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ON THE WEAK LOWER SEMICONTINUITY OF ENERGIES WITH POLYCONVEX INTEGRANDS

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ABSTRACT. – Let $f: \Omega \times \mathbf{R}^N \times \mathbf{R}^{N \times N} \to [0, \infty)$ be a Borel measurable function such that $f(x, u, \xi) = a(x, u) g(x, \xi)$ and $g(x, \cdot)$ is polyconvex in the last variable ξ for almost every $x \in \Omega$. It is shown that if f is continuous, if a is bounded away from zero and if $F(u) := \int_\Omega a(x, u) g(x, \nabla u(x))\, dx$, $u \in W^{1, N}(\Omega, \mathbf{R}^N)$, then F is weakly lower semicontinuous in $W^{1,p}$, $p > N-1$, in the sense that $F(u) \leqq \lim \inf_{\nu \to \infty} F(u_\nu)$ for $u_\nu, u \in W^{1, N}(\Omega, \mathbf{R}^N)$ such that $u_\nu \rightharpoonup u$ in $W^{1,p}$. On the contrary if g is only a Carathéodory function then in general F is not weakly lower semicontinuous in $W^{1,p}$ for $N > p > N-1$. Precisely, it is shown that if $F(u) := \int_K |\det(\nabla u(x))|\, dx$ where K is a compact set, then F is weakly lower semicontinuous in $W^{1,p}$, $N > p > N-1$ if and only if $meas(\partial K) = 0$.

1. Introduction

Let $N \geqq 2$ be an integer number, let $\Omega \subset \mathbf{R}^N$ be an open bounded set and let $f: \Omega \times \mathbf{R}^N \times \mathbf{R}^{N \times N} \to [0, \infty)$ be a Borel measurable function. We set

$$F(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx, \qquad u \in W^{1,p}(\Omega, \mathbf{R}^N) := W^{1,p}.$$

If one uses the direct method of the calculus of variations to obtain existence of minima for F, one needs to show that F is weakly lower semicontinuous in $W^{1,p}$. Since Morrey's works ([Mo1], [Mo2]) and later Acerbi-Fusco ([AF], Marcellini [Ma2]) and others, it is well known that if $1 \leqq p < \infty$ and if

$$0 \leqq f(x, u, \xi) \leqq a + b\, |\xi|^p, \qquad \forall (x, u, \xi) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N \times N} \tag{1.1}$$

then F is weakly lower semicontinuous in $W^{1,p}$ if only if f is quasiconvex with respect to the last variable ξ. We recall that f is said to be quasiconvex if it verifies the following Jensen's inequality

$$\frac{1}{|\Omega|} \int_\Omega f(x_0, u_0, \xi + \nabla u(x))\, dx \geqq f(x_0, u_0, \xi_0)$$

for almost every $x_0 \in \Omega$, for every $(u_0, \xi_0) \in \mathbf{R}^N \times \mathbf{R}^{N \times N}$ and for every $u \in W_0^{1,\infty}(\Omega, \mathbf{R}^N)$. As it is very hard to check whether or not a given function is quasiconvex,

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We give some definitions relevant for this work.

DEFINITION 1.1. – *Let N, $M \geqq 1$ be two integer numbers and let $\Omega \subset \mathbb{R}^M$ be an open set. A function $f: \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times M} \to \mathbb{R}$ is said to be a* **Carathéodory function** *if $f(\cdot, u, \psi)$ is measurable for every $(u, \psi) \in \mathbb{R}^N \times \mathbb{R}^{N \times M}$ and $f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.*

DEFINITION 1.2. – *(See* [Da]*).*

Let $f: \mathbb{R}^{N \times M} \to \mathbb{R}$ be a Borel measurable function defined on the set of the $N \times M$ real matrices.

- *f is said to be* **convex** *if $f(\lambda\xi + (1-\lambda)\eta) \leqq \lambda f(\xi) + (1-\lambda) f(\eta)$ for every $\xi, \eta \in \mathbb{R}^{N \times M}$ and every $\lambda \in (0, 1)$.*
- *f is said to be* **polyconvex** *if there exists a function $h: \mathbb{R}^{\tau(N, M)} \to \mathbb{R}$ convex such that $f(\xi) = h(T(\xi))$ for every $\xi \in \mathbb{R}^{N \times M}$, where $\tau(N, M) = \sum_{1 \leqq s \leqq \min(N, M)} \binom{M}{s}\binom{N}{s}$, $T(\xi) = (adj_1 \xi, \ldots, adj_{\min(N, M)} \xi)$ and $adj_s \xi$ stands for the matrix of all $s \times s$ minors of ξ. When $N = M = 2$ then $T(\xi) = (\xi, \det(\xi))$.*
- *f is said to be* **quasiconvex** *if $\frac{1}{|\Omega|} \int_\Omega f(\xi + \nabla \phi) \geqq f(\xi)$ for every $\xi \in \mathbb{R}^{N \times M}$, for every $\Omega \subset \mathbb{R}^N$ open bounded set and for every $\phi \in W_0^{1, \infty}(\Omega)^M$ (it is equivalent to assume that the previous inequality holds for one fixed open, bounded, $\Omega \subset \mathbb{R}^N$).*

For completeness we state the following well known result.

PROPOSITION 1.3. – *Let N, $M \geqq 2$ be two integer numbers, let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $f: \Omega \times \mathbb{R}^M \times \mathbb{R}^{N \times M} \to \mathbb{R}$ be a continuous function such that $f(x, u, \cdot)$ is quasiconvex for each $(x, u) \in \Omega \times \mathbb{R}^M$. Furthermore assume that f satisfies*

$$-\alpha(|u|^q + |\xi|^q) - \gamma(x) \leqq f(x, u, \xi) \leqq \alpha(|u|^p + |\xi|^p) + \gamma(x),$$

where $\alpha > 0$, $\gamma \in L^1(\Omega)$, $1 \leqq q < p < \infty$,

$$|f(x, u, \xi) - f(x, v, \eta)| \leqq \beta(1 + |u|^{p-1} + |v|^{p-1} + |\xi|^{p-1} + |\eta|^{p-1}) \times (|u - v| + |\xi - \eta|)$$

where $\beta > 0$ and

$$|f(x, u, \xi) - f(y, u, \xi)| \leqq \nu(|x - y|)(1 + |u|^p + |\xi|^p),$$

where ν is a continuous increasing function with $\nu(0) = 0$. Let

$$F(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx, \qquad u \in W^{1, p}(\Omega, \mathbb{R}^M).$$

Then F is weakly lower semicontinuous in $W^{1, p}$.

Proof. – For the proof we refer the reader to Theorem 2.4 in [Da].

LEMMA 1.4. – *Let $\phi: \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $\psi \in C_0^\infty(\Omega, \mathbb{R}^\tau)$. If $p > N-1$, if $u_\nu, u \in W^{1,N}(\Omega, \mathbb{R}^N)$ and if $u_\nu \rightharpoonup u$ in $W^{1,p}$ then*

$$\lim_{\nu\to\infty} \int_\Omega \phi'(u_\nu^1)\ldots\phi'(u_\nu^N)\langle\psi; T(\nabla u_\nu)\rangle dx = \int_\Omega \phi'(u^1)\ldots\phi'(u^N)\langle\psi; T(\nabla u)\rangle dx.$$

Moreover the results stands for $p = N-1$, $N = 2$. Here $\langle\, ;\, \rangle$ is the scalar product in $\mathbb{R}^\tau$ and $\tau = \sum_{1\leqq s\leqq N} \binom{N}{s}^2$.

Proof. – Lemma 1.4 is obtained as a slight modification of the proof of Lemma 1 in [DM].

2. The case of continuous integrands

Let us first state the main result of this section.

THEOREM 2.1. – *Let $N \geqq 2$ be an integer number, let $\gamma > 0$, let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $\tau = \sum_{1\leqq s\leqq N} \binom{N}{s}^2$. Let $a: \Omega\times\mathbb{R}^N \to [0, \infty)$ and $g: \mathbb{R}^N\times\mathbb{R}^\tau \to [0, \infty)$ be two continuous functions such that $g(x, \cdot)$ is convex for each $x \in \Omega$. Let:*

$$F(u) := \int_\Omega a(x, u(x))\, g(x, T(\nabla u(x)))\, dx, \qquad u \in W^{1,p}(\Omega, \mathbb{R}^N).$$

Then,

$$F(u) \leqq \lim \inf_{\nu\to\infty} F(u_\nu), \tag{2.2}$$

if $u_\nu, u \in W^{1,N}(\Omega, \mathbb{R}^N)$ and $u_\nu \rightharpoonup u$ in $W^{1,p}$, $p > N-1$. Moreover, if $N = 2$ the result is true even if $p = N-1 = 1$.

We recall that $T(\nabla u)$ stands for the matrix of all minors of ∇u.

Remark 2.2. – 1. – If $p \geqq N$ it is easy to prove Theorem 2.1 even in the general case where $F(u) := \int_\Omega f(x, u(x), T(\nabla u(x)))\, dx$, f continuous, $f(x, u, T) \geqq 0$, $f(x, u, \cdot)$ convex. Indeed, h being a fixed real number, we truncate the sequence u_ν, u and get a sequence v_ν, v so that $|v_\nu(x)|$, $|v(x)| < h$ for almost every $x \in \Omega$. As f is convex in the last variable T, using Lemma 2.3 we can approximate f on $\Omega \times [-h, h]^N \times \mathbb{R}^\tau$ by a non decreasing sequence of smooth functions f_j such that

$$0 \leqq f_j(x, u, T) \leqq C_j(x, u)(1 + |T|),$$

where $C_j(x, u) = 0$ for every $x \in \Omega$ such that $dist(x, \partial\Omega) < \dfrac{1}{l_j}$, for suitable $l_j \in \mathbb{N}$. Then we apply Proposition 1.3 to $f_j(x, u, T)$ and to the sequence v_ν, v. Letting j go to infinity and then h go to infinity, we obtain

$$F(u) \leqq \lim \inf_{\nu\to\infty} F(u_\nu).$$

2. – In the general case $F(u) := \int_\Omega f(x, u(x), T(\nabla u(x)))\,dx$, (2.2) would be true if we knew that the sequence $\{\det(\nabla u_\nu)\}_\nu$ is bounded in L^1. Indeed, as indicated above, by De Giorgi's Lemma, we approximate $f(x, u, T)$ from below by a function $g(x, u, T)$ which is smooth and grows lineary in the variable T. We can assume without loss of generality that there eixsts a constant $h > 0$ such that $|u_\nu(x)|$, $|u(x)| \leqq h$ for almost every $x \in \Omega$. Then, we fix a compact set K in $\Omega \times [-h, h]^N$ and by Weierstrass's Approximation Lemma we obtain:

$$(2.3) \qquad \begin{cases} |g(x, u, T) - g_n(x, u, T)| \leqq \varepsilon\,(1 + \max\limits_{(y,\,v)\in K} g(y, v, T)) \\ \qquad\qquad \forall T \in \mathbb{R}^\tau, \quad \forall (x, u) \in K, \end{cases}$$

where $g_n(x, u, T)$ has the form

$$g_n(x, u, T) = \sum_{k=0}^{n} a_k^{1,n}(u_1)\ldots a_k^{N,n}(u_N)\,h_k^n(x, T).$$

Then we can show with the relaxed assumptions

$$a_k^{1,n}(u_1), \ldots, a_k^{N,n}(u_N),\ h_k^n(x, T) \geqq 0$$

that if $F_n(u) := \displaystyle\int_\Omega g_n(x, u(x), T(\nabla u)(x)))\,dx$, then,

$$F_n(u) \leqq \lim\inf_{\nu\to\infty} F_n(u_\nu),$$

which together with (2.3) and the fact that $\{\det(\nabla u_\nu)\}_\nu$ is bounded in L^1, yields

$$F(u) \leqq \lim\inf_{\nu\to\infty} F(u_\nu).$$

However $\{\det(\nabla u_\nu)\}_\nu$ is not necessarily bounded in L^1. [DM] provides an example where $u_\nu \rightharpoonup u$ in $W^{1,p}$, $N > p > N-1$ and $\{\det(\nabla u_\nu)\}_\nu$ is not bounded in L^1 (cf. also [BM]). For instance if $N = 2$, $\Omega = (0, 1)^2$, $1 < p \leqq 2$,

$$u_\nu \equiv \nu^{\frac{1}{p}-1}(1-y)^\nu(\sin\nu x, \cos\nu x) \rightharpoonup (0, 0) \quad \text{in } W^{1,p}.$$

Then $\det(\nabla u_\nu) = -\nu^{\frac{2}{p}}(1-y)^{2\nu-1}$ is not bounded in L^1 if $p < 2$.

3. – The assumption that u_ν, $u \in W^{1,N}$ is important. It can be useful to extend the definition of $F(u)$ to functions $u \in W^{1,p}$, $p < N$ (cf. [Mal]). Also Theorem 2.1 is false if one omits this assumption (cf. [BM]).

4. – If $1 \leqq p < N-1$, and if $N \geqq 3$, then F is not necessarily weakly lower semicontinuous (cf. [Mal]). But if $p = N-1$, $N \geqq 3$, the question to know whether or not F is weakly lower semicontinuous is still open. However Malý proved in [Mal] that if u, $u_\nu \in W^{1,N-1}$ are sense preserving diffeomorphisms such that $u_\nu \rightharpoonup u$ in $W^{1,N-1}$, then $F(u) \leqq \lim\inf_{\nu\to\infty} F(u_\nu)$.

5. – The basic idea to prove Theorem 2.1 is the following: in the first step, we approximate f from below by a sum of functions of the form $c(x)\,b^1(u^1)\cdots b^N(u^N)\,g(x, T(\nabla u))$, with $c(x)$, $b^1(u^1), \ldots, b^N(u^N) \geqq 0$. This can be done using Weierstrass's Approximation Theorem (*see* Lemma 2.5). In the second step, changing variables we write $c(x)\,b^1(u^1)\cdots b^N(u^N)\;g(x, T(\nabla u))$ in the form

$h = h(x, T(\nabla v))$. Then, following the idea of Dacorogna and Marcellini in their study of integrands of the form $h = h(T(\nabla v))$ (*see* [DM]), we conclude the Theorem.

LEMMA 2.3. (De Giorgi's Lemma). – *Let* N, $\tau \geqq 1$ *be two integer numbers, let* $\Omega \in \mathbb{R}^N$ *be a open bounded set, and let* $g: \quad \Omega \times \mathbb{R}^\tau \to [0, \infty)$ *be a continuous function such that* $g(x, \cdot)$ *is convex for each* $x \in \Omega$. *There exists a non decreasing sequence of functions* $(g_l)_l$ *of class* $C^\infty(\Omega \times \mathbb{R}^\tau)$ *such that:*

i) $g_l \geqq -1$;

ii) $(g_l)_l$ *converges uniformly to* g *in every compact subset of* $\Omega \times \mathbb{R}^\tau$;

iii) $g_l(x, \cdot)$ *is convex;*

iv) $g_l(x, T) = 0$ *if* $dist(x, \partial\Omega) \leqq \frac{1}{l}$;

v) On every compact subset K *of* Ω, $D_T\, g_l(x, T)$ *is bounded in* $K \times \mathbb{R}^\tau$ *by a constant which depend only on* l, g *and* K, *where* $D_T\, g_l = \left(\frac{\partial}{\partial T_1} g_l, \ldots, \frac{\partial}{\partial T_r} g_l\right)$.

Proof. – For the proof we refer the reader to [Ma2].

Remark 2.4. – One can deduce from Lemma 2.3 that there exists a constant $C \equiv C(l, g)$ such that $|D_T\, g_l(x, T)| \leqq C$ for every $(x, T) \in \Omega \times \mathbb{R}^\tau$.

LEMMA 2.5. – (Weierstrass's Approximation Theorem)
Let $f: \quad [0, 1] \to \mathbb{R}$ *be a continuous function. Then, for every* $\varepsilon > 0$, *there exists* $n_0(\varepsilon) \in \mathbb{N}$ *such that* $n \geqq n_0(\varepsilon)$ *implies*

$$\left| f(u) - \sum_{0 \leqq k \leqq n} \binom{n}{k} f\left(\frac{k}{n}\right) u^k (1-u)^{n-k} \right| \leqq \varepsilon \left(1 + \max_{0 \leqq t \leqq 1} |f(t)|\right),$$

for every $u \in [0, 1]$.

Proof. – For the proof we refer the reader to [Kl]).

Proof of Theorem 2.1. – We give the proof of Theorem 2.1 only in the case where $N > p > N - 1$ since the case $p \geqq N$ is easily obtained (*see* Remark 2.2). In the first step of the proof, we truncate the functions $(u_\nu)_\nu$ and u to get a new sequence which is uniformly bounded in L^∞. Then we write f as a sum of functions of the form $c(x)\, b^1(u^1) \cdots b^N(u^N)\, g(x, T(\nabla u))$, where c and $b^1, \ldots, b^N$ are smooth. In the second step we study the particular case where f has the form $c(x)\, b^1(u^1) \cdots b^N(u^N)\, g(x, T(\nabla u))$. In the last step we study the general case where f satisfies the hypotheses of Theorem 2.1. Clearly (2.2) is true if

$$\lim \inf_{\nu \to \infty} \int_{\Omega'} a(x, u_\nu)\, g(x, T(\nabla(u_\nu))) = \infty.$$

Assume that

$$M := \lim \inf_{\nu \to \infty} \int_{\Omega'} a(x, u_\nu)\, g(x, T(\nabla(u_\nu))) < +\infty.$$

Fix $h > 0$, $E = [-h, h]^N$, $l_0 \in \mathbb{N}$ and $\Omega' = \left\{x \in \Omega,\ dist(x, \partial\Omega) > \frac{1}{l_0 + 1}\right\}$.

First step.

a) Factorization of $a(x, u)$

Using explicitly Weierstrass's Approximation Theorem and the fact that $a(x, u) \geqq \gamma > 0$, it is easy to deduce that there are two sequences $(b_k^n)_{k \leqq n}$ and $(c_k^n)_{k \leqq n}$ such that for every $\varepsilon > 0$, there is $n(\varepsilon) \in \mathbb{N}$ depending only on ε, Ω' and h verifying

$$(2.4)\quad 0 \leqq a(x, u) - \sum_{k=0}^{n} c_k^n(x)\, b_k^n(u) \leqq \varepsilon \quad \forall (x, u) \in \bar{\Omega}' \times E := K, \qquad \forall n \geqq n(\epsilon),$$

$$b_k^n(x, u_1, \ldots, u_N) = b_k^{1,n}(u_1) \cdots b_k^{N,n}(u_N) \quad k = 1, \ldots, n, \qquad n \geqq 0.$$

$$b_k^{j,n} \in C^\infty(\mathbb{R}), \qquad b_k^{j,n} \geqq 0 \quad j = 1, \ldots, N \quad k = 1, \ldots, n, \quad n \geqq 1,$$

$$c_k^n \in C^\infty(\bar{\Omega}'), \qquad c_k^n \geqq 0 \quad k = 1, \ldots, n, \quad n \geqq 1.$$

$$c_0^n(0) = 1, \qquad b_0^n(u) = -\varepsilon.$$

b) Truncation of u and u_ν.

Fix $\delta(h) \ll 1$. Truncate u and u_ν by considering $\phi(u)$ and $\phi(u_\nu)$ respectively where ϕ is given by

$$(2.5)\qquad \phi(u) = \prod_{i=1}^{N} \psi(u^i), \qquad \phi'(u) = \prod_{i=1}^{N} \psi'(u^i) \qquad \text{with} \quad \psi'(t) = \frac{d\psi}{dt}(t),$$

and $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ is defined in the following way

$$\psi(t) = \begin{cases} -h & \text{if} \quad t < -h - \delta(h), \\ t & \text{if} \quad |t| \leqq h, \\ h & \text{if} \quad t > h + \delta(h), \end{cases}$$

$0 \leqq \psi'(t) \leqq 1$ for every $t \in \mathbb{R}$ and $\psi'(t) = 0$ if and only if $|t| \geqq h + \delta(h)$.

c) Regularization of $g(x, T)$.

We apply Lemma 2.3 to $g:\ \Omega \times \mathbb{R}^\tau \to [0, \infty)$. We obtain a sequence $(g_l)_l$ which has properties i), ..., v) of Lemma 2.3. Recall that

$$(2.6)\qquad g(x, T) = \lim_{l \to \infty} g_l(x, T) \qquad \forall (x, T) \in \Omega \times \mathbb{R}^\tau.$$

Since $(g_l)_l$ is uniformly bounded below on $\Omega \times \mathbb{R}^\tau$, we can assume without loss of generality that $g_l \geqq 0$.

Second step. For $l = l_0$, $k \geqq 1$ we show that

$$(2.7)\qquad \liminf_{\nu \to \infty} \int_{\Omega'} \phi'(u_\nu)\, c_k^n(x)\, b_k^n(u_\nu)\, g_l(x, T(\nabla u_\nu)) \geqq \int_{\Omega'} \phi'(u)\, c_k^n(x)\, b_k^n(u)\, g_l(x, T(\nabla u)).$$

– If $\liminf_{\nu \to \infty} \int_{\Omega'} \phi'(u_\nu)\, c_k^n(x)\, b_k^n(u_\nu)\, g_l(x, T(\nabla u_\nu)) = \infty$ then (2.7) is trivial.

– Assume that $\inf_{\nu \to \infty} \int_{\Omega'} \phi'(u_\nu)\, c_k^n(x)\, b_k^n(u_\nu)\, g_l(x, T(\nabla u)) < \infty$. We may assume without loss of generality that

$$u \in C^\infty(\Omega, \mathbf{R}^N).$$

If this wasn't the case then it would suffice to replace u by $u_\varepsilon \in C^\infty(\Omega, \mathbf{R}^N)$ such that $\| u_\varepsilon - u \|_{W^{1,N}} \leqq \epsilon$, following the proof with necessary modifications. Since

$$g_l(x, \cdot) \equiv 0 \quad \text{if} \quad dist(x, \partial\Omega) \leqq \frac{1}{l}, \tag{2.8}$$

$$| D_T\, g_l(x, T) | \leqq C \equiv C(l, h) \;\text{ for every }\; (x, T) \in \Omega \times \mathbf{R}^\tau,$$
$$g_l \in C^\infty(\Omega \times \mathbf{R}^\tau, [0, \infty)) \quad \text{and} \quad g_l(x, \cdot) \text{ is convex,}$$

$$c_k^n \in C^\infty(\bar{\Omega}'),$$

$$b_k^n \in C^\infty(\mathbf{R}^N),$$

and

$$\phi' \in C^\infty(\mathbf{R}, \mathbf{R})$$

we deduce that

$$\begin{aligned}
&\lim \inf_{\nu \to \infty} \int_{\Omega'} c_k^n(x)\, b_k^n(u_\nu)\, \phi'(u_\nu)\, g_l(x, T(\nabla u_\nu)) \\
&\quad \geqq \lim \inf_{\nu \to \infty} c_k^n(x)\, b_k^n(u_\nu)\, \phi'(u_\nu)\, g_l(x, T(\nabla u)) \\
&\quad + \lim \inf_{\nu \to \infty} \int_{\Omega'} c_k^n(x)\, b_k^n(u_\nu)\, \phi'(u_\nu) \langle D_T\, g_l(x, T(\nabla u));\; T(\nabla u_\nu) - T(\nabla u) \rangle \\
&\quad \geqq \int_{\Omega'} c_k^n(x)\, b_k^n(u)\, \phi'(u)\, g_l(x, T(\nabla u)) \\
&\quad + \lim \inf_{\nu \to \infty} \int_{\Omega'} c_k^n(x)\, b_k^n(u_\nu)\, \phi'(u_\nu) \langle D_T\, g_l(x, T(\nabla u));\; T(\nabla u_\nu) - T(\nabla u) \rangle,
\end{aligned}$$

where we used Fatou's Lemma and the fact that

$$c_k^n(x)\, b_k^n(u_\nu)\, \phi'(u_\nu) \to c_k^n(x)\, b_k^n(u)\, \phi'(u) \quad \text{a.e.}$$

For $T \in \mathbf{R}^\tau$, we set $T = (\bar{T}, t)$, $t \in \mathbf{R}$. For fixed $x \in \Omega$, let $D_{\bar{T}}\, g_l(x, \cdot)$ denote the matrix of the partial derivatives of $g_l(x, \cdot)$ with respect to the $\tau - 1$ first variables in $\mathbf{R}^\tau$. Let H be the functional defined on $\Omega \times \mathbf{R}^N \times \mathbf{R}^{N \times N}$ by

$$H(x, v, \xi) = c_k^n(x)\, b_k^n(v)\, \phi'(v)\, \langle D_{\bar{T}}\, g_l(x, T(\nabla u(x)));\; \bar{T}(\xi) - \bar{T}(\nabla u) \rangle.$$

It is easy to see that H and $-H$ are quasiconvex in the last variable. Using the fact that $u \in C^\infty(\Omega, \mathbf{R}^N)$, (2.8) and the fact that $| \phi'(u_\nu) | \leqq 1$, we get that H and $-H$ verify the assumptions of Proposition 1.3. We deduce that

$$\lim \inf_{\nu \to \infty} \int_{\Omega'} c(x)_k^n\, b_k^n(u_\nu)\, \phi'(u_\nu)\, \langle D_{\bar{T}}\, g_l(x, T(\nabla u));\; \bar{T}(\nabla u_\nu) - \bar{T}(\nabla u) \rangle = 0. \tag{2.9}$$

On the other hand, setting

$$v_\nu^i = B_k^i(\psi(u_\nu^i)), \qquad v^i = B_k^i(\psi(u^i)),$$

where $B_k^i(t) = \int_{-h-\delta(h)}^{t} b_k^{i,n} \circ \psi^{-1}(s)\, ds,, \ |t| \leqq h + \delta(h),$

then we obtain

$$v_\nu^i \to v^i \quad \text{in } W^{1,p},$$
$$b_k^{i,n}(u_\nu^i)\,\psi'(u_\nu^i) \to b_k^{i,n}(u^i)\,\psi'(u^i) \text{ a.e.}$$

As

$$\frac{\partial}{\partial t} g_l(x, T\nabla u) \in C_0^\infty(\Omega),$$

by Lemma 1.4 we obtain:

$$\liminf_{\nu\to\infty} \int_{\Omega'} c_k^n(x)\, b_k^n(u_\nu)\,\phi'(u_\nu)\frac{\partial}{\partial t} g_l(x, T(\nabla u))(\det(\nabla u_\nu) - \det(\nabla u))$$
$$= \liminf_{\nu\to\infty}\left(\int_{\Omega'} c_k^n(x)\frac{\partial}{\partial t} g_l(x, T\nabla u)(\det(\nabla v_\nu) - \det(\nabla v)\right)$$
$$- \int_{\Omega'} c_k^n(x)\left(b_k^n(u_\nu)\,\phi'(u_\nu) - b_k^n(u)\,\phi'(u))\frac{\partial}{\partial t} g_l(x, T\nabla u)\det(\nabla u)\right) = 0$$

which together with (2.9), yields (2.7).

Third step. We conclude that

$$\int_\Omega a(x, u(x))\, g(x, T(\nabla u(x)))\, dx \leqq \liminf_{\nu\to\infty} \int_\Omega a(x, u_\nu(x))\, g(x, T(\nabla u_\nu(x)))\, dx.$$

Since $M := \liminf_{\nu\to\infty} \int_{\Omega'} a(x, u_\nu)\, g(x, T(\nabla u_\nu)) < \infty$ and $a(x, u) \geqq \gamma > 0$, by steps 1 and 2 we obtain

$$\liminf_{\nu\to\infty} \int_\Omega a(x, u_\nu(x))\, g(x, T(\nabla u_\nu(x)))\, dx$$
$$\geqq \liminf_{\nu\to\infty} \int_{\Omega'} \left(\phi'(u_\nu)\sum_{k=0}^{n(\varepsilon)} c_k^{n(\varepsilon)}(x)\, b_k^{n(\varepsilon)}(u_\nu(x))\, g_{l_0}(x, T(\nabla u_\nu))\right) dx$$
$$\geqq \sum_{k=0}^{n(\varepsilon)} \int_{\Omega'} \phi'(u)\, c_k^{n(\varepsilon)}(x)\, b_k^{n(\varepsilon)}(u)\, g_{l_0}(x, T(\nabla u))\, dx - \varepsilon S,$$

where $S = \dfrac{M+1}{\gamma} + 3\, meas(\Omega) + \int_\Omega g_{l_0}(x, T(\nabla u))\, dx.$

In the previous inequalities, we used the second step to prove that

$$\liminf_{\nu\to\infty} \int_{\Omega'} \phi'(u_\nu)\, c_k^n(x)\, b_k^n(u_\nu) g_l(x, T(\nabla u_\nu)) \geqq \int_{\Omega'} \phi'(u)\, c_k^n(x)\, b_k^n(u)\, g_l(x, T(\nabla u))$$

for $k \neq 0$. For $k = 0$, we used the fact that $a(x, u) \geqq \gamma > 0$, and $M < \infty$. Letting ε go to zero, l_0 go to infinity and then h go to infinity in the previous inequality we obtain (2.2). ■

3. The case of Carathéodory integrands

We state the main result of this section.

THEOREM 3.1. – *Let $N \geqq 2$ be an integer number, $N-1 < p < N$, let $\Omega \subset \mathbf{R}^N$ be an open bounded set, and let $K \subset \Omega$ be a compact set. The two following assertions are equivalent:*

$$meas\,(\partial K) \neq 0, \tag{3.10}$$

$$\liminf_{\nu \to \infty} \int_K |\det(\nabla u_\nu(x))|\,dx < \int_K |\det(\nabla u(x))|\,dx \tag{3.11}$$

for a suitable u_ν, $u \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that $u_\nu \rightharpoonup u$ in $W^{1,p}$.
Before proving Theorem 3.1 we begin with some remarks.

Remark 3.2. – Let us recall that if $F(u) = \int_K |\det(\nabla u(x))|\,dx$ and if K is a compact set then, for $p \geqq N$, F is weakly lower semicontinuous on $W^{1,p}$ even if $meas\,(\partial K) \neq 0$ (see [AF]). For $p < N-1$ then F is not weakly lower semicontinuous on $W^{1,p}$ even if $meas\,(\partial K) = 0$ (see [Mal]).
The following lemma will be used to prove that (3.10) implies (3.11).

LEMMA 3.3. – *Let $N, r \geqq 2$ be two integer numbers, let $\Omega \subset \mathbf{R}^N$ be an open bounded set and let $K \subset \Omega$ be a compact set such that $meas\,(\partial K) > 0$. Let $p < N$ be a real number. Then there is a sequence $u_k \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that*

$$\text{(i)} \quad u_k \rightharpoonup u = \mathrm{id} \quad \text{in } W^{1,p}(\Omega, \mathbf{R}^N) \qquad \text{with } \mathrm{id}(x) := x,$$

$$\text{(ii)} \quad |\det(\nabla u_k(x))| \leqq 1 \quad \text{on } K,$$

$$\text{(iii)} \quad meas\,\{x \in \partial K : \det(\nabla u_k(x)) \neq 0\} < \frac{1}{2^k}.$$

Proof. – We divide the proof into five steps. We assume without loss of generality that $\Omega = (0, 1)^N$.

First step. We construct the sequence u_k. Let $k \in \mathbf{N}$ be fixed. Using Vitali's Covering Theorem we find two sequences $(x_i^k)_i \subset \partial K$, $(\beta_i^k)_i \subset \left(0, \frac{1}{2^k}\right)$ such that

$$\begin{cases} \partial K \subset \tilde{N}_k \bigcup (\bigcup_{i=1} B(x_i^k, \beta_i^k)), \\ B(x_i^k, \beta_i^k) \cap B(x_j^k, \beta_j^k) = \varnothing \qquad \text{for} \quad i \neq j, \quad i, j = 1, \ldots, \infty, \\ meas\,(\tilde{N}_k) \leqq \dfrac{meas\,(\partial K)}{2^{k+1}}, \end{cases} \tag{3.12}$$

$$\begin{cases} meas\,(\bigcup_{i=1} B(x_i^k, \beta_i^k) \backslash \partial K) \leqq \dfrac{meas\,(\partial K)}{2^{k+1}}, \\ B(x_i^k, \beta_i^k) \subset \Omega \qquad \text{for} \quad i = 1, \ldots, \infty, \end{cases} \tag{3.13}$$

where $B(x, \beta)$ stands for the open ball in $\mathbb{R}^N$ with center x and radius β and $\tilde{N}_k$ is an open set. Since K is a compact set we have

$$\partial K \subset \tilde{N}_k \cup \left(\bigcup_{i=1}^{T(k)} B(x_i^k, \beta_i^k) \right), \tag{3.14}$$

where $T(k)$ is a constant depending on k. Now we want to change the centers x_i^k by other centers which belong to the complementary of K. Using (3.12), (3.13), (3.14) *and the fact that $x_i^k \in \partial K$, we deduce that there are an open set N_k and two sequences $a_i^k \in B(x_i^k, \beta_i^k)\backslash K$, $0 < \varepsilon_i^k < \beta_i^k$, such that*

$$\begin{cases} \partial K \subset N_k \bigcup \left(\bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k) \right), \\ B(a_i^k, \varepsilon_i^k) \subset B(x_j^k, \beta_j^k) \quad i = 1, \dots, T(k), \end{cases} \tag{3.15}$$

$$meas(N_k) \leqq \frac{meas(\partial K)}{2^k}, \tag{3.16}$$

$$meas\left(\bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k)\backslash \partial K \right) \leqq \frac{meas(\partial K)}{2^k}. \tag{3.17}$$

Since $\Omega\backslash K$ is an open set and $a_i^k \in B(x_i^k, \beta_i^k)\backslash K$, there is $\delta_i^k > 0$ such that

$$\delta_i^k < \left(\frac{1}{T(k)(2^k \cdot \varepsilon_i^k)^p} \right)^{\frac{1}{N-p}} \quad i = 1, \dots, T(k) \tag{3.18}$$

and

$$B(a_i^k, \delta_i^k) \subset \Omega\backslash K \quad i = 1, \dots, T(k). \tag{3.19}$$

We define

$$u_k(x) = \begin{cases} a_i^k + \frac{\varepsilon_i^k}{\delta_i^k}(x - a_i^k) & x \in B(a_i^k, \delta_i^k), \\ a_i^k + \frac{\varepsilon_i^k}{|x-a_i^k|}(x - a_i^k) & x \in B(a_i^k, \varepsilon_i^k)\backslash B(a_i^k, \delta_i^k), \\ x & x \in \Omega\backslash \left(\bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k) \right). \end{cases}$$

It is easy to see that u_k is a diffeomorphism from $B(a_i^k, \delta_i^k)$ into $B(a_i^k, \varepsilon_i^k)$ and u_k maps $B(a_i^k, \varepsilon_i^k)\backslash B(a_i^k, \delta_i^k)$ into $\partial B(a_i^k, \varepsilon_i^k)$.

Second step. In this step we show that $u_k \in W^{1,\infty}(\Omega, \mathbf{R}^N)$. As

$$u_k \in C^1(\bar{B}(a_i^k, \delta_i^k), \mathbf{R}^N),$$

$$u_k \in C^1(\bar{B}(a_i^k, \varepsilon_i^k)\backslash B(a_i^k, \delta_i^k), \mathbf{R}^N)$$

and

$$u_k \text{ is continuous on } \bar{B}(a_i^k, \varepsilon_i^k),$$

we have

$$u_k \in W^{1,\infty}(B(a_i^k, \varepsilon_i^k), \mathbf{R}^N) \tag{3.20}$$

and since

$$u_k(x) = x \quad \text{on } \partial B(a_i^k, \varepsilon_i^k) \tag{3.21}$$

we conclude that

$$u_k \in C^0(\Omega, \mathbf{R}^N). \tag{3.22}$$

Using the definition of u_k on $\Omega \backslash \left(\bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k)\right)$ it is obvious that

$$u_k \in W^{1,\infty}\left(\Omega \backslash \bigcup_{i=1}^{T(k)} \bar{B}(a_i^k, \varepsilon_i^k)\right), \tag{3.23}$$

which together with (3.20) and (3.22) yields

$$u_k \in W^{1,\infty}(\Omega, \mathbf{R}^N). \tag{3.24}$$

Third step. We show that, up to a subsequence, $u_k \rightharpoonup u = \text{id}$ in $W^{1,p}(\Omega, \mathbf{R}^N)$. Using the definition of u_k, we obtain:

$$|u_k(x) - x| \leqq \frac{1}{2^k} \quad \text{for every } x \in \Omega \tag{3.25}$$

and

$$\nabla u_k(x) = \begin{cases} \frac{\varepsilon_i^k}{\delta_i^k} I_N & x \in B(a_i^k, \delta_i^k), \\ \frac{\varepsilon_i^k}{|x - a_i^k|}\left(I_N - \frac{(x-a_i^k)\otimes(x-a_i^k)}{|x-a_i^k|^2}\right) & x \in B(a_i^k, \varepsilon_i^k) \backslash B(a_i^k, \delta_i^k), \\ I_N & x \in \Omega \backslash \left(\bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k)\right), \end{cases}$$

where I_N is the identity matrix in $\mathbf{R}^{N \times N}$. If $a, b \in \mathbf{R}^N$, $a \otimes b$ denotes the $N \times N$ matrix with component $a_i b_j$ and $|a| = \sqrt{a_1^2 + \cdots + a_N^2}$. Cleary, there exists a constant $C = C(N)$ such that

$$|\nabla u_k(x)| \leqq \begin{cases} C \frac{\varepsilon_i^k}{\delta_i^k} & x \in B(a_i^k, \delta_i^k), \\ C \frac{\varepsilon_i^k}{|x - a_i^k|} & x \in B(a_i^k, \varepsilon_i^k) \backslash B(a_i^k, \delta_i^k), \\ C & x \in \Omega \backslash \left(\bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k)\right). \end{cases}$$

Thus by (3.17) and (3.18) we have:

$$\int_\Omega |\nabla u_k(x)|^p\,dx \leqq C^p\left(1+\sum_{i=1}^{T(k)}\left(\int_{B(a_i^k,\,\varepsilon_i^k)}\left(\frac{\varepsilon_i^k}{|x-a_i^k|}\right)^p dx+\int_{B(a_i^k,\,\delta_i^k)}\left(\frac{\varepsilon_i^k}{\delta_i^k}\right)^p dx\right)\right.$$

$$\leqq w_N\,C^p\left(1+\sum_{i=1}^{T(k)} N\,\frac{(\varepsilon_i^k)^N}{N-p}+\frac{1}{2^k}\right),$$

where $w_N = meas\,B(0,\,1)$. Recalling that $B(a_i^k,\,\varepsilon_i^k)$ does not intersect $B(a_j^k,\,\varepsilon_j^k)$ for $i\neq j$ and $B(a_i^k,\,\varepsilon_i^k)\subset\Omega=(0,\,1)^N$ we conclude that

$$\int_\Omega |\nabla u_k(x)|^p\,dx \leqq w_N\,C^p\left(1+\frac{N}{w_N(N-p)}+\frac{1}{2^k}\right). \tag{3.26}$$

Therefore $(u_k)_k$ is bounded in $W^{1,p}$ and by (3.25) we deduce that, up to a subsequence,

$$u_k \rightharpoonup u=\mathrm{id}\quad\text{in } W^{1,p}(\Omega,\,\mathbf{R}^N).$$

Fourth step. We show that $|\det(\nabla u_k(x))|\leqq 1$ a.e. on K. Indeed (Σ) implies that

$$\det(\nabla u_k(x))=1\ \text{ a.e. } x\in\Omega\backslash\left(\bigcup_{i=1}^{T(k)} B(a_i^k,\,\varepsilon_i^k)\right). \tag{3.27}$$

We know that $u_k\in C^1(\bar B(a_i^k,\,\varepsilon_i^k)\backslash B(a_i^k,\,\delta_i^k),\,\mathbf{R}^N)$ and

$$|u_k(x)-a_i^k|=\varepsilon_i^k\quad\forall x\in\bar B(a_i^k,\,\varepsilon_i^k)\backslash B(a_i^k,\,\delta_i^k).$$

As u_k is the identity on $\partial B(a_i^k,\,\varepsilon_i^k)$ we obtain

$$u_k(\bar B(a_i^k,\,\varepsilon_i^k)\backslash B(a_i^k,\,\delta_i^k))=\partial B(a_i^k,\,\varepsilon_i^k).$$

Therefore u_k is not locally invertible at any point $x\in B(a_i^k,\,\varepsilon_i^k)\backslash B(a_i^k,\,\delta_i^k)$. We conclude that

$$\det(\nabla u_k(x))=0\ \text{ a.e. } x\in B(a_i^k,\,\varepsilon_i^k)\backslash B(a_i^k,\,\delta_i^k), \tag{3.28}$$

which, together with (3.19) and (3.27) implies that

$$0\leqq\det(\nabla u_k(x))\leqq 1\ \text{ a.e. } x\in K. \tag{3.29}$$

Fifth step. We claim that $meas\,(x\in\partial K:\ \ \det(\nabla u_k(x))\neq 0\}\leqq\dfrac{meas\,(\partial K)}{2^k}$.
By (3.15), (3.19), (3.27) and (3.28) we have

$$\{x\in\partial K:\ \ \det(\nabla u_k(x))\neq 0\}\subset N_k \tag{3.30}$$

and the result follows now from (3.16). ■

Proof of Theorem 3.1. – We prove that (3.10) implies (3.11). Assume that $meas\,(\partial K)\neq 0$. By Lemma 3.3 there exists a sequence $u_k\in W^{1,N}(\Omega,\,\mathbf{R}^N)$ such that:

$$\text{(i)}\quad u_k\rightharpoonup u\quad in\ W^{1,p}(\Omega,\,\mathbf{R}^N),\qquad u(x):=x,$$

$$\text{(ii)}\quad |\det(\nabla u_k(x))|\leqq 1\ \text{ a.e. on } K, \tag{3.31}$$

$$\text{(3.32)} \qquad \text{(iii)} \quad meas\,\{x \in \partial K : \quad \det(\nabla u_k(x)) \neq 0\} < \frac{1}{2^k}.$$

Then (3.31) and (3.32) imply that

$$\int_K |\det(\nabla u_k(x))|\,dx = \int_{\partial K} |\det(\nabla u_k(x))|\,dx + \int_{K\setminus\partial K} |\det(\nabla u_k(x))|\,dx$$
$$\leqq \frac{meas\,(\partial K)}{2^k} + meas\,(K\setminus\partial K)$$

and so

$$\lim_{k\to\infty}\inf \int_K |\det(\nabla u_k(x))|\,dx \leqq meas\,(K\setminus\partial K)$$
$$< meas\,(K) = \int_K |\det(\nabla u(x))|\,dx$$

and we conclude (3.11).

In order to prove that (3.11) implies (3.10), we assume that $meas\,(\partial K) = 0$. It is easy to construct a sequence $a_n \in C^0(\Omega, \mathbf{R}^N)$ such that (*see* [Ga])

$$\text{(3.33)} \qquad a_n(x) \to 1_K(x) \quad \text{a.e.} \quad x \in \Omega,$$

$$\text{(3.34)} \qquad 0 \leqq a_n(x) \leqq a_{n+1}(x) \leqq 1_K(x) \quad \text{a.e.} \quad x \in \Omega.$$

Let u_k, $u \in W^{1,N}(\Omega, \mathbf{R}^N)$ be such that $u_k \rightharpoonup u\, W^{1,p}(\Omega, \mathbf{R}^N)$. Setting in Theorem 2.1

$$a(x, u) \equiv 1, \qquad g(x, \bar{T}, t) = a_n(x)\,|t|,$$

we obtain

$$\int_\Omega a_n(x)\,|\det(\nabla u(x))|\,dx \leqq \lim_{k\to\infty}\inf \int_\Omega a_n(x)\,|\det(\nabla u_k(x))|\,dx$$
$$\leqq \lim_{k\to\infty}\inf \int_K |\det(\nabla u_k(x))|\,dx,$$

for each fixed n. Using (3.33), (3.34) and Fatou's Lemma we conclude that

$$\int_K |\det(\nabla u(x))|\,dx \leqq \lim_{k\to\infty}\inf \int_K |\det(\nabla u_k(x))|\,dx. \quad \blacksquare$$

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REFERENCES

[AF] E. Acerbi, N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Rat Mech. Anal.*, *86*, 1984, pp. 125-145.

[BM] J. M. Ball, F. Murat, Quasiconvexity and variational problems for multiple integrals, *J. Funct. Anal.*, *58*, 1984, pp. 337-403.

[Da] B. Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, 1989.

[DM] B. Dacorogna, P. Marcellini, Semicontinuité pour des intégrandes polyconvexes sans continuité des déterminants, *C. R. Acad. Sci. Paris*, *t. 311*, Série I, 1990, pp. 393-396.

[Ga] W. Gangbo, *Thesis*, Swiss Federal Institute of Technology, 1992.

[Kl] G. Klambauer, *Real analysis*, Elsevier.

[Mal] J. Maly, *Weak lower semicontinuity of polyconvex integrals*, to appear.

[Ma1] P. Marcellini On the definition and the lower semicontiuity of certain quasiconvex integrals, *Ann. Inst. H Poincaré, Anal. Non lin.*, *3*, 1986, pp. 385-392.

[Ma2] P. Marcellini, Approximation of quasiconvex functions and lower semicontinuity of multiple integrals, *Manus. math.*, *51*, 1985, pp. 1-28.

[Mo1] C. B. Morrey, Quasiconvexity and semicontinuity of multiple integrales, *Pacific J. Math.*, *2*, 1952, pp. 25-53.

[Mo2] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer, 1966.

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