + \tau \mathbf{u} \quad \text{and} \quad \bar{\mathbf{r}}^\tau := \text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau)$$

are optimal transport maps pushing $\varrho \mathcal{L}^d$ and $\varrho^\tau \mathcal{L}^d$ forward to $\hat{\varrho}^\tau \mathcal{L}^d$. Their inverse maps $\mathbf{q}^\tau := (\mathbf{r}^\tau)^{-1}$ and $\bar{\mathbf{q}}^\tau := (\bar{\mathbf{r}}^\tau)^{-1}$ are given by the formulas

$$\begin{aligned} \mathbf{q}^\tau &:= \text{id} - \tau \mathbf{u} \circ (\text{id} + \tau \mathbf{u})^{-1}, \\ \bar{\mathbf{q}}^\tau &:= \text{id} - \frac{2\tau^2}{3} \nabla U'(\varrho^\tau) \circ \left(\text{id} + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau) \right)^{-1}. \end{aligned}$$

Consider now the family of measures defined by

$$\varrho_s := \left((1-s) \bar{\mathbf{q}}^\tau + s \mathbf{q}^\tau \right) \# (\hat{\varrho}^\tau \mathcal{L}^d) \quad \text{for all } s \in [0, 1].$$

In the terminology of Definition 9.2.2 in [4], the map $s \mapsto \varrho_s$ is called a generalized geodesic. Moreover, according to Lemma 9.2.1 (see in particular equation (9.2.7b)) and Proposition 9.3.9 in [4], the functional Φ is convex along this curve, and

$$\Phi(\varrho_s) \leq (1-s)\Phi(\varrho^\tau) + s\Phi(\varrho) - \frac{s(1-s)}{2\tau^2} \int_{\mathbf{R}^d} |\mathbf{q}^\tau \circ \bar{\mathbf{r}}^\tau - \text{id}|^2 \varrho^\tau dx$$

for all $s \in [0, 1]$. Rearranging terms, we obtain

$$\Phi(\varrho^\tau) \leq \Phi(\varrho) - \lim_{s \rightarrow 0} \frac{\Phi(\varrho_s) - \Phi(\varrho^\tau)}{s} - \frac{1}{2\tau^2} \int_{\mathbf{R}^d} |\mathbf{q}^\tau \circ \bar{\mathbf{r}}^\tau - \text{id}|^2 \varrho^\tau dx.$$

Using the explicit formula for the optimal transport $\bar{\mathbf{r}}^\tau$, we can now write

$$\Phi(\varrho^\tau) = \frac{2\tau^2}{9} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx + \mathcal{U}(\varrho^\tau).$$

Moreover, using Proposition 3.2 above and Proposition 7.3.6 in [4], we obtain

$$-\lim_{s \rightarrow 0} \frac{\Phi(\varrho_s) - \Phi(\varrho^\tau)}{s} \leq \frac{\tau}{3} \int_{\mathbf{R}^d} \tilde{\mathbf{u}}^\tau \cdot \nabla U'(\varrho^\tau) \varrho^\tau dx - \frac{2\tau^2}{9} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx$$

since $\mathbf{q}^\tau \circ \bar{\mathbf{r}}^\tau - \text{id} = -\tau \tilde{\mathbf{u}}^\tau + \frac{2\tau^2}{3} \nabla U'(\varrho^\tau)$ with $\tilde{\mathbf{u}}^\tau := \mathbf{u} \circ \mathbf{q}^\tau \circ \bar{\mathbf{r}}^\tau$. We have

$$\begin{aligned} & -\frac{1}{2\tau^2} \int_{\mathbf{R}^d} |\mathbf{q}^\tau \circ \bar{\mathbf{r}}^\tau - \text{id}|^2 \varrho^\tau dx \\ &= -\frac{1}{2} \int_{\mathbf{R}^d} |\tilde{\mathbf{u}}^\tau|^2 \varrho^\tau dx + \frac{2\tau}{3} \int_{\mathbf{R}^d} \tilde{\mathbf{u}}^\tau \cdot \nabla U'(\varrho^\tau) \varrho^\tau dx - \frac{2\tau^2}{9} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx. \end{aligned}$$

Collecting terms, we arrive at the following estimate

$$\mathcal{E}(\varrho^\tau, \tilde{\mathbf{u}}^\tau - \tau \nabla U'(\varrho^\tau)) + \frac{\tau^2}{6} \int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau dx \leq \mathcal{E}(\varrho, \mathbf{u}),$$

which is the desired result since $\tilde{\mathbf{u}}^\tau - \tau \nabla U'(\varrho^\tau) = \mathbf{u}^\tau$. It is quite remarkable that the intermediate velocity $\mathbf{u}^\tau$, which we obtained previously by solving a minimization problem for the internal energy, appears naturally in the above computation.

To prove Statement (1) of Proposition 4.10, we observe first that we only need to consider the change in kinetic energy since $\varrho^\tau = \varrho$, so the internal energy remains unchanged in the velocity projection. Let $\hat{\varrho}^\tau \mathcal{L}^d := \mathbf{r}^\tau \# (\varrho \mathcal{L}^d)$ with $\mathbf{r}^\tau := \text{id} + \tau \mathbf{u}$. Suppose also that there exists another optimal transport map $\bar{\mathbf{r}}$ pushing $\varrho \mathcal{L}^d$ forward to $\bar{\varrho} \mathcal{L}^d := \bar{\mathbf{r}} \# (\varrho \mathcal{L}^d)$. Then $\bar{\mathbf{r}}$ is unique and essentially injective. We now define

$$\Psi(\varrho_*) := \frac{\mathbf{W}(\varrho, \varrho_*)^2}{2\tau^2} \quad \text{for all } \varrho_* \in \mathcal{P}(\mathbf{R}^d),$$

which implies that $\mathcal{K}(\varrho^\tau, \mathbf{u}^\tau) = \Psi(\hat{\varrho}^\tau)$. Consider the family of measures

$$\varrho_s := \left((1-s)\bar{\mathbf{r}} + s\mathbf{r}^\tau \right) \# (\varrho \mathcal{L}^d) \quad \text{for all } s \in [0, 1].$$

By Theorem 7.3.2 in [4], we have the estimate

$$\Psi(\varrho_s) \geq (1-s)\Psi(\bar{\varrho}) + s\Psi(\hat{\varrho}^\tau) - \frac{s(1-s)}{2\tau^2} \int_{\mathbf{R}^d} |\mathbf{r}^\tau - \bar{\mathbf{r}}|^2 \varrho dx \quad (4.29)$$

for all $s \in [0, 1]$. Rearranging terms, we obtain

$$\Psi(\bar{\varrho}) \geq \Psi(\hat{\varrho}^\tau) - \lim_{s \rightarrow 0} \frac{\Psi(\varrho_s) - \Psi(\bar{\varrho})}{s} - \frac{1}{2\tau^2} \int_{\mathbf{R}^d} |\mathbf{r}^\tau - \bar{\mathbf{r}}|^2 \varrho dx.$$

Since $\bar{\mathbf{r}}$ is an optimal transport map, we can write

$$\Psi(\bar{\varrho}) = \frac{1}{2\tau^2} \int_{\mathbf{R}^d} |\bar{\mathbf{r}} - \text{id}|^2 \varrho \, dx.$$

Moreover, using Proposition 7.3.6 in [4], we obtain

$$-\lim_{s \rightarrow 0} \frac{\Psi(\varrho_s) - \Psi(\hat{\varrho})}{s} = -\frac{1}{\tau} \int_{\mathbf{R}^d} (\bar{\mathbf{r}} - \text{id}) \cdot \mathbf{u} \varrho \, ds + \frac{1}{\tau^2} \int_{\mathbf{R}^d} |\bar{\mathbf{r}} - \text{id}|^2 \varrho \, dx.$$

Recall that $\mathbf{r}^\tau = \text{id} + \tau \mathbf{u}$. Finally, we have

$$\begin{aligned} & -\frac{1}{2\tau^2} \int_{\mathbf{R}^d} |\mathbf{r}^\tau - \bar{\mathbf{r}}|^2 \varrho \, dx \\ &= -\frac{1}{2} \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho \, dx + \frac{1}{\tau} \int_{\mathbf{R}^d} (\bar{\mathbf{r}} - \text{id}) \cdot \mathbf{u} \varrho \, ds - \frac{1}{2\tau^2} \int_{\mathbf{R}^d} |\bar{\mathbf{r}} - \text{id}|^2 \varrho \, dx. \end{aligned}$$

Collecting all terms, we arrive at the inequality $\mathcal{K}(\varrho^\tau, \mathbf{u}^\tau) \leq \mathcal{K}(\varrho, \mathbf{u})$. This computation shows how the decrease in kinetic energy in the velocity projection is related to the semiconcavity of the Wasserstein distance. As explained in [4], the inequality (4.29) expresses the fact that the Wasserstein space has positive curvature.

5. MEASURE-VALUED SOLUTIONS

We now consider the initial-value problem for the isentropic Euler equations.

5.1. Time Discretization. In this section, we propose a new time discretization for the isentropic Euler equations (1.1). The approximate solution is constructed by solving a sequence of minimization problems as defined in Section 4.

Definition 5.1 (Time Discretization). Let $\delta > 0$ and $(\bar{\varrho}, \bar{\mathbf{u}}) \in \mathbb{T}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ be given. Then we define a sequence of timesteps $\tau_k^\delta > 0$ and of functions

$$(\varrho_k^\delta, \mathbf{u}_k^\delta) \in \mathbb{T}\mathcal{P}_{\text{reg}}(\mathbf{R}^d, \tau_k^\delta) \quad \text{for all } k \in \mathbf{N} \cup \{0\},$$

by executing the following program:

- (1) Let $(\bar{\varrho}_0^\delta, \bar{\mathbf{u}}_0^\delta) := (\bar{\varrho}, \bar{\mathbf{u}})$ and set $k = 0$.
- (2) Pick a number $\tau_k^\delta \in [\delta/2, \delta]$ such that the push-forward $(\text{id} + \tau_k^\delta \bar{\mathbf{u}}_k^\delta) \# (\bar{\varrho}_k^\delta \mathcal{L}^d)$ is absolutely continuous with respect to the Lebesgue measure.
- (3) Velocity Projection: Let

$$(\varrho_k^\delta, \mathbf{u}_k^\delta) := \mathbf{P}[\bar{\varrho}_k^\delta, \bar{\mathbf{u}}_k^\delta | \tau_k^\delta].$$

- (4) Energy Minimization: Let

$$(\bar{\varrho}_{k+1}^\delta, \bar{\mathbf{u}}_{k+1}^\delta) := \mathbf{M}[\varrho_k^\delta, \mathbf{u}_k^\delta | \tau_k^\delta].$$

- (5) Increase k by one and continue with (2).

We refer the reader to Definitions 4.9 and 4.6 for more details on $\mathbf{M}$ and $\mathbf{P}$. Then we define a piecewise constant curve $(\varrho^\delta, \mathbf{u}^\delta): [0, \infty) \rightarrow \mathbb{T}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ by

$$(\varrho^\delta, \mathbf{u}^\delta)(t) := (\varrho_k^\delta, \mathbf{u}_k^\delta) \quad \text{for all } t \in [t_k^\delta, t_{k+1}^\delta) \text{ and } k \in \mathbf{N} \cup \{0\},$$

where $t_k^\delta := \sum_{l=0}^{k-1} \tau_l^\delta$. Finally, let $\mathbf{m}^\delta(t) := (\varrho^\delta, \mathbf{u}^\delta)(t)$ for all $t \in [0, \infty)$.

We consider now a sequence $\delta^n \rightarrow 0$ and construct a corresponding sequence of densities/velocities as in Definition 5.1. We will use the superscript n instead of δ^n in the following, to simplify notation. We obtain a sequence of functions

$$(\varrho^n, \mathbf{m}^n): [0, \infty) \times \mathbf{R}^d \longrightarrow \mathbb{H} \quad \text{for all } n \in \mathbf{N},$$

taking values in the set $\mathbb{H} := ((0, \infty) \times \mathbf{R}^d) \cup \{(0, 0)\}$.

Our goal is to show that there exists a subsequence of $\{(\varrho^n, \mathbf{m}^n)\}$ that converges to a solution of the isentropic Euler equations (1.1), in a sense to be specified below. Let us first establish precompactness of the sequence in a suitable topology. Consistency with (1.1) will be considered in Section 5.3.

The only uniform bound on $\{(\varrho^n, \mathbf{m}^n)\}$ that is readily available, is the bound on the total energy provided by Proposition 4.10. We have that

$$\sup_n \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx < \infty, \quad (5.1)$$

with $E(r, m) := \frac{1}{2}|m|^2/r + U(r)$ for all $(r, m) \in \mathbb{H}$. The energy dissipation estimate

$$\sum_{k=1}^{\infty} \frac{1}{6} (\tau_k^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \leq \sum_{k=1}^{\infty} \left(E(\varrho_{k-1}^n, \mathbf{m}_{k-1}^n) - E(\varrho_k^n, \mathbf{m}_k^n) \right), \quad (5.2)$$

which also follows from Proposition 4.10, is too weak to enforce strong convergence of $\{\varrho^n\}$ in some Lebesgue space. We therefore try to identify a notion of convergence that relies only on the energy bound (5.1). Let us assume for the moment that the internal energy U is given by the power-law (1.8). Then (5.1) implies that

$$\sup_n \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} \left((\varrho^n)^\gamma + |\mathbf{m}^n|^p \right)(t, x) dx < \infty \quad (5.3)$$

for $\gamma > 1$ and $p := 2\gamma/(\gamma + 1)$. By Banach-Alaoglu theorem, there exists a subsequence (which we still denote by $\{(\varrho^n, \mathbf{m}^n)\}$ for simplicity) such that

$$\begin{aligned} \varrho^n &\rightharpoonup \varrho \quad \text{weak* in } \mathcal{L}^\infty([0, \infty), \mathcal{L}^\gamma(\mathbf{R}^d)), \\ \mathbf{m}^n &\rightharpoonup \mathbf{m} \quad \text{weak* in } \mathcal{L}^\infty([0, \infty), \mathcal{L}^p(\mathbf{R}^d)), \end{aligned} \quad (5.4)$$

for suitable limit density/momentum $(\varrho, \mathbf{m})$. By lower semicontinuity of the total energy (see Section 2.6 in [2]), we find that $\mathbf{m}$ is absolutely continuous with respect to the measure $\varrho \mathcal{L}^d$, so that there exists a unique velocity field $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$ with $\mathbf{m} = \varrho \mathbf{u}$. Moreover, for a.e. $0 \leq a < b < \infty$ we have the estimate

$$\int_{[a, b] \times \mathbf{R}^d} E(\varrho, \mathbf{m})(t, x) dx dt \leq \liminf_{n \rightarrow \infty} \int_{[a, b] \times \mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt. \quad (5.5)$$

For more general internal energies, the above argument can be modified suitably.

Note that by Proposition 4.10, the map

$$t \mapsto \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \quad \text{for a.e. } t \in [0, \infty) \quad (5.6)$$

is nonincreasing in time, thus a function of bounded variation. Using Helly's theorem and extracting another subsequence if necessary, we may therefore assume that the sequence of maps (5.6) converges pointwise a.e. to a bounded, nonincreasing limit function. Using Fatou's lemma in (5.5), we obtain that

$$\int_{[a, b] \times \mathbf{R}^d} E(\varrho, \mathbf{m})(t, x) dx dt \leq \int_{[a, b]} \left(\lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(x, t) dx \right) dt$$

for a.e. $0 \leq a < b < \infty$, which implies the estimate

$$\int_{\mathbf{R}^d} E(\varrho, \mathbf{m})(t, x) dx \leq \lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx$$

for a.e. $t \in [0, \infty)$. In general, this inequality can be strict. Estimate (5.1) does not rule out the possibility of energy being lost due to the following mechanisms:

(1) *Leakage to Infinity.* There exists a sequence of subsets of $\mathbf{R}^d$ that carry a certain fraction of the total energy, and that move to infinity as $n \rightarrow \infty$.

(2) *Concentrations.* It is possible that the sequence of total energy densities concentrates energy on a sequence of subsets in $\mathbf{R}^d$ whose Lebesgue measure converge to zero. In particular, it can happen that the sequence $\{E(\varrho^n, \mathbf{m}^n)\}$ converges weak* to a singular measure. On the other hand, the density/momentum ϱ and $\mathbf{m}$ in (5.4) are Lebesgue measurable, so $E(\varrho, \mathbf{m})$ does not have singular parts.

Concentration of energy also occurs when the density ϱ^n converges to zero in some set, while the velocity $\mathbf{u}^n$ grows without bound in such a way that the kinetic energy stays finite. In that case, the weak* limit of $\{E(\varrho^n, \mathbf{m}^n)\}$ in the measure sense might not be absolutely continuous with respect to $\varrho \mathcal{L}^d$.

(3) *Oscillations.* The sequence $\{(\varrho^n, \mathbf{m}^n)\}$ oscillates wildly as $n \rightarrow \infty$.

In order to show that these effects do not occur for the sequence of approximate solutions $\{(\varrho^n, \mathbf{m}^n)\}$ we need stronger information in addition to (5.1). In the next section, we will discuss sufficient conditions that prevent concentration effects. We will also introduce a Young measure that is capable of capturing oscillations. It will allow us to represent limits of nonlinear functions of density and momentum, such as those relevant for the momentum equation in (1.1). Since we are interested in distributional solutions of the isentropic Euler equations, for which test functions are compactly supported, we are not concerned about leakage to infinity.

Remark 5.2. The time discretization of Definition 5.1 can also be used for numerics: In [33], we introduced a fully discrete version of the variational scheme above for the *one-dimensional* case. We showed that the method captures very well the nonlinear features of the flow, such as rarefaction waves and shock discontinuities.

5.2. Young Measures. Our goal is to find a representation of the weak limits of sequences of the form $\{F(\varrho^n, \mathbf{m}^n)\}$ for a sufficiently large class of functions F . This class should contain all nonlinear maps relevant to the momentum equation in (1.1). We will denote by $\mathcal{W}(\mathbb{H})$ the space of all functions defined as

$$F(r, m) := \varphi(r, m) \left(1 + E(r, m) \right) + \phi \left(\frac{m}{|m|} \right) \frac{|m|^2}{2r} + cU(r) \quad (5.7)$$

for all $(r, m) \in \mathbb{H}$, with weights

$$\varphi \in \mathcal{C}_0(\mathbb{H}), \quad \phi \in \mathcal{C}(S^{d-1}), \quad \text{and} \quad c \in \mathbf{R}.$$

Here $\mathcal{C}_0(\mathbb{H})$ denotes the closure of the space of continuous functions with compact support in $\mathbb{H}$, with respect to the sup-norm. We have the following relations:

$F(r, m)$	$\varphi(r, m)$	$\phi\left(\frac{m}{ m }\right)$	c
ϱ	$r\left(1 + E(r, m)\right)^{-1}$	0	0
m	$m\left(1 + E(r, m)\right)^{-1}$	0	0

Note that $E(r, m) \rightarrow \infty$ superlinearly in r and quadratically in m , as either $r \rightarrow 0$ with $m \neq 0$, or as $r + |m| \rightarrow \infty$. This implies that in both cases $\varphi \in \mathcal{C}_0(\mathbb{H})$.

We can represent more complicated nonlinear terms as follows:

$F(r, m)$	$\varphi(r, m)$	$\phi\left(\frac{m}{ m }\right)$	c
$\frac{ m ^2}{2r}$	0	1	0
$\frac{m \otimes m}{r}$	0	$\frac{2m \otimes m}{ m ^2}$	0
$U(r)$	0	0	1
$P(r)$	0	0	$\gamma - 1$

Here we assumed the equation of state of a polytropic gas; see (1.4). For more general pressures a combination of nonzero c and φ is necessary. In the isothermal case with $U(r) = \kappa r \log r$, for which $P(r) = \kappa r$, we can even choose $c = 0$.

To simplify notation, we will often use the short-hand $\mathbf{z} \equiv (r, m) \in \mathbb{H}$ in the following. We denote by $\mathcal{P}_E(\mathbb{H})$ the space of all probability measures μ on $\mathbb{H}$ with the property that $\int_{\mathbb{H}} E(\mathbf{z}) \mu(d\mathbf{z}) < \infty$. Let $Q \subset [0, \infty) \times \mathbf{R}^d$ be a compact subset. For any n we then define a weak*-measurable map $\nu^n \in \mathcal{L}_w^\infty(Q, \mathcal{P}_E(\mathbb{H}))$ by

$$\int_Q \int_{\mathbb{H}} \zeta(t, x) F(\mathbf{z}) \nu^n(d\mathbf{z}|t, x) dx dt := \int_Q \zeta(t, x) F(\varrho^n, \mathbf{m}^n)(t, x) dx dt$$

for all $\zeta \in \mathcal{L}^1(Q)$ and $F \in \mathcal{C}_b(\mathbb{H})$. Note that the sublevels $\{\mathbf{z} \in \mathbb{H} : E(\mathbf{z}) \leq R\}$ for all $R \geq 0$ are compact and convex. Since (5.1) implies that

$$\sup_n \int_Q \int_{\mathbb{H}} E(\mathbf{z}) \nu^n(d\mathbf{z}|t, x) dx dt < \infty,$$

we can use Remark 5.1.5 in [4] to conclude that the sequence of measures $\{\nu^n\}$ is tight, thus precompact in the narrow topology, by Prokhorov's theorem. Therefore there exists a subsequence of $\{\nu^n\}$ (still denoted by $\{\nu^n\}$ for simplicity) and a limit function $\nu \in \mathcal{L}_w^\infty(Q, \mathcal{P}_E(\mathbb{H}))$ such that

$$\lim_{n \rightarrow \infty} \int_Q \int_{\mathbb{H}} \zeta(t, x) F(\mathbf{z}) \nu^n(d\mathbf{z}|t, x) dx dt = \int_Q \int_{\mathbb{H}} \zeta(t, x) F(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt \quad (5.8)$$

for all $\zeta \in \mathcal{L}^1(Q)$ and $F \in \mathcal{C}_b(\mathbb{H})$. Our goal is to extend this convergence to all F from the class $\mathcal{W}(\mathbb{H})$. To achieve this goal, we need additional assumptions. Our first assumption exclude the possibility of energy concentrating on small sets:

Assumption 5.3 (Uniform Integrability). For all $0 \leq a < b < \infty$ we have

$$\lim_{R \rightarrow \infty} \int_{[a,b] \times \mathbf{R}^d} \mathbf{1}_{\{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt = 0 \quad \text{uniformly in } n \in \mathbf{N}.$$

We claim that if Assumption 5.3 holds, then (5.8) remains true for all $\zeta \in \mathcal{L}^\infty(Q)$ and all $F \in \mathcal{W}(\mathbb{H})$ of the form (5.7) with $\phi = 0$. Note that these F are continuous in $\mathbb{H}$. We first prove that the sequence $\{F(\varrho^n, \mathbf{m}^n)\}$ is equi-integrable, and thus weakly precompact in $\mathcal{L}^1(Q)$. For any $\varepsilon > 0$, there exists $R \geq 0$ with

$$\sup_n \int_{Q \cap \{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt \leq \frac{\varepsilon}{4L}, \quad (5.9)$$

where $L := \|\varphi\|_{\mathcal{L}^\infty(\mathbb{H})} + |c|$. For all n and all $A \subset Q$ Borel we can then estimate

$$\int_A |F(\varrho^n, \mathbf{m}^n)(t, x)| dx dt \leq \|\varphi\|_{\mathcal{L}^\infty(\mathbb{H})} |A| + L \int_A E(\varrho^n, \mathbf{m}^n)(t, x) dx dt.$$

We decompose the latter integral and obtain

$$\begin{aligned} \int_A E(\varrho^n, \mathbf{u}^n)(t, x) dx dt &= \int_{A \cap \{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{u}^n)(t, x) dx dt \\ &\quad + \int_{A \cap \{E(\varrho^n, \mathbf{m}^n) \leq R\}} E(\varrho^n, \mathbf{u}^n)(t, x) dx dt \leq \frac{\varepsilon}{4L} + R|A| \end{aligned}$$

for all $n \in \mathbf{N}$, using (5.9). This implies that

$$\int_A |F(\varrho^n, \mathbf{u}^n)(t, x)| dx dt \leq \frac{\varepsilon}{4} + \left(\|\varphi\|_{\mathcal{L}^\infty(\mathbb{H})} + LR \right) |A|$$

for all $n \in \mathbf{N}$, which is less than ε for all $A \subset Q$ with sufficiently small Lebesgue measure. We conclude that the sequence $\{F(\varrho^n, \mathbf{m}^n)\}$ is equi-integrable.

We now define $E_R(\mathbf{z}) := \min\{E(\mathbf{z}), R\}$ and $U_R(\mathbf{z}) := \min\{U(r), R\}$, and set

$$F_R(\mathbf{z}) := \varphi(\mathbf{z}) \left(1 + E_R(\mathbf{z}) \right) + cU_R(\mathbf{z})$$

for all $\mathbf{z} = (r, m) \in \mathbb{H}$ and $R \geq 0$. Then we have $F_R \in \mathcal{C}_b(\mathbb{H})$, and so (5.8) holds for F_R and all $\zeta \in \mathcal{L}^\infty(Q)$. By Assumption 5.3, there exists $R \geq 0$ such that

$$\sup_n \int_{Q \cap \{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt \leq \frac{\varepsilon}{4L'},$$

where $L' := L\|\zeta\|_{\mathcal{L}^\infty(Q)}$, which implies that

$$\int_Q \int_{\mathbb{H} \cap \{E(\mathbf{z}) > R\}} E(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt \leq \frac{\varepsilon}{4L'}.$$

For all $\zeta \in \mathcal{L}^\infty(Q)$, we can then estimate

$$\begin{aligned} & \left| \int_Q \int_{\mathbb{H}} \zeta(t, x) F(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt - \int_Q \zeta(t, x) F(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right| \\ & \leq \left| \int_Q \int_{\mathbb{H}} \zeta(t, x) F_R(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt - \int_Q \zeta(t, x) F_R(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right| \\ & \quad + L \|\zeta\|_{\mathcal{L}^\infty(Q)} \left(\int_Q \int_{\mathbb{H} \cap \{E(\mathbf{z}) > R\}} E(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt \right) \\ & \quad + L \|\zeta\|_{\mathcal{L}^\infty(Q)} \left(\sup_n \int_{Q \cap \{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right), \end{aligned}$$

which is less than ε if n is large enough. We get that the map $\langle F(r, m) \rangle$ defined by

$$\langle F(r, m) \rangle(t, x) := \int_{\mathbb{H}} F(\mathbf{z}) \nu(d\mathbf{z}|t, x) \quad \text{for a.e. } (t, x) \in Q \quad (5.10)$$

is in $\mathcal{L}^1(Q)$, and that

$$F(\varrho^n, \mathbf{m}^n) \rightharpoonup \langle F(r, m) \rangle \quad \text{weakly in } \mathcal{L}^1(Q) \quad (5.11)$$

for all $F \in \mathcal{W}(\mathbb{H})$ of the form (5.7) with $\phi = 0$.

For given $\phi \in \mathcal{C}(S^{d-1})$, we now consider the map

$$F_\phi(r, m) := \phi\left(\frac{m}{|m|}\right) K(r, m) \quad \text{for all } (r, m) \in \mathbb{H},$$

with kinetic energy defined as $K(0, 0) := 0$ and $K(r, m) := \frac{1}{2}|m|^2/r$ otherwise. Note that $K(r, 0) = 0$ for all $r \geq 0$, so the function F_ϕ is continuous in $(0, \infty) \times \mathbf{R}^d$. It is, however, not continuous at the vacuum $(r, m) = (0, 0)$ because K is only lower semicontinuous there. In order to have (5.11) for all $F \in \mathcal{W}(\mathbb{H})$, we will assume that no kinetic energy is concentrated in vacuum, in the following sense:

Assumption 5.4 (Absolute Continuity). For all $0 \leq a < b < \infty$ we have

$$\lim_{R \rightarrow \infty} \int_{[a, b] \times \mathbf{R}^d} \mathbf{1}_{\{\varrho^n < 1/R\}} K(\varrho^n, \mathbf{m}^n)(t, x) dx dt = 0 \quad \text{uniformly in } n \in \mathbf{N}.$$

Note first that since the kinetic energy can be bounded above by the total energy, we can repeat the argument above to show that the sequence $\{F_\phi(\varrho^n, \mathbf{m}^n)\}$ converges weakly in $\mathcal{L}^1(Q)$ (after extracting another subsequence if necessary). We only need to show that the limit can be represented by the Young measure ν as in (5.11) and (5.10). By Assumption 5.4, there exists $R \geq 0$ such that

$$\sup_n \int_{Q \cap \{\varrho^n < 1/R\}} K(\varrho^n, \mathbf{m}^n)(t, x) dx dt \leq \frac{\varepsilon}{8L'},$$

where $L := \|\phi\|_{\mathcal{C}(S^{d-1})}$ and $L' := L\|\zeta\|_{\mathcal{L}^\infty(Q)}$, which implies that

$$\int_Q \int_{\mathbb{H} \cap \{r < 1/R\}} K(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt \leq \frac{\varepsilon}{8L'}.$$

Recall that $\mathbf{z} = (r, m) \in \mathbb{H}$. By Assumption 5.3, we can choose $R \geq 0$ such that

$$\begin{aligned} \sup_n \int_{Q \cap \{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt &\leq \frac{\varepsilon}{8L}, \\ \int_Q \int_{\mathbb{H} \cap \{E(\mathbf{z}) > R\}} E(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt &\leq \frac{\varepsilon}{8L}. \end{aligned}$$

We now choose $\psi \in \mathcal{C}(\mathbb{H}, [0, 1])$ with compact support and

$$\psi(\mathbf{z}) = 1 \quad \text{for all } \mathbf{z} \in \mathbb{H} \text{ with } E(\mathbf{z}) \leq R \text{ and } r \geq 1/R.$$

Let $F_\phi = F_0 + F_1$ with $F_0 := \psi F_\phi \in \mathcal{C}_b(\mathbb{H})$. For all $\zeta \in \mathcal{L}^\infty(Q)$, we then have

$$\begin{aligned} &\left| \int_Q \int_{\mathbb{H}} \zeta(t, x) F_\phi(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt - \int_Q \zeta(t, x) F_\phi(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right| \\ &\leq \left| \int_Q \int_{\mathbb{H}} \zeta(t, x) F_0(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt - \int_Q \zeta(t, x) F_0(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right| \\ &\quad + L \|\zeta\|_{\mathcal{L}^\infty(Q)} \left(\int_Q \int_{\mathbb{H} \cap \{E(\mathbf{z}) > R\}} E(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt \right) \\ &\quad + L \|\zeta\|_{\mathcal{L}^\infty(Q)} \left(\int_Q \int_{\mathbb{H} \cap \{r < 1/R\}} K(\mathbf{z}) \nu(d\mathbf{z}|t, x) dx dt \right) \\ &\quad + L \|\zeta\|_{\mathcal{L}^\infty(Q)} \left(\sup_n \int_{Q \cap \{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right) \\ &\quad + L \|\zeta\|_{\mathcal{L}^\infty(Q)} \left(\sup_n \int_{Q \cap \{\varrho^n < 1/R\}} K(\varrho^n, \mathbf{m}^n)(t, x) dx dt \right), \end{aligned}$$

which is less than ε if n is large enough, because of (5.8). This proves (5.11) for all functions $F \in \mathcal{W}(\mathbb{H})$, with $\langle F(r, m) \rangle \in \mathcal{L}^1(Q)$ defined in (5.10).

Consider now an increasing sequence $\{Q_l\}$ of compact subset $Q_l \subset [0, \infty) \times \mathbf{R}^d$ that converges to the whole space $[0, \infty) \times \mathbf{R}^d$ as $l \rightarrow \infty$. Repeating the argument above successively for all l , we obtain the following result:

Proposition 5.5 (Young Measure). *Suppose that the sequence $\{(\varrho^n, \mathbf{m}^n)\}$ of functions constructed by the time discretization of Definition 5.1 satisfies*

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{[a, b] \times \mathbf{R}^d} \mathbf{1}_{\{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt &= 0, \\ \lim_{R \rightarrow \infty} \int_{[a, b] \times \mathbf{R}^d} \mathbf{1}_{\{\varrho^n < 1/R\}} K(\varrho^n, \mathbf{m}^n)(t, x) dx dt &= 0 \end{aligned}$$

uniformly in n , for all $0 \leq a < b < \infty$. Then there exist a subsequence (which we still denote by $\{(\varrho^n, \mathbf{m}^n)\}$) and a map $\nu \in \mathcal{L}_w^\infty([0, \infty) \times \mathbf{R}^d, \mathcal{P}_E(\mathbb{H}))$ with

$$F(\varrho^n, \mathbf{m}^n) \rightharpoonup \langle F(r, m) \rangle \quad \text{weakly in } \mathcal{L}_{\text{loc}}^1([0, \infty) \times \mathbf{R}^d),$$

where $F \in \mathcal{W}(\mathbb{H})$ and $\langle F(r, m) \rangle \in \mathcal{L}_{\text{loc}}^1([0, \infty) \times \mathbf{R}^d)$ is defined by

$$\langle F(r, m) \rangle(t, x) := \int_{\mathbb{H}} F(\mathbf{z}) \nu(d\mathbf{z}|t, x) \quad \text{for a.e. } (t, x) \in [0, \infty) \times \mathbf{R}^d.$$

Proposition 5.5 gives sufficient conditions for the absence of concentration effects. Moreover, the Young measure constructed above allows us to represent the limits of nonlinear compositions $\{F(\varrho^n, \mathbf{m}^n)\}$ for all test functions $F \in \mathcal{W}(\mathbb{H})$. Note that after extracting another subsequence if necessary, we may assume that $\langle r \rangle = \varrho$ and $\langle m \rangle = \mathbf{m}$, with density ϱ and momentum $\mathbf{m}$ given by (5.4).

Proposition 5.6 (Strong Convergence). *Suppose that the sequence $\{(\varrho^n, \mathbf{m}^n)\}$ satisfies the assumptions of Proposition 5.5 and generates a Young measure ν as explained there. Then we have $(\varrho^n, \mathbf{m}^n) \longrightarrow (\varrho, \mathbf{m})$ strongly in $\mathcal{L}_{\text{loc}}^1([0, \infty) \times \mathbf{R}^d)$ for limit functions $(\varrho, \mathbf{m})$ defined in (5.4), if and only if*

$$\int_{[0, \infty) \times \mathbf{R}^d} \zeta(t, x) \langle F(r, m) \rangle(t, x) dx dt = \int_{[0, \infty) \times \mathbf{R}^d} \zeta(t, x) F(\varrho, \mathbf{m})(t, x) dx dt \quad (5.12)$$

for all $\zeta \in \mathcal{D}([0, \infty) \times \mathbf{R}^d)$ and $F \in \mathcal{W}(\mathbb{H})$, with $\langle F(r, m) \rangle$ defined in (5.10).

Proof. It is known that property (5.12) of the Young measure is equivalent to

$$(\varrho^n, \mathbf{m}^n) \longrightarrow (\varrho, \mathbf{m}) \quad \text{locally in measure;}$$

see Lemma 5.4.1 in [4]. Together with (5.4) this implies strong convergence. $\square$

5.3. Conservation Laws. In this section, we study the consistency of the sequence of functions $\{(\varrho^n, \mathbf{m}^n)\}$ constructed in Definition 5.1, with the system of isentropic Euler equations (1.1). Recall that we have the uniform energy bound

$$\sup_n \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx < \infty, \quad (5.13)$$

and the energy dissipation bound

$$\sum_{k=1}^{\infty} \frac{1}{6} (\tau_k^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \leq \sum_{k=1}^{\infty} \left(E(\varrho_{k-1}^n, \mathbf{u}_{k-1}^n) - E(\varrho_k^n, \mathbf{u}_k^n) \right). \quad (5.14)$$

The right-hand side of (5.14) is uniformly bounded by the initial total energy. We will also assume in the following that assumptions of Proposition 5.5 are satisfied. That is, for all $0 \leq a < b < \infty$ we assume that

$$\lim_{R \rightarrow \infty} \int_{[a, b] \times \mathbf{R}^d} \mathbf{1}_{\{E(\varrho^n, \mathbf{m}^n) > R\}} E(\varrho^n, \mathbf{m}^n)(t, x) dx dt = 0, \quad (5.15)$$

$$\lim_{R \rightarrow \infty} \int_{[a, b] \times \mathbf{R}^d} \mathbf{1}_{\{\varrho^n < 1/R\}} K(\varrho^n, \mathbf{m}^n)(t, x) dx dt = 0, \quad (5.16)$$

uniformly in n . After extracting a subsequence if necessary, the sequence $\{(\varrho^n, \mathbf{m}^n)\}$ generates a Young measure ν as explained in Proposition 5.5. Note that assumptions 5.15 and (5.16) are quite natural, but we do not yet have a proof for them. This requires further investigation. Our goal is to show that

$$\begin{aligned} \partial_t \varrho + \nabla \cdot \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \nabla \cdot \langle r^{-1} m \otimes m \rangle + \nabla \langle P(r) \rangle &= 0 \end{aligned} \quad (5.17)$$

in the sense of distributions, with density $\varrho = \langle r \rangle$ and momentum $\mathbf{m} = \langle m \rangle$ defined by (5.4). If the Young measure is concentrated in the sense of Proposition 5.6, then $(\varrho, \mathbf{m})$ is a weak solution of (1.1). Let us first consider the conservation of mass.

Proposition 5.7 (Continuity Equation). *With the notation above, we have*

$$\partial_t \varrho + \nabla \cdot \mathbf{m} = 0 \quad \text{and} \quad \varrho(0, \cdot) = \bar{\varrho} \quad (5.18)$$

in the sense of distributions, and the curve $t \mapsto \varrho(t) \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ with $t \in [0, \infty)$ is absolutely continuous with respect to the Wasserstein distance.

Moreover, the following energy inequality holds for a.e. $0 \leq t_1 < t_2 < \infty$:

$$\int_{\mathbf{R}^d} \langle E(r, m) \rangle(t_2, x) dx \leq \int_{\mathbf{R}^d} \langle E(r, m) \rangle(t_1, x) dx. \quad (5.19)$$

Proof of Proposition 5.7. Note first that for the approximate solution $(\varrho^n, \mathbf{m}^n)$

$$\text{the map } t \mapsto \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \text{ is nonincreasing in time,}$$

as a consequence of Proposition 4.10. It is bounded above by the initial total energy, and below by zero. We can therefore apply Helly's theorem to obtain that (after extracting another subsequence if necessary) the limit curve

$$t \mapsto \int_{\mathbf{R}^d} \langle E(r, m) \rangle(t, x) dx \quad \text{with } t \in [0, \infty)$$

is again nonincreasing, with the same bounds. Then inequality (5.19) follows.

The continuity of the curve $t \mapsto \varrho(t, \cdot)$ with respect to the Wasserstein distance follows from Proposition 8.3.1 in [4]: Note first that for a.e. $t \in [0, \infty)$ there exists a velocity field $\mathbf{u}(t, \cdot) \in \mathcal{L}^2(\mathbf{R}^d, \varrho(t, \cdot))$ with $\mathbf{m} = \varrho \mathbf{u}$ a.e. This follows from the lower semicontinuity of the kinetic energy functional; see Section 2.6 in [2]. Assuming that (5.18) holds, we then obtain that $t \mapsto \varrho(t, \cdot)$ is continuous with respect to the narrow topology: For a.e. $0 \leq t_1 < t_2 < \infty$ and any $\varphi \in \mathcal{C}_b(\mathbf{R}^d)$ we have

$$\begin{aligned} & \int_{\mathbf{R}^d} \varphi(x) (\varrho(t_2, x) - \varrho(t_1, x)) dx \\ &= \int_{\mathbf{R}^d} (\varphi(x) - \varphi_\delta(x)) (\varrho(t_2, x) - \varrho(t_1, x)) dx + \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \nabla \varphi_\delta(x) \cdot \mathbf{m}(t, x) dx dt, \end{aligned}$$

where $\varphi_\delta \in \mathcal{D}(\mathbf{R}^d)$. We can therefore estimate

$$\begin{aligned} \left| \int_{\mathbf{R}^d} \varphi(x) (\varrho(t_2, x) - \varrho(t_1, x)) dx \right| &\leq 2 \|\varphi - \varphi_\delta\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} \varrho(t, x) dx \\ &\quad + |t_2 - t_1| \|\nabla \varphi_\delta\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} |\mathbf{m}|(t, x) dx, \end{aligned}$$

which can be made arbitrarily small by choosing φ_δ close to φ .

It remains to establish the continuity equation (5.18). Let $\varphi \in \mathcal{D}([0, \infty) \times \mathbf{R}^d)$ be any given test function. In view of Definition 5.1, we can write

$$\begin{aligned} & \int_{[0, \infty) \times \mathbf{R}^d} \partial_t \varphi(t, x) \varrho^n(t, x) dx dt + \int_{\mathbf{R}^d} \varphi(0, x) \bar{\varrho}(x) dx \\ &= \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\int_{t_{k-1}^n}^{t_k^n} \partial_t \varphi(t, x) dt \right) \varrho_{k-1}^n(x) dx + \int_{\mathbf{R}^d} \varphi(0, x) \bar{\varrho}(x) dx \\ &= \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \varphi(t_k^n, x) (\varrho_{k-1}^n(x) - \varrho_k^n(x)) dx + \int_{\mathbf{R}^d} \varphi(0, x) (\bar{\varrho}(x) - \varrho_0^n(x)) dx. \end{aligned}$$

Since $\varrho_0^n = \bar{\varrho}$ by construction, the last integral is equal to zero.

Recall that the density update involves a free transport in the direction of the velocity field, followed by a minimization step to decrease the internal energy. The velocity projection does not affect the densities, so $\varrho_k^n = \varrho_k^n$ for all k . We have

$$\left(\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n\right) \# (\varrho_{k-1}^n \mathcal{L}^d) = \left(\text{id} + \frac{2}{3}(\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n)\right) \# (\varrho_k^n \mathcal{L}^d); \quad (5.20)$$

see Proposition 4.5. Writing $\varphi_k^n := \varphi(t_k^n, \cdot)$, we decompose

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi_k^n (\varrho_{k-1}^n - \varrho_k^n) dx &= \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \varrho_{k-1}^n dx \\ &\quad + \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\text{id} + \frac{2}{3}(\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n dx \end{aligned} \quad (5.21)$$

and estimate the two terms on the right-hand side separately.

Step 1. To estimate the first term, we define

$$\psi^n(t, x) := \int_0^1 \nabla \varphi(t, x + \theta \tau_{k-1}^n \mathbf{u}_{k-1}^n(x)) d\theta \quad \text{for a.e. } x \in \mathbf{R}^d,$$

for all $t \in (t_{k-1}^n, t_k^n]$ and $k \in \mathbf{N}$. Then we can write

$$\int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \varrho_{k-1}^n dx = -\tau_{k-1}^n \int_{\mathbf{R}^d} \left(\psi_k^n \cdot \mathbf{u}_{k-1}^n \right) \varrho_{k-1}^n dx,$$

where $\psi_k^n := \psi^n(t_k^n, \cdot)$. We use the mean value theorem to estimate

$$\left| \tau_{k-1}^n \psi_k^n(x) - \int_{t_{k-1}^n}^{t_k^n} \psi^n(t, x) dt \right| \leq (\tau_{k-1}^n)^2 \|\partial_t \nabla \varphi(\cdot, x)\|_{\mathcal{L}^\infty([t_{k-1}^n, t_k^n])}$$

for a.e. $x \in \mathbf{R}^d$, which implies that

$$\begin{aligned} &\left| \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \varrho_{k-1}^n dx + \int_{t_{k-1}^n}^{t_k^n} \int_{\mathbf{R}^d} \left(\psi^n \cdot \mathbf{u}^n \right) \varrho^n dx dt \right| \\ &\leq \sqrt{2}(\tau_{k-1}^n)^2 \|\partial_t \nabla \varphi\|_{\mathcal{L}^\infty([t_{k-1}^n, t_k^n] \times \mathbf{R}^d)} \left(\int_{\mathbf{R}^d} \frac{1}{2} |\mathbf{u}_{k-1}^n|^2 \varrho_{k-1}^n dx \right)^{1/2}. \end{aligned}$$

We now sum in k and get

$$\begin{aligned} &\left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \varrho_{k-1}^n dx + \int_{[0, \infty) \times \mathbf{R}^d} \left(\psi^n \cdot \mathbf{u}^n \right) \varrho^n dx dt \right| \\ &\leq \delta^n \sqrt{2} T \|\partial_t \nabla \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \left(\text{ess sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \right)^{1/2}, \end{aligned}$$

where $T > 0$ is chosen large enough such that $\text{spt } \varphi \subset [0, T) \times \mathbf{R}^d$. The right-hand side converges to zero as $n \rightarrow \infty$ because of the uniform energy bound (5.13).

Using another Taylor-expansion, we obtain the identity

$$\begin{aligned} &\nabla \varphi(t, x) - \psi^n(t, x) \\ &= \tau_{k-1}^n \left(\int_0^1 \left(\int_0^1 D^2 \varphi(t, x + \theta \tau_{k-1}^n \mathbf{u}_{k-1}^n(x)) d\theta' \right) \theta d\theta \right) \mathbf{u}_{k-1}^n(x) \end{aligned}$$

for a.e. $x \in \mathbf{R}^d$ and all $t \in (t_{k-1}^n, t_k^n], k \in \mathbf{N}$. This implies the estimate

$$\begin{aligned} & \left| \int_{t_{k-1}^n}^{t_k^n} \int_{\mathbf{R}^d} (\psi^n \cdot \mathbf{u}^n) \varrho^n dx dt - \int_{t_{k-1}^n}^{t_k^n} \int_{\mathbf{R}^d} (\nabla \varphi \cdot \mathbf{u}^n) \varrho^n dx dt \right| \\ & \leq (\tau_{k-1}^n)^2 \|D^2 \varphi\|_{\mathcal{L}^\infty([t_{k-1}^n, t_k^n] \times \mathbf{R}^d)} \left(\int_{\mathbf{R}^d} \frac{1}{2} |\mathbf{u}_{k-1}^n|^2 \varrho_{k-1}^n dx \right). \end{aligned}$$

We now sum in k and get

$$\begin{aligned} & \left| \int_{[0, \infty) \times \mathbf{R}^d} (\psi^n \cdot \mathbf{u}^n) \varrho^n dx dt - \int_{[0, \infty) \times \mathbf{R}^d} (\nabla \varphi \cdot \mathbf{u}^n) \varrho^n dx dt \right| \\ & \leq \delta^n T \|D^2 \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \left(\operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \right), \end{aligned}$$

which converges to zero as $n \rightarrow \infty$. We conclude that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\operatorname{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \varrho_{k-1}^n dx \\ & = - \int_{[0, \infty) \times \mathbf{R}^d} \nabla \varphi(t, x) \cdot \mathbf{m}(t, x) dx dt \quad \text{for all } \varphi \in \mathcal{D}([0, \infty) \times \mathbf{R}^d). \end{aligned}$$

Step 2. To estimate the second term on the right-hand side of (5.21), we write

$$\begin{aligned} & \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\operatorname{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n dx \\ & = \frac{2}{3} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} \left(\int_0^1 \nabla \varphi_k^n \circ \left(\operatorname{id} + \theta \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) d\theta \right) \cdot \nabla U'(\varrho_k^n) \varrho_k^n dx, \end{aligned}$$

which implies the estimate

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\operatorname{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n dx \right| \\ & \leq \sqrt{\frac{8}{3}} \tau_{k-1}^n \|\nabla \varphi(t_k^n, \cdot)\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \left(\frac{1}{6} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \right)^{1/2}. \end{aligned}$$

We now sum in k and obtain

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\operatorname{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n dx \right| \\ & \leq \sqrt{\delta^n} \sqrt{\frac{8T}{3}} \|\nabla \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \left(\sum_{k=1}^{\infty} \frac{1}{6} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \right)^{1/2}, \end{aligned} \tag{5.22}$$

using the Cauchy-Schwarz inequality. The sum on the right-hand side can be controlled using the energy dissipation estimate in Proposition 4.10; see (5.2). The constant T is chosen as above. Then (5.22) converges to zero as $n \rightarrow \infty$.

Collecting all estimates we find that

$$\begin{aligned} & \int_{[0,\infty) \times \mathbf{R}^d} \partial_t \varphi(t, x) \varrho(t, x) dx dt + \int_{\mathbf{R}^d} \varphi(0, x) \bar{\varrho}(x) dx \\ &= - \int_{[0,\infty) \times \mathbf{R}^d} \nabla \varphi(t, x) \cdot \mathbf{m}(t, x) dx dt \quad \text{for all } \varphi \in \mathcal{D}([0, \infty) \times \mathbf{R}^d), \end{aligned}$$

and thus $(\varrho, \mathbf{m})$ satisfies the continuity equation (5.18) in distributional sense. $\square$

Establishing the momentum equation in (5.17) is more delicate, and so far we have not been able to give a rigorous proof of it. In the following, we will give some heuristic. We intend to address the issue in a future publication.

Conjecture 5.1 (Momentum Equation). *With the notation above, we have*

$$\partial_t \mathbf{m} + \nabla \cdot \langle r^{-1} m \otimes m \rangle + \nabla \langle P(r) \rangle = 0 \quad \text{and} \quad \mathbf{m}(0, \cdot) = \bar{\mathbf{m}} \quad (5.23)$$

in the distribution sense. Moreover, the curve $t \mapsto \mathbf{m}(t) \in \mathcal{M}(\mathbf{R}^d)$ with $t \in [0, \infty)$ is absolutely continuous with respect to the narrow topology of measures.

Proof. Let us first prove the narrow continuity of the curve $t \mapsto \mathbf{m}(t, \cdot)$, assuming that (5.23) holds. For a.e. $0 \leq t_1 < t_2 < \infty$ and any $\varphi \in \mathcal{C}_b(\mathbf{R}^d)$ we write

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi(x) (\mathbf{m}(t_2, x) - \mathbf{m}(t_1, x)) dx &= \int_{\mathbf{R}^d} (\varphi(x) - \varphi_\delta(x)) (\mathbf{m}(t_2, x) - \mathbf{m}(t_1, x)) dx \\ &\quad + \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \nabla \varphi_\delta(x) \cdot \langle r^{-1} m \otimes m \rangle(t, x) dx dt \\ &\quad + \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \nabla \varphi_\delta(x) \langle P(r) \rangle(t, x) dx dt, \end{aligned}$$

where $\varphi_\delta \in \mathcal{D}(\mathbf{R}^d)$. We can therefore estimate

$$\begin{aligned} \left| \int_{\mathbf{R}^d} \varphi(x) (\mathbf{m}(t_2, x) - \mathbf{m}(t_1, x)) dx \right| &\leq 2 \|\varphi - \varphi_\delta\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} |\mathbf{m}|(t, x) dx \\ &\quad + C |t_2 - t_1| \|\nabla \varphi_\delta\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \left(|\operatorname{spt} \varphi_\delta| + \operatorname{ess\,sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} \langle E(r, m) \rangle(t, x) dx \right), \end{aligned}$$

which can be made arbitrarily small by choosing φ_δ close to φ . Here we used the estimate $P(r) \leq C(1 + U(r))$ for all $r \in [0, \infty)$, with $C > 0$ some constant.

It remains to prove (5.23). Given $\varphi \in \mathcal{D}([0, \infty) \times \mathbf{R}^d)$, we can write

$$\begin{aligned} & \int_{[0,\infty) \times \mathbf{R}^d} \partial_t \varphi(t, x) \mathbf{m}^n(t, x) dx dt + \int_{\mathbf{R}^d} \varphi(0, x) \bar{\mathbf{m}}(x) dx \\ &= \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\int_{t_{k-1}^n}^{t_k^n} \partial_t \varphi(t, x) dt \right) \mathbf{m}_{k-1}^n(x) dx + \int_{\mathbf{R}^d} \varphi(0, x) \bar{\mathbf{m}}(x) dx \\ &= \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \varphi(t_k^n, x) (\mathbf{m}_{k-1}^n(x) - \mathbf{m}_k^n(x)) dx + \int_{\mathbf{R}^d} \varphi(0, x) (\bar{\mathbf{m}}(x) - \mathbf{m}_0^n(x)) dx. \end{aligned}$$

Let us first consider the second integral. We have $\bar{\mathbf{m}} = \bar{\varrho} \bar{\mathbf{u}}$ with $(\bar{\varrho}, \bar{\mathbf{u}}) \in \mathbb{T} \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$. Moreover, we can write $\mathbf{m}_0^n = \varrho_0^n \mathbf{u}_0^n$ with $\varrho_0^n = \bar{\varrho}$ and $\mathbf{u}_0^n = \nabla \phi_0^n$, where

$$\bar{\mathbf{u}} = \nabla \phi_0^n \circ (\operatorname{id} + \tau_0^n \mathbf{w}_0^n) + \mathbf{w}_0^n$$

is the polar factorization of the initial velocity $\bar{\mathbf{u}}$. Proposition 2.3 shows that

$$\nabla \phi_0^n \longrightarrow \bar{\mathbf{u}} \quad \text{weakly in } \mathcal{L}^2(\mathbf{R}^d, \bar{\varrho}),$$

which implies that $\mathbf{m}$ attains the initial data $\bar{\mathbf{m}}$ in distributional sense.

Proposition 4.5 describes how the density and velocity are updated in the energy minimization. For any $k \in \mathbf{N}$ let $\tilde{\mathbf{u}}_k^n \in \mathcal{L}^2(\mathbf{R}^d, \varrho_k^n)$ be defined a.e. by

$$\tilde{\mathbf{u}}_k^n := \mathbf{u}_{k-1}^n \circ \left(\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n \right)^{-1} \circ \left(\text{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right). \quad (5.24)$$

Writing $\varphi_k^n := \varphi(t_k^n, \cdot)$, we decompose

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi_k^n (\mathbf{m}_{k-1}^n - \mathbf{m}_k^n) dx &= \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \mathbf{m}_{k-1}^n dx \\ &\quad + \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\text{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \tilde{\mathbf{u}}_k^n \varrho_k^n dx \\ &\quad + \int_{\mathbf{R}^d} \varphi_k^n (\tilde{\mathbf{u}}_k^n - \bar{\mathbf{u}}_k^n) \varrho_k^n dx \\ &\quad + \int_{\mathbf{R}^d} \varphi_k^n (\bar{\mathbf{u}}_k^n - \mathbf{u}_k^n) \varrho_k^n dx; \end{aligned} \quad (5.25)$$

see (5.20) and (5.24). We discuss each term in (5.25) individually.

Step 1. To estimate the first term, we define

$$\psi^n(t, x) := \int_0^1 \nabla \varphi(t, x + \theta \tau_{k-1}^n \mathbf{u}_{k-1}^n(x)) d\theta \quad \text{for a.e. } x \in \mathbf{R}^d,$$

for all $t \in (t_{k-1}^n, t_k^n]$ and $k \in \mathbf{N}$. Then we can write

$$\int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \mathbf{m}_{k-1}^n dx = -\tau_{k-1}^n \int_{\mathbf{R}^d} (\psi_k^n \cdot \mathbf{u}_{k-1}^n) \mathbf{m}_{k-1}^n dx,$$

where $\psi_k^n := \psi^n(t_k^n, \cdot)$. We now use the mean value theorem to estimate

$$\left| \tau_{k-1}^n \psi_k^n(x) - \int_{t_{k-1}^n}^{t_k^n} \psi^n(t, x) dt \right| \leq (\tau_{k-1}^n)^2 \|\partial_t \nabla \varphi(\cdot, x)\|_{\mathcal{L}^\infty([t_{k-1}^n, t_k^n])} \quad (5.26)$$

for a.e. $x \in \mathbf{R}^d$, which implies that

$$\begin{aligned} &\left| \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \mathbf{m}_{k-1}^n dx + \int_{t_{k-1}^n}^{t_k^n} \int_{\mathbf{R}^d} (\psi^n \cdot \mathbf{u}^n) \mathbf{m}^n dx dt \right| \\ &\leq 2(\tau_{k-1}^n)^2 \|\partial_t \nabla \varphi\|_{\mathcal{L}^\infty([t_{k-1}^n, t_k^n] \times \mathbf{R}^d)} \left(\int_{\mathbf{R}^d} \frac{1}{2} |\mathbf{u}_{k-1}^n|^2 \varrho_{k-1}^n dx \right). \end{aligned}$$

We now sum in k and get

$$\begin{aligned} &\left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \right) \mathbf{m}_{k-1}^n dx + \int_{[0, \infty) \times \mathbf{R}^d} (\psi^n \cdot \mathbf{u}^n) \mathbf{m}^n dx dt \right| \\ &\leq \delta^n 2T \|\partial_t \nabla \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \left(\text{ess sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \right), \end{aligned}$$

where $T > 0$ is chosen large enough such that $\text{spt } \varphi \subset [0, T) \times \mathbf{R}^d$. The right-hand side converges to zero as $n \rightarrow \infty$ because of the uniform energy bound (5.13).

We now claim that

$$\lim_{n \rightarrow \infty} \int_{[0, \infty) \times \mathbf{R}^d} (\psi^n \cdot \mathbf{u}^n) \mathbf{m}^n dx dt = \int_{[0, \infty) \times \mathbf{R}^d} \nabla \varphi \cdot \langle r^{-1} m \otimes m \rangle dx dt.$$

Indeed, using the mean value theorem, we obtain

$$\left| \psi^n(t, x) - \nabla \varphi(t, x) \right| \leq \frac{1}{2} \tau_{k-1}^n |\mathbf{u}_{k-1}^n(x)| \|D^2 \varphi(t, \cdot)\|_{\mathcal{L}^\infty(\mathbf{R}^d)}$$

for a.e. $(t, x) \in (t_{k-1}^n, t_k^n] \times \mathbf{R}^d$ and all $k \in \mathbf{N}$. This implies the estimate

$$\begin{aligned} & \left| \int_{[0, \infty) \times \mathbf{R}^d} (\psi^n \cdot \mathbf{u}^n) \mathbf{m}^n dx dt - \int_{[0, \infty) \times \mathbf{R}^d} (\nabla \varphi \cdot \mathbf{u}^n) \mathbf{m}^n dx dt \right| \\ & \leq \delta^n R \|D^2 \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \int_{[0, T] \times \mathbf{R}^d} \mathbf{1}_{\{|\mathbf{u}^n| \leq R\}} K(\varrho^n, \mathbf{m}^n) dx dt \\ & \quad + 4 \|\nabla \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \int_{[0, T] \times \mathbf{R}^d} \mathbf{1}_{\{|\mathbf{u}^n| > R\}} K(\varrho^n, \mathbf{m}^n) dx dt, \end{aligned} \quad (5.27)$$

where $T > 0$ as above. The first integral on the right-hand side is bounded uniformly in n by the energy inequality (5.13). For the second integral, we have

$$\begin{aligned} \int_{[0, T] \times \mathbf{R}^d} \mathbf{1}_{\{|\mathbf{u}^n| > R\}} K(\varrho^n, \mathbf{m}^n) dx dt & \leq \int_{[0, T] \times \mathbf{R}^d} \mathbf{1}_{\{E(\varrho^n, \mathbf{m}^n) > \frac{1}{2} R\}} E(\varrho^n, \mathbf{m}^n) dx dt \\ & \quad + \int_{[0, T] \times \mathbf{R}^d} \mathbf{1}_{\{\varrho^n < 1/R\}} K(\varrho^n, \mathbf{m}^n) dx dt. \end{aligned}$$

Let $\varepsilon > 0$ be given. By choosing $R \geq 0$ large enough, we can make the right-hand side less than $\varepsilon/2$ uniformly in n , because of assumptions (5.15) and (5.16). Hence (5.27) converges to zero as $\delta^n \rightarrow 0$. On the other hand, we have

$$(\varrho^n)^{-1} \mathbf{m}^n \otimes \mathbf{m}^n \rightharpoonup \langle r^{-1} m \otimes m \rangle \quad \text{weakly in } \mathcal{L}_{\text{loc}}^1([0, \infty) \times \mathbf{R}^d),$$

by Proposition 5.5, and so our claim follows.

Step 2. To estimate the second term on the right-hand side of (5.25), we write

$$\begin{aligned} & \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\text{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \tilde{\mathbf{u}}_k^n \varrho_k^n dx \\ & = \frac{2}{3} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} \left(\int_0^1 \nabla \varphi_k^n \circ \left(\text{id} + \theta \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) d\theta \right) \cdot \nabla U'(\varrho_k^n) \tilde{\mathbf{u}}_k^n \varrho_k^n dx, \end{aligned}$$

which implies the estimate

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\text{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \tilde{\mathbf{u}}_k^n \varrho_k^n dx \right| \\ & \leq \sqrt{\frac{16}{3}} \tau_{k-1}^n \|\nabla \varphi(t_k^n, \cdot)\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \left(\frac{1}{6} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \right)^{1/2} \\ & \quad \times \left(\int_{\mathbf{R}^d} \frac{1}{2} |\tilde{\mathbf{u}}_k^n|^2 \varrho_k^n dx \right)^{1/2}. \end{aligned}$$

Now notice that because of (5.20) and (5.24) we have

$$\int_{\mathbf{R}^d} \frac{1}{2} |\tilde{\mathbf{u}}_k^n|^2 \varrho_k^n dx = \int_{\mathbf{R}^d} \frac{1}{2} |\mathbf{u}_{k-1}^n|^2 \varrho_{k-1}^n dx,$$

which is uniformly bounded by (5.13). We sum in k and obtain

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\text{id} + \frac{2}{3} (\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \tilde{\mathbf{u}}_k^n \varrho_k^n dx \right| \\ & \leq \sqrt{\delta^n} \sqrt{\frac{16T}{3}} \|\nabla \varphi\|_{\mathcal{L}^\infty([0,\infty) \times \mathbf{R}^d)} \left(\sum_{k=1}^{\infty} \frac{1}{6} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \right)^{1/2} \\ & \quad \times \left(\text{ess sup}_{t \in [0,\infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \right)^{1/2}, \end{aligned} \quad (5.28)$$

using the Cauchy-Schwarz inequality. The sum on the right-hand side is controlled by the energy dissipation estimate in Proposition 4.10 (see (5.2)), with T chosen as above. Therefore (5.28) converges to zero as $n \rightarrow \infty$.

Step 3. Using Proposition 4.5, we can write

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi_k^n (\tilde{\mathbf{u}}_k^n - \bar{\mathbf{u}}_k^n) \varrho_k^n dx &= -\tau_{k-1}^n \int_{\mathbf{R}^d} \varphi_k^n \nabla U'(\varrho_k^n) \varrho_k^n dx. \\ &= \tau_{k-1}^n \int_{\mathbf{R}^d} \nabla \varphi_k^n P(\varrho_k^n) dx. \end{aligned}$$

We used the fact that $P(\varrho_k^n) \in \mathcal{W}^{1,1}(\mathbf{R}^d)$ and $\nabla U'(\varrho_k^n) \varrho_k^n = \nabla P(\varrho_k^n)$, which follows from Proposition 4.10 above and Theorem 10.4.6 in [4]. From (5.26), we get

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} \varphi_k^n (\tilde{\mathbf{u}}_k^n - \bar{\mathbf{u}}_k^n) \varrho_k^n dx - \int_{t_{k-1}^n}^{t_k^n} \int_{\mathbf{R}^d} \nabla \varphi P(\varrho^n) dx dt \right| \\ & \leq (\tau_k^n)^2 \|\partial_t \nabla \varphi\|_{\mathcal{L}^\infty([t_{k-1}^n, t_k^n] \times \mathbf{R}^d)} \left(\int_K P(\varrho_k^n) dx \right), \end{aligned}$$

where K is a bounded set with $\text{spt } \varphi(t, \cdot) \subset K$ for all $t \in [0, \infty)$. Now we use again the estimate $P(r) \leq C(1 + U(r))$ for all $r \in [0, \infty)$, with $C > 0$ some constant. We can then sum in k and arrive at

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \varphi_k^n (\tilde{\mathbf{u}}_k^n - \bar{\mathbf{u}}_k^n) \varrho_k^n dx - \int_{[0,\infty) \times \mathbf{R}^d} \nabla \varphi P(\varrho^n) dx dt \right| \\ & \leq \delta^n CT \|\partial_t \nabla \varphi\|_{\mathcal{L}^\infty([0,\infty) \times \mathbf{R}^d)} \left(|K| + \text{ess sup}_{t \in [0,\infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \right), \end{aligned}$$

with $T > 0$ chosen as above. The right-hand side vanishes as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} \int_{[0,\infty) \times \mathbf{R}^d} \nabla \varphi P(\varrho^n) dx dt = \int_{[0,\infty) \times \mathbf{R}^d} \nabla \varphi \langle P(r) \rangle(t, x) dx dt.$$

Step 4. Remember the polar factorization of the velocity $\bar{\mathbf{u}}_k^n$ given by

$$\bar{\mathbf{u}}_k^n = \nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) + \mathbf{w}_k^n;$$

see Definition 4.6. Using this identity in the last integral in (5.25), we obtain

$$\begin{aligned} & \int_{\mathbf{R}^d} \varphi_k^n \left(\bar{\mathbf{u}}_k^n - \mathbf{u}_k^n \right) \varrho_k^n dz \\ &= \int_{\mathbf{R}^d} \varphi_k^n \left(\nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) - \nabla \phi_k^n \right) \varrho_k^n dz + \int_{\mathbf{R}^d} \varphi_k^n \mathbf{w}_k^n \varrho_k^n dz. \end{aligned}$$

The first term on the right-hand side can be rewritten as

$$\begin{aligned} & \int_{\mathbf{R}^d} \varphi_k^n \left(\nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) - \nabla \phi_k^n \right) \varrho_k^n dz \\ &= \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) \right) \nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) \varrho_k^n dz, \end{aligned}$$

where we used that the map $\text{id} + \tau_k^n \mathbf{w}_k^n$ preserves the measure $\varrho_k^n \mathcal{L}^d$. Let now

$$\psi_k^n(x) := \int_0^1 \nabla \varphi_k^n(x + \theta \tau_k^n \mathbf{w}_k^n(x)) d\theta \quad \text{for a.e. } x \in \mathbf{R}^d.$$

Then we can proceed with

$$\begin{aligned} & \int_{\mathbf{R}^d} \left(\varphi_k^n - \varphi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) \right) \nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) \varrho_k^n dz \\ &= \tau_k^n \int_{\mathbf{R}^d} \left(\psi_k^n \cdot \mathbf{w}_k^n \right) \nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) \varrho_k^n dz, \end{aligned}$$

which in turn implies the estimate

$$\begin{aligned} & \left| \int_{\mathbf{R}^d} \varphi_k^n \left(\nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) - \nabla \phi_k^n \right) \varrho_k^n dz \right| \\ & \leq \tau_k^n \|\nabla \varphi_k^n\|_{\mathcal{L}^\infty(\mathbf{R}^d)} \left(\int_{\mathbf{R}^d} |\mathbf{w}_k^n|^2 \varrho_k^n dz \right)^{1/2} \left(\int_{\mathbf{R}^d} |\nabla \phi_k^n|^2 \varrho_k^n dz \right)^{1/2}. \end{aligned} \quad (5.29)$$

Note that both parts of the polar factorization are bounded by the kinetic energy of $(\varrho_k^n, \bar{\mathbf{u}}_k^n)$, which is uniformly bounded because of the energy inequality (5.13). The factor τ_k^n on the right-hand side of (5.29) allows us to sum in k to obtain

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \varphi_k^n \left(\nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) - \nabla \phi_k^n \right) \varrho_k^n dx \right| \\ & \leq CT \|\nabla \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \left(\text{ess sup}_{t \in [0, \infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t, x) dx \right), \end{aligned} \quad (5.30)$$

where the time $T > 0$ is chosen such that $\text{spt } \varphi \subset [0, T) \times \mathbf{R}^d$. We conclude that the contribution to the residual coming from the gradient part of the velocity projection is bounded in the space $\mathcal{M}([0, \infty), \text{Lip}'(\mathbf{R}^d))$ uniformly in n .

To proceed we need the following formal argument.

Heuristic 5.8. Consider a density $\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and a velocity $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$. For any $\tau > 0$ we can then compute the polar factorization of $\mathbf{u}$ in the form

$$\mathbf{u} = \nabla \phi^\tau \circ (\text{id} + \tau \mathbf{w}^\tau) + \mathbf{w}^\tau; \quad (5.31)$$

see Section 2.2. As shown in Proposition 2.3, we have that

$$(\nabla \phi^\tau, \mathbf{w}^\tau) \rightharpoonup (\mathbf{v}^0, \mathbf{w}^0) \quad \text{weakly in } \mathcal{L}^2(\mathbf{R}^d, \varrho),$$

where $\mathbf{v}^0 := \mathbb{P}\mathbf{u}$ and $\mathbf{w}^0 := (\text{id} - \mathbb{P})\mathbf{u}$, with $\mathbb{P}: \mathcal{L}^2(\mathbf{R}^d, \varrho) \longrightarrow \mathbb{T}_\varrho \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ denoting the orthogonal projection of (2.4). Differentiating (5.31), we find that *formally*

$$0 = \nabla \dot{\phi}^\tau \circ (\text{id} + \tau \mathbf{w}^\tau) + \left(D^2 \phi^\tau \circ (\text{id} + \tau \mathbf{w}^\tau) \right) [\mathbf{w}^\tau + \tau \dot{\mathbf{w}}^\tau] + \dot{\mathbf{w}}^\tau,$$

where the dot indicates the partial derivative with respect to τ . Setting $\tau = 0$, we obtain the identity $0 = \dot{\mathbf{v}}^0 + D\mathbf{v}^0[\mathbf{w}^0] + \dot{\mathbf{w}}^0$, which implies that

$$\dot{\mathbf{v}}^0 = -\mathbb{P}D\mathbf{v}^0[\mathbf{w}^0] \quad \text{and} \quad \dot{\mathbf{w}}^0 = -(\text{id} - \mathbb{P})D\mathbf{v}^0[\mathbf{w}^0]. \quad (5.32)$$

A formal Taylor expansion yields

$$\begin{aligned} \nabla \phi^\tau &= \mathbf{v}^0 - \tau \mathbb{P}D\mathbf{v}^0[\mathbf{w}^0] + o(\tau), \\ \mathbf{w}^\tau &= \mathbf{w}^0 - \tau(\text{id} - \mathbb{P})D\mathbf{v}^0[\mathbf{w}^0] + o(\tau). \end{aligned} \quad (5.33)$$

Therefore, using the orthogonality of the decomposition (2.4), we obtain

$$\begin{aligned} \int_{\mathbf{R}^d} |\nabla \phi^\tau|^2 \varrho \, dx &= \int_{\mathbf{R}^d} |\mathbf{v}^0|^2 \varrho \, dx - 2\tau \int_{\mathbf{R}^d} \mathbf{v}^0 \cdot D\mathbf{v}^0[\mathbf{w}^0] \varrho \, dx + o(\tau), \\ \int_{\mathbf{R}^d} |\mathbf{w}^\tau|^2 \varrho \, dx &= \int_{\mathbf{R}^d} |\mathbf{w}^0|^2 \varrho \, dx - 2\tau \int_{\mathbf{R}^d} \mathbf{w}^0 \cdot D\mathbf{v}^0[\mathbf{w}^0] \varrho \, dx + o(\tau). \end{aligned}$$

This implies the following identity:

$$\begin{aligned} \int_{\mathbf{R}^d} |\mathbf{w}^\tau|^2 \varrho \, dx &= \int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho \, dx - \int_{\mathbf{R}^d} |\nabla \phi^\tau|^2 \varrho \, dx \\ &\quad - 2\tau \int_{\mathbf{R}^d} \mathbf{u} \cdot D\mathbf{v}^0[\mathbf{w}^0] \varrho \, dx + o(\tau). \end{aligned} \quad (5.34)$$

Note that the first two terms on the right-hand side of (5.34) is just the difference in kinetic energies before and after the velocity projection. Moreover, if we assume that $\mathbf{v}^0$ is regular enough such that $D\mathbf{v}^0$ is bounded, then we can estimate

$$\left| 2 \int_{\mathbf{R}^d} \mathbf{u} \cdot D\mathbf{v}^0[\mathbf{w}^0] \varrho \, dx \right| \leq C \left(\int_{\mathbf{R}^d} |\mathbf{u}|^2 \varrho \, dx \right)^{1/2} \left(\int_{\mathbf{R}^d} |\mathbf{w}^0|^2 \varrho \, dx \right)^{1/2},$$

which is bounded above by the total energy. Here $C > 0$ is some constant. Notice also that in the one-dimensional case, there are no non-vanishing, divergence free vector fields in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$. Hence $\mathbf{w}^0 = 0$ and the linear term in (5.34) vanishes.

We can now improve (5.29) as follows: By Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \varphi_k^n \left(\nabla \phi_k^n \circ (\text{id} + \tau_k^n \mathbf{w}_k^n) - \nabla \phi_k^n \right) \varrho_k^n \, dx \right| \\ &\leq \sqrt{\delta^n} \sqrt{T} \|\nabla \varphi\|_{\mathcal{L}^\infty([0, \infty) \times \mathbf{R}^d)} \left(\sup_k \int_{\mathbf{R}^d} |\nabla \phi_k^n|^2 \varrho_k^n \, dx \right)^{1/2} \\ &\quad \times \left(\sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(|\bar{\mathbf{u}}_k^n|^2 - |\nabla \phi_k^n|^2 \right) \varrho_k^n \, dx + T \left(C \sup_k \int_{\mathbf{R}^d} |\bar{\mathbf{u}}_k^n|^2 \varrho_k^n \, dx + o(1) \right) \right)^{1/2}. \end{aligned} \quad (5.35)$$

The telescope sum of kinetic energies is bounded above by the initial total energy (see also (5.2)), and $T > 0$ is chosen as above. Here we assumed that the sequence of $(\varrho_k^n, \bar{\mathbf{u}}_k^n)$ is regular enough so that the argument of Heuristic 5.8 can be applied. Then the right-hand side of (5.35) converges to zero as $\delta^n \rightarrow 0$.

Heuristic 5.9. Consider a density $\varrho \in \mathcal{P}_{\text{reg}}(\mathbf{R}^d)$ and a velocity $\mathbf{u} \in \mathcal{L}^2(\mathbf{R}^d, \varrho)$. Suppose that for some $\tau > 0$, the map $\mathbf{r} := \text{id} + \frac{2\tau^2}{3}\nabla U'(\varrho)$ is an optimal transport map pushing $\varrho\mathcal{L}^d$ forward to the measure $\hat{\varrho}\mathcal{L}^d := \mathbf{r}\#(\varrho\mathcal{L}^d)$, and that

$$\mathbf{u} = \hat{\mathbf{u}} \circ \left(\text{id} + \frac{2\tau^2}{3}\nabla U'(\varrho) \right) - \tau\nabla U'(\varrho)$$

ϱ -a.e., with $\hat{\mathbf{u}} \in \mathbb{T}_{\hat{\varrho}}\mathcal{P}_{\text{reg}}(\mathbf{R}^d)$. Then a formal Taylor expansion yields

$$\mathbf{u} = \left(\hat{\mathbf{u}} - \tau\nabla U'(\varrho) \right) + D\hat{\mathbf{u}} \left[\frac{2\tau^2}{3}\nabla U'(\varrho) \right] + o\left(\frac{2\tau^2}{3}\nabla U'(\varrho) \right).$$

Consider the polar factorization (5.31) of $\mathbf{u}$ and its expansion (5.33), where now

$$\begin{aligned} \mathbf{v}^0 &= \left(\hat{\mathbf{u}} - \tau\nabla U'(\varrho) \right) + \mathbb{P}D\hat{\mathbf{u}} \left[\frac{2\tau^2}{3}\nabla U'(\varrho) \right] + o\left(\frac{2\tau^2}{3}\nabla U'(\varrho) \right), \\ \mathbf{w}^0 &= (\text{id} - \mathbb{P})D\hat{\mathbf{u}} \left[\frac{2\tau^2}{3}\nabla U'(\varrho) \right] + o\left(\frac{2\tau^2}{3}\nabla U'(\varrho) \right). \end{aligned}$$

If we assume that the velocity $\hat{\mathbf{u}}$ is sufficiently regular, we obtain

$$\mathbf{w}^\tau = \mathcal{O}\left(\frac{2\tau^2}{3}\nabla U'(\varrho) \right) + o(\tau). \quad (5.36)$$

In the one-dimensional case, there exists no non-vanishing, divergence free vector fields in $\mathcal{L}^2(\mathbf{R}^d, \varrho)$, and therefore we can replace $\mathcal{O}$ by small o in (5.36).

We assume now that the sequence of $(\varrho_k^n, \bar{\mathbf{u}}_k^n)$ is regular enough so that the argument of Heuristic 5.9 can be applied. By Cauchy-Schwarz, we have

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \varphi_k^n \mathbf{w}_k^n \varrho_k^n dx \right| &\leq \|\varphi\|_{\mathcal{L}^\infty([0,\infty) \times \mathbf{R}^d)} \\ &\times \left(\sqrt{\delta^n} \sqrt{\frac{8T}{3}} \left(\sum_{k=1}^{\infty} \frac{1}{6} (\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n dx \right)^{1/2} + o(1) T \right). \end{aligned} \quad (5.37)$$

The sum on the right-hand side is controlled by the energy dissipation estimate in Proposition 4.10 (see (5.2)), with T chosen as above. Therefore (5.37) converges to zero as $n \rightarrow \infty$. This concludes the (formal) proof of (5.23). $\square$

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