L^p Approximation of maps by diffeomorphisms

Yann Brenier

CNRS, Laboratoire Dieudonné et Institut non linéaire de Nice

Parc Valrose, 06108 Nice, France

(en détachement de l'Université Paris 6)

yann.brenier@math.unice.fr

Wilfrid Gangbo*

School of Mathematics

Georgia Institute of Technology

Atlanta, GA 30332, USA

gangbo@math.gatech.edu

September 18, 2001

Abstract

It is shown that if $d \geq 2$, then every map $\phi : \Omega \subset \mathbf{R}^d \to \mathbf{R}^d$ of class L^∞ can be approximated in the L^p-norm by a sequence of orientation-preserving diffeomorphims $\phi_n : \bar{\Omega} \to \phi_n(\bar{\Omega})$. These conclusions hold provided that $\Omega \subset \mathbf{R}^d$ is open, bounded, and that $1 \leq p < +\infty$. In addition, $\phi_n(\bar{\Omega})$ is contained in the $1/n$-neighborhood of the convex hull of $\phi(\Omega)$. All these conclusions fail for $\Omega \subset \mathbf{R}$. The main ingredients of the proof are the polar factorization of maps [4] and an approximation result for measure-preserving maps on the unit cube for which we provide a new proof based on the concept of doubly stochastic measures (corollary 1.5).

*WG gratefully acknowledges the support of National Science Foundation grants DMS-99-70520, and DMS-00-74037.

Contents

Introduction

The purpose of this paper is to prove that every map $\phi : \Omega \to \mathbf{R}^d$ of class L^∞, is the limit in the L^p-norm of a sequence $\phi_n : \bar{\Omega} \to \phi_n(\bar{\Omega})$ of orientation-preserving diffeomorphims. These diffeomorphims can be chosen such that $\phi_n(\bar{\Omega})$ is contained in $[\operatorname{conv}\phi(\Omega)]_{1/n}$. Here, $1 \leq p < +\infty$, $d \geq 2$ and $\Omega \subset \mathbf{R}^d$ is open and bounded. If $A \subset \mathbf{R}^d$ and $\epsilon > 0$, $\operatorname{conv}(A)$ stands for the convex hull of A, and A_ϵ denotes the set of $x \in \mathbf{R}^d$ such that the distance between x and A is less than or equal to ϵ. An analogous result can be readily derived for vector-valued maps ϕ of class $L^p(\Omega)$. These approximation results are interesting from a purely mathematical point of view, but they are also useful in approximating functionals occuring in variational problems (e.g. [16] theorem 3.2). These conclusions fail when $d = 1$. Indeed, a diffeomorphism in $(0, 1)$ is e! ither increasing or decreasing and so, in general a map $\phi \in L^\infty(0, 1)$ cannot be the limit in any L^p-norm of a sequence $\phi_n : [0, 1] \to \phi_n([0, 1])$ of diffeomorphisms.

Our result uses two main ingredients. The first one is the following *approximation result on measure-preserving maps* (corollary 1.5): if $Q \subset \mathbf{R}^d$ is an open cube and $\mathbf{s} : Q \to Q$ is measure-preserving with respect to $\mathcal{H}^d$, the d–dimensional Lebesgue measure, then $\mathbf{s}$ can be approximated in the L^p-norm by a sequence of orientation-preserving diffeomorphisms $\mathbf{s}_n : \bar{Q} \to \bar{Q}$. Furthermore for each n, $\mathbf{s}_n$ is measure-preserving and $\mathbf{s}_n(x) = x$ for all x in a neighborhood of ∂Q. These conclusions hold provided that $d \geq 2$ and $1 \leq p < +\infty$. This result has been known for a while and can be found, presumably for the first time, in A.I. Shnirelman's seminal paper on groups of volume preserving maps [26]. There are many related results on approximations of measure-preserving maps by permutation maps or measure-preserving homeomorphisms. See for instance the works of A. B. Katok [20], P. Lax [21] and A.I. Shnirelman [27]. It is also worthy to mention a result by Fonseca & Tartar [12] asserting that every permutation is the L^p limit of a sequence of measure-preserving diffeomorphisms that leave invariant a neighborhood of the boundary of Q. Our proof differs from the one used by Shnirelman [26] and was introduced in an unpublished lecture notes by the first author [3]. It is based on a classical result by G. Birkhoff [2] that characterizes the extreme points of the set of bistochastic matrices m. To describe our approach and the use of the *Birkhoff theorem*, we first introduce needed terminologies.

A $N \times N$ real-valued matrix is said to be a bistochastic matrix if is entries satisfy $m_{ij} \geq 0$ and

$$\sum_{i=1}^{N} m_{ij_o} = \sum_{j=1}^{N} m_{i_oj} = 1,$$

for all $i_o, j_o = 1, \cdots, N$. By analogy we say that a Borel measure γ on $Q \times Q$ is a bistochastic measure if it has $\mu_1 = \mathcal{H}^d$ and $\mu_2 = \mathcal{H}^d$ as its marginals:

$$\gamma[B \times Q] = \gamma[Q \times B] = \mathcal{H}^d[B].$$

for all $B \subset Q$ Borel. A permutation matrix is a matrix obtained by permuting the rows of the identity matrix. Analogously if n is an integer, we divide $\bar{Q} = [-1/2, 1/2]^d$ into $N := 2^{nd}$ parallel cubes $Q_{n,i}$, of the same size and of center $x_{n,i}$ $(i = 1, \cdots, N)$. To each permutation σ of $\{1, 2, \cdots, N\}$ we associate the *n-permutation map* $\mathbf{p}_\sigma : \bar{Q} \to \bar{Q}$ defined by

$$\mathbf{p}_\sigma(x) = x - x_{n,i} + x_{n,\sigma(i)}, \quad (x \in Q_{n,i}).$$

Let P_n be the set of *n-permutation maps*. We define a *permutation map* to be any element of

$$P := \cup_{n=1}^{\infty} P_n.$$

We say that $\mathbf{p}_\sigma$ is a *transposition of adjacent cubes* whenever σ is also a transposition of two cubes that intersect the same hyperplane. Note that any permutation map can be obtained as a finite composition of *transposition of adjacent cubes*. An improved version of the *Birkhoff theorem* asserts that every $N \times N$ bistochastic matrix is a convex combination of $K \leq N^2$ permutation matrices (see [23] pp 117–119). In theorem 1.4 we use *Birkhoff theorem* to deduce that every bistochastic measure is contained in the weak $*$ closure of the set $\{\mu_p \mid p \text{ permutation}\}$. Here, if $\mathbf{s} : Q \to Q$ is a Borel map, $\mu_{\mathbf{s}}$ is the Borel measure on $\bar{Q} \times \bar{Q}$ defined by

$$\mu_{\mathbf{s}}[B] = \mathcal{H}^d[\{x \in Q \mid (x, \mathbf{s}(x)) \in B\}],$$

for $B \subset \bar{Q}$ Borel. In particular if $\mathbf{s} : Q \to Q$ is measure-preserving with respect to $\mathcal{H}^d$ then $\mathbf{s}$ is contained in the L^p closure of P even if $\mathbf{s}$ in not one-to-one. Similar results were used in [14] (see Proposition A.3). In the sequel we denote by S_Q the set of (Lebesgue) measure-preserving maps from the closed cube $\bar{Q}$ onto itself. The definition of measure-preserving maps is given in Definition 0.2.

The second main ingredient in this work is *the polar factorization of maps*, a result obtained by the first author of this paper [4]. See also [5], [6], [13], [15] and [22] for variants and extensions. The statement on *the polar factorization of maps* is the following. Assume that $\Omega \subset \mathbf{R}^d$ is open, bounded, that $\phi : \Omega \to \mathbf{R}^d$ is of class L^∞ and nondegenerate (see Definition 0.1). Then there exists a Lipschitz continuous convex function $\psi : \mathbf{R}^d \to \mathbf{R}$ and a (Lebesgue) measure-preserving map $\mathbf{s} : \Omega \to \Omega$ such that

$$\phi = (D\psi) \circ \mathbf{s}.$$

Here D stands for the a.e. derivative of a Lipschitz function. One can readily show that $D\psi$ is the limit in $L^p_{loc}(\Omega)$ of a sequence of diffeomorphisms that are orientation-preserving. We conclude with the help of approximation results on measure-preserving maps obtained in Section 1 that $(D\psi) \circ \mathbf{s}$ is in the L^p–closure of the set of diffeomorphisms defined on Ω. In lemma 2.3 we show that every L^∞–map $\bar{\phi} : \Omega \to \mathbf{R}^d$ can be approximated in the L^p norm by a sequence of nondegenerate maps defined on Ω and conclude that $\bar{\phi}$ must be in the L^p–closure of the set of diffeomorphisms defined on Ω. Results parallel to ours were obtained by H.E. White in 1969 [28]. He used approximation lemmas by Morse and Heubsch [18], [19], to conclude that every map ϕ that is *"differentiable in a weak sense"* with a nonnegative jacobian can be approximated by a sequence of diffeomorphisms. Our approach neither overlaps nor is a consequence of White's approach and our conclusions are somehow stronger. We also refer the reader to approximation results in the literature related to dynamical systems, by P.R. Halmos [17].

Notations and definitions

For the convenience of the reader we collect together some of the notations introduced throughout the text.

• If $\Omega \subset \mathbf{R}^d$ then $\overline{\Omega}$ denotes the closure of Ω.

• $B_R(x)$ is the open ball of center x and radius $R > 0$. When $x = 0$ we write B_R instead of $B_R(0)$.

• $\mathcal{H}^d[A]$ stands for the d-dimensional Lebesgue measure of the set $A \subset \mathbf{R}^d$. For $\epsilon > 0$ A_ϵ is the set $\{x \in \mathbf{R}^d \,:\, \mathrm{dist}\,(x, A) \leq \epsilon\}$. The characteristic function of $A \subset \mathbf{R}^d$ is denoted by χ_A.

• If $Q \subset \mathbf{R}^d$ we denote by $\Gamma_Q(\mathcal{H}^d)$ the set of all Borel measure on $\bar{Q} \times \bar{Q}$ such that

$$\gamma[\bar{Q} \times B] = \gamma[B \times \bar{Q}] = \mathcal{H}^d[B],$$

for all Borel $B \subset \bar{Q}$.

• If $\psi : \mathbf{R}^d \to \mathbf{R}$ then the Legendre-Fenchel transform of ψ is the convex, lower semicontinuous function $\psi^* : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\psi^*(y) := \sup_{x \in \mathbf{R}^d} \{x \cdot y - \psi(x)\}. \tag{1}$$

• The subdifferential of a convex function $\psi : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ is the set $\partial\psi \subset \mathbf{R}^d \times \mathbf{R}^d$ consisting of all (x, y) satisfying

$$\psi(z) - \psi(x) \geq y \cdot (z - x), \qquad \forall\, z \in \mathbf{R}^d.$$

If $(x, y) \in \partial\psi$ we may also write $y \in \partial\psi(x)$. Recall $x \in \partial\psi^*(y)$ whenever $y \in \partial\psi(x)$, while the converse also holds true if ψ is convex lower semicontinuous. $\mathrm{domD}\psi$ stands for the set where ψ is differentaible.

• $\mathbf{id}$ stands for the identity map $\mathbf{id}(x) = x$.

• We denote the set of all $d \times d$ matrices whose entries are real numbers by $\mathbf{R}^{d \times d}$.

• If $Q \subset \mathbf{R}^d$ we denote by S_Q the set of measure-preserving maps from $\bar{Q}$ onto itself. We define

$$V_Q := \{\mathbf{v} \in C_o^\infty((0,1) \times Q)^d \,|\, \mathrm{div}(\mathbf{v}) = 0\}.$$

If $\mathbf{v} \in V_Q$, we set $j(\mathbf{v}) := \mathbf{g}(1, \cdot)$ where $\mathbf{g}$ is the unique solution of the initial value problem

$$\begin{cases} \frac{\partial \mathbf{g}}{\partial t}(t, x) &= \mathbf{v}(t, \mathbf{g}(t, x)) \qquad x \in \bar{Q},\ t \in [0, 1] \\ \mathbf{g}(0, x) &= x. \qquad\qquad x \in \bar{Q}. \end{cases} \tag{2}$$

• We define G_Q to be the set of all maps $j(\mathbf{v})$ for $v \in V_Q$. We denote by $\bar{G}_Q^{L^p}$ the closure of G_Q in $L^p(Q)$.

Definition 0.1 *Let $A, B \subset \mathbf{R}^d$. We say that a Borel map $\mathbf{v} : A \to B$ is* nondegenerate *if $\mathcal{H}^d[\mathbf{v}^{-1}(N)] = 0$ whenever $\mathcal{H}^d[N] = 0$.*

Definition 0.2 *Let $A \subset \mathbf{R}^d$ and let $\mathbf{s} : A \to A$ be a Borel map. We say that $\mathbf{s}$ is (Lebesgue)* measure-preserving *if $\mathcal{H}^d[\mathbf{s}^{-1}(B)] = \mathcal{H}^d[B]$ for all Borel sets $B \subset A$.*

The first author would like to thank the Newton Institute of Mathematical Sciences for its hospitality during the writing of this paper. The second author would like to thank A. Swiech who provided fruitful discussions. Both authors are grateful to T. Stoyanov for drawing figures 2 and 3 of this paper. Figure 1 has been reproduced from M. Roesch's PhD dissertation [24].

1 Approximating measure–preserving maps by diffeomorphisms

Throughout this section we assume that $1 \leq p < +\infty$ and that $d \geq 2$.

Lemma 1.1 (Properties of G_Q and $\bar{G}_Q^{L^p}$) *If $Q \subset \mathbf{R}^d$ is a cube then*
(i) G_Q is a group for the usual composition law of maps $\circ$ and $\bar{G}_Q^{L^p}$ is a subset of S_Q which is itself just a semi-group.
(ii) If $\mathbf{s}_1, \mathbf{s}_2 \in \bar{G}_Q^{L^p}$, then $\mathbf{s}_1 \circ \mathbf{s}_2 \in \bar{G}_Q^{L^p}$.

Proof: G_Q is stable for the composition rule. Indeed, assume that two fields $\mathbf{v}_1$, $\mathbf{v}_2$ respectively generate two elements $\mathbf{s}_1 = j(\mathbf{v}_1)$ and $\mathbf{s}_2 = j(\mathbf{v}_2)$. Then $\mathbf{s}_2 \circ \mathbf{s}_1$ is generated by the vector field

$$\mathbf{w}(t, x) = \begin{cases} 2\mathbf{v}_1(2t, x) & \text{if} \quad 0 \leq t \leq 1/2 \\ 2\mathbf{v}_2(2t-1, x) & \text{if} \quad 1/2 \leq t \leq 1, \end{cases}$$

which is still divergence free, smooth and compactly supported in $(0,1) \times Q$. The unit element of G_Q is $j(0)$ and the inverse of $j(v)$ is generated by time reversal of v. From its very definition S_Q is closed for the strong L^p topology, for all $p \geq 1$. Furthermore, it is stable for the composition rule. However it is only a semi-group since many elements are not one-to-one even in the almost everywhere sense. For instance the map $\mathbf{s}$ defined on $\bar{Q} = [-1, +1]^2$ by $\mathbf{s}(x_1, x_2) = (2x_1 \, mod \, 1, x_2)$ is not one-to-one. So, the proof of (i) is complete.

Let us now prove (ii). Suppose now that $\mathbf{s}_1, \mathbf{s}_2 \in \bar{G}_Q^{L^p}$, and $\epsilon > 0$. We choose first $\mathbf{g}_1 \in G_Q$ and then $\mathbf{g}_2 \in G_Q$ such that

$$||\mathbf{g}_1 - \mathbf{s}_1||_{L^p(Q)} < \epsilon/2, \quad ||\mathbf{g}_2 - \mathbf{s}_2||_{L^p(Q)} < \frac{\epsilon}{2Lip(\mathbf{g}_1)}. \tag{3}$$

By the triangle inequality we have that

$$||\mathbf{s}_1 \circ \mathbf{s}_2 - \mathbf{g}_1 \circ \mathbf{g}_2||_{L^p(Q)} \leq ||\mathbf{s}_1 \circ \mathbf{s}_2 - \mathbf{g}_1 \circ \mathbf{s}_2||_{L^p(Q)} + ||\mathbf{g}_1 \circ \mathbf{s}_2 - \mathbf{g}_1 \circ \mathbf{g}_2||_{L^p(Q)}.$$

This, together with the fact that by (i) $\mathbf{s}_2 \in \bar{G}_Q^{L^p} \subset S_Q$ implies that

$$||\mathbf{s}_1 \circ \mathbf{s}_2 - \mathbf{g}_1 \circ \mathbf{g}_2||_{L^p(Q)} \leq ||\mathbf{s}_1 - \mathbf{g}_1||_{L^p(Q)} + Lip(\mathbf{g}_1)||\mathbf{s}_2 - \mathbf{g}_2||_{L^p(Q)}. \tag{4}$$

Combining (3), and (4) we deduce that

$$||\mathbf{s}_1 \circ \mathbf{s}_2 - \mathbf{g}_1 \circ \mathbf{g}_2||_{L^p(Q)} < \epsilon. \tag{5}$$

Since $\epsilon > 0$ is any arbitrary number in (5), and (i) gives that $\mathbf{g}_1 \circ \mathbf{g}_2 \in G_Q$, we conclude the proof of (ii). QED.

Lemma 1.2 (A special diffeomorphism) *If* $Q := [-1, +1]^2$ *then* $\mathbf{s}_o : \bar{Q} \to \bar{Q}$ *defined by* $\mathbf{s}_o(x) = -x$*, belongs to* $\bar{G}_Q^{L^p}$.

Proof: Observe that Q can be expressed in polar coordinates (r, θ) as

$$Q = \{(r\cos\theta, r\sin\theta),\ r^2 f(\theta) \leq 2\}$$

where

$$f(\theta) = 2\max(\cos^2\theta, \sin^2\theta) = 1 + |\cos(2\theta)|.$$

Let us approximate f by

$$f_\epsilon(\theta) = 1 + \sqrt{\epsilon^2 + \cos^2(2\theta)} > f(\theta)$$

and define

$$Q_\epsilon = \{(r\cos\theta, r\sin\theta),\ \ \epsilon < r^2 f_\epsilon(\theta) < 2\},$$

which is an open subset of the interior of Q differing from Q by a set of vanishing Lebesgue measure as ϵ approaches 0. For each fixed ϵ, we can choose a smooth function ψ_ϵ compactly supported in the interior of Q such that

$$\psi_\epsilon(x) = \lambda_\epsilon \frac{r^2}{2} f_\epsilon(\theta), \quad \forall x \in Q_\epsilon,$$

where

$$\lambda_\epsilon = \int_0^\pi \frac{d\theta}{f_\epsilon(\theta)} > 0.$$

Then,

$$v_\epsilon(x) = (-\partial_{x_2}\psi_\epsilon(x), \partial_{x_1}\psi_\epsilon(x))$$

define a smooth divergence free vector field compactly supported in the interior of Q. Let us integrate the ODE

$$x'(t) = v_\epsilon(x(t)), \;\; x(0) = x_0.$$

Since ψ_ϵ is preserved along each trajectory, Q_ϵ is an invariant domain. Thus for each initial point $x_0 \in Q_\epsilon$, the solution $x(t)$, written in polar coordinates $(r(t), \theta(t))$, satisfies

$$r'(t) = -\lambda_\epsilon \frac{r}{2} f'_\epsilon(\theta(t)), \quad \theta'(t) = \lambda_\epsilon f_\epsilon(\theta(t)).$$

(Indeed the rotated gradient of ψ_ϵ and $x'(t)$ can be respectively written in polar coordinates $(-r\partial_\theta\psi_\epsilon, \partial_r\psi_\epsilon)$ and $(r'(t), r(t)\theta'(t))$.) So, in polar coordinates, the ODE decouples. The angle $\theta(t)$ can be solved, as a monotonic function of t, by the simple quadrature

$$\int_{\theta(0)}^{\theta(t)} \frac{d\phi}{f_\epsilon(\phi)} = \lambda_\epsilon t.$$

Thus, using the definition of λ_ϵ and the $\pi-$ periodicity of f_ϵ, we deduce that $\theta(t = 1) = \theta(0) + \pi$. Next, using the conservation of ψ_ϵ, we get,

$$r^2(t) f_\epsilon(\theta(t)) = r^2(0) f_\epsilon(\theta(0)),$$

and, therefore, $r(t = 1) = r(t = 0)$. So, the map $j(v_\epsilon)(x)$, generated by v_ϵ at time $t = 1$, which belongs to G by construction, does not differ from $-x$ on Q_ϵ. Since the measure of Q minus Q_ϵ vanishes with ϵ, it follows that the map $-x$ on Q belongs to the (strong) L^p closure of G for all $1 \leq p < +\infty$. QED.

Lemma 1.3 (Approximating permutations) *We have that*
(i) Every ϕ_σ transposition of adjacent cubes *belongs to* $\bar{G}_Q^{L^p}$*.*
(ii) Every permutation ϕ_σ *belongs to* $\bar{G}_Q^{L^p}$*.*

Proof: Let ϕ_σ be a *transposition of adjacent cubes.* By rotating coordinates and translating the origin if necessary, we may substitute Q by the cube

$$C := [-1,1] \times [0,1]^{d-1}$$

and set

$$\phi_\sigma(x) = \begin{cases} x + \mathbf{e}_1 & \text{if} \quad x \in C_1 \\ x - \mathbf{e}_1 & \text{if} \quad x \in C_2 \\ x & \text{if} \quad x \notin C_1 \cup C_2. \end{cases}$$

Here $\mathbf{e}_1 := (1, 0, \cdots, 0)$,

$$C_1 := [-1,0] \times [0,1]^{d-1} \quad \text{and } C_1 := [0,1] \times [0,1]^{d-1}.$$

Since we have reduced the proof of (i) to the particular case where $d = 2$ we assume in the sequel that $d = 2$. Let $\mathbf{s}$ be the map defined on $C = [-1,1] \times [0,1]$ by

$$\frac{\mathbf{s}(x) + x}{2} = O := (0, 1/2).$$

We call $\mathbf{s}$ the central symmetry of center O.

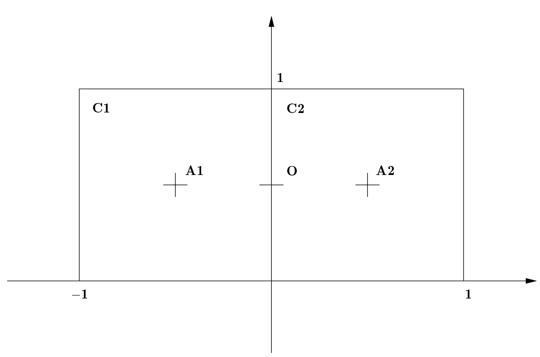

Fig 1

Let $\mathbf{s}_1$ be the unique map defined over C, whose restriction to C_2 coincides with the identity map and whose restriction to C_1 coincides with the central symmetry of center $A_1 := (-1/2, 1/2)$. Similarly, we define $\mathbf{s}_2$ to be the unique map defined over C, whose restriction to C_1 coincides with the identity map and whose restriction to C_2 coincides with the symmetry of center $A_2 := (1/2, 1/2)$. By lemma 1.2 we have that

$$\mathbf{s} \in \bar{G}_C^{L^p}, \quad \mathbf{s}_1 \in \bar{G}_{C_1}^{L^p} \subset \bar{G}_C^{L^p}, \quad \mathbf{s}_2 \in \bar{G}_{C_2}^{L^p} \subset \bar{G}_C^{L^p}.$$

Hence, using lemma 1.1 (ii) and the fact that $\phi_\sigma = \mathbf{s}_2 \circ \mathbf{s}_1 \circ \mathbf{s}$, we obtain that ϕ_σ belongs to $\bar{G}_C^{L^p}$. This concludes the proof of (i).

Assume that $\sigma : \{1, 2, \cdots N\} \to \{1, 2, \cdots N\}$. Then, ϕ_σ is a finite composition of *transposition of adjacent cubes*. Using lemma 1.1 and (i) we conclude that ϕ_σ belongs to $\bar{G}_Q^{L^p}$. QED.

We recall that $\Gamma_Q(\mathcal{H}^d)$ is the set of all bistochastic measures on $\bar{Q} \times \bar{Q}$. For each measure preserving map $s \in S_Q$, we define a corresponding bistochastic measure $\mu_\mathbf{s}$ by

$$\mu_\mathbf{s}[B] := \mathcal{H}^d[\{x \in Q \,|\, (x, \mathbf{s}(x)) \in B\}],$$

for every Borel set $B \subset \bar{Q} \times \bar{Q}$.

Theorem 1.4 (Approximation of bistochastic measures) *(i) For every $\mu \in \Gamma_Q(\mathcal{H}^d)$ there exists a sequence $\{\mathbf{p}_n\}_{n=1}^\infty \subset P$ such that $\mu_{\mathbf{p}_n}$ converges weak $*$ to μ as n tends to $+\infty$. In other words*

$$\lim_{n \to +\infty} \int_Q f(x, \mathbf{p}_n(x))dx = \int_{Q \times Q} f(x, y) d\mu(x, y),$$

for all $f \in C(\bar{Q} \times \bar{Q})$.
(ii) In particular if $\mathbf{s} \in S_Q$, then there exists a sequence $\{\mathbf{p}_n\}_{n=1}^\infty \subset P$ that converges to $\mathbf{s}$ in $L^p(Q)$.

Proof: Let m be an integer and divide Q into $N_m := 2^{md}$ parallel cubes $Q_{m,i}$, of same volumes $\mathcal{H}^d[Q_{m,i}] = 1/N_m$ and of centers $x_{m,i}$. The measure

$$\gamma_m := \sum_{i,j=1}^{N_m} \nu_{i,j} \mu[Q_{m,i} \times Q_{m,j}] \delta_{(x_{m,i}, x_{m,j})}$$

that approximates μ, as m tends to $+\infty$ will be identified with the $N_m \times N_m$ matrix ν defined by

$$\nu_{i,j} := N_m \mu[Q_{m,i} \times Q_{m,j}].$$

Observe that ν is a bistochastic matrix, i.e., $\nu_{i,j} \geq 0$ and

$$\sum_{i=1}^{N_m} \nu_{i,j_o} = \sum_{j=1}^{N_m} \nu_{i_o,j} = 1$$

for each $i_o, j_o = 1, \cdots, N_m$. By *Birkhoff theorem* there exists an integer $K \leq N_m^2$, depending on N_m, such that ν can be written as a convex combination of K permutation matrices (see [23] pp 117–119). Hence, there exist non-negative numbers $\theta_1, \cdots \theta_K$ and permutations $\sigma_1, \cdots, \sigma_K : \{1, 2, \cdots N_m\} \to \{1, 2, \cdots N_m\}$ such that

$$\nu_{i,j} = \sum_{k=1}^{K} \theta_k \delta_{\sigma_k(i),j}, \quad \sum_{k=1}^{K} \theta_k = 1.$$

Let $[\cdot]$ be the greatest integer function. To substitute θ_k by rational numbers, we choose $L := 2^{ld} > N_m$, where l will be specified later and choose $\epsilon_k \in \{0, 1\}$ such that the rational numbers

$$\theta'_k := \frac{[L\theta_k] + \epsilon_k}{L},$$

satisfy

$$\sum_{k=1}^{K} \theta'_k = 1 \quad \text{and} \ \sup_k |\theta'_k - \theta_k| \leq 1/L.$$

Define the matrix ν' whose entries are

$$\nu'_{i,j} := \sum_{k=1}^{K} \theta'_k \delta_{\sigma_k(i),j}. \tag{6}$$

Note that

$$\sum_{i,j} |\nu_{i,j} - \nu'_{i,j}| \leq \frac{KN_m}{L}.$$

Up to a relabelling of the list of permutations with possible repetitions, we may assume that all coefficients θ'_k to be equal to $1/L$ and get a new expression

$$\nu'_{i,j} := \sum_{k=1}^{K} \theta'_k \delta_{\sigma_k(i),j}.$$

We subdivise each $Q_{m,i}$ into cubes $Q_{m+l,i,m'}$ of centers $x_{m+l,i,m'}$ and of the same volume $2^{-(m+l)d}$, where, $i = 1, \cdots, N_m$ and $m' = 1, \cdots, L$. Then for m and l fixed, we define the map $\mathbf{p}_{m,l} : \bar{Q} \to \bar{Q}$ by

$$\mathbf{p}_{m,l}(x) = x - x_{m+l,i,m'} + x_{m+l,\sigma_{m'}(i),m'}, \qquad x \in Q_{m+l,i,m'}.$$

It is straightforward to check that $(i, m') \to (\sigma_{m'}(i), m')$ is one-to-one, and so, $\mathbf{p}_{m,l} \in P_{m+l}$ holds. Let $f \in C(\bar{Q} \times \bar{Q})$. We have to estimate $I_1 - I_2$, where

$$I_1 := \int_{Q\times Q} f(x, y) d\mu(x, y), \quad I_2 := \int_Q f(x, \mathbf{p}_{m,l}(x)) dx.$$

We have that

$$|I_1 - I_2| \le |I_1 - I_3| + |I_3 - I_4| + |I_4 - I_5| + |I_5 - I_2|,$$

where,

$$I_3 := \frac{1}{N_m} \sum_{i,j=1}^{N_m} f(x_{m,i}, x_{m,j}) \nu_{i,j}, \quad I_4 := \frac{1}{N_m} \sum_{i,j=1}^{N_m} f(x_{m,i}, x_{m,j}) \nu'_{i,j},$$

$$I_5 := \frac{1}{N_m L} \sum_{i=1}^{N_m} \sum_{m'=1}^{L} f(x_{m+l,i,m'}, x_{m+l,\sigma_{m'}(i),m'}).$$

Let η be the modulus of continuity of f. We have that

$$|I_1 - I_3| \le \eta(2^{-m+d/2}), \quad |I_3 - I_4| \le ||f||_{C(\bar{Q}^2)} \frac{K}{L},$$

$$|I_4 - I_5| \le \eta(2^{-m+d/2}), \quad |I_5 - I_2| \le \eta(2^{-m-l+d/2}).$$

We have shown that

$$|I_1 - I_2| \le ||f||_{C(\bar{Q}^2)} \eta(2^{(2m-l)d}) + 3\eta(2^{-m+d/2}), \tag{7}$$

because $L = 2^{ld}$ and $K = N_m^2 = 2^{2md}$. Given $\epsilon > 0$ we may choose first l and then m large enough in (7) so that $|I_1 - I_2| \le \epsilon$. Reordering the set $\{\mathbf{p}_{m,l}\}_{m,l}$ we have shown that there exists a sequence $\{\mathbf{p}_n\}_{n=1}^\infty \subset P$ such that $\mu_{\mathbf{p}_n}$ converges weak $*$ to μ as n tends to $+\infty$. This concludes the proof of (i).

To prove (ii), we use that by (i) there exists a sequence $\{\mathbf{p}_n\}_{n=1}^\infty \subset P$ which converges weak $*$ to $\mu_{\mathbf{s}}$. Since the sequence $\{\mathbf{p}_n\}_{n=1}^\infty$ is bounded in

$L^\infty(Q)$, then its converges to $\mathbf{s}$ in $L^p(Q)$ if and only if its converges to $\mathbf{s}$ in $L^2(Q)$. Note that when $f(x,y)$ is of the form $g(x)\cdot y$, the fact that

$$\lim_{n\to+\infty}\int_Q f(x,\mathbf{p}_n(x))dx = \int_{Q\times Q} f(x,y)d\mu_{\mathbf{s}}(x,y)$$

implies that $\{\mathbf{p}_n\}_{n=1}^\infty \subset P$ converges weakly to $\mathbf{s}$ in $L^2(Q)$. So, exploiting the fact that $\mathbf{p}_n, \mathbf{s} \in S_Q$, we deduce that

$$\lim_{n\to+\infty} ||\mathbf{p}_n - \mathbf{s}||_{L^2(Q)} = 2\lim_{n\to+\infty}\int_Q \mathbf{s}\cdot(\mathbf{s}-\mathbf{p}_n)dx = 0. \tag{8}$$

This conclude the proof of (ii). QED.

Example 1: The following figure illustrates how a (Lebesgue) measure-preserving map $\mathbf{s}_o : [-1/2, 1/2]^d \to [-1/2, 1/2]^d$ can be approximated by a sequence of permutation maps in the case $d = 1$. Note that although $\mathbf{s}_o$ fails to be one-to-one, the permutation map $\mathbf{p}_m$ is one-to-one.

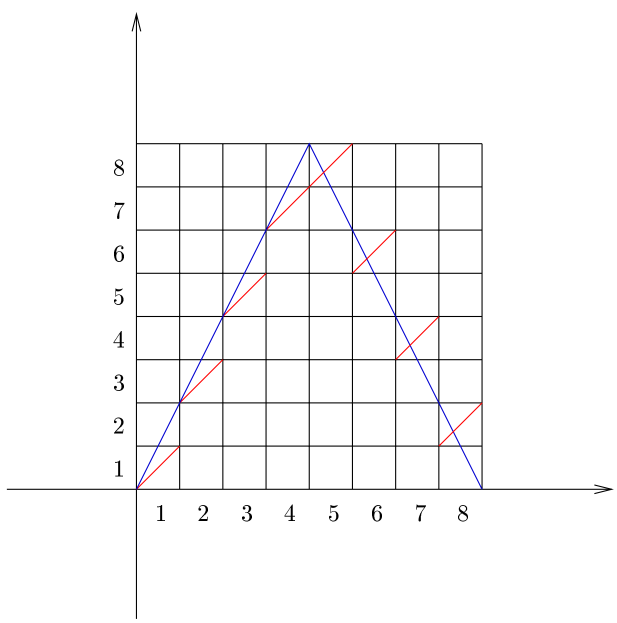

Fig 2: The blue graph represents the map $\mathbf{s}_o$ and the red

one represents a permutation map $\mathbf{p}_m$ that approximates $\mathbf{s}_o$.

Example 2: The following figure illustrates how a bistochastic measure γ_o defined on $[-1/2, 1/2]^d \times [-1/2, 1/2]^d$ can be approximated by a sequence of permutation maps $\{\mathbf{p}_m\}$ in the case $d = 1$. Note that permutation maps are one-to-one although the support of γ_o does not lie on the graph of a map.

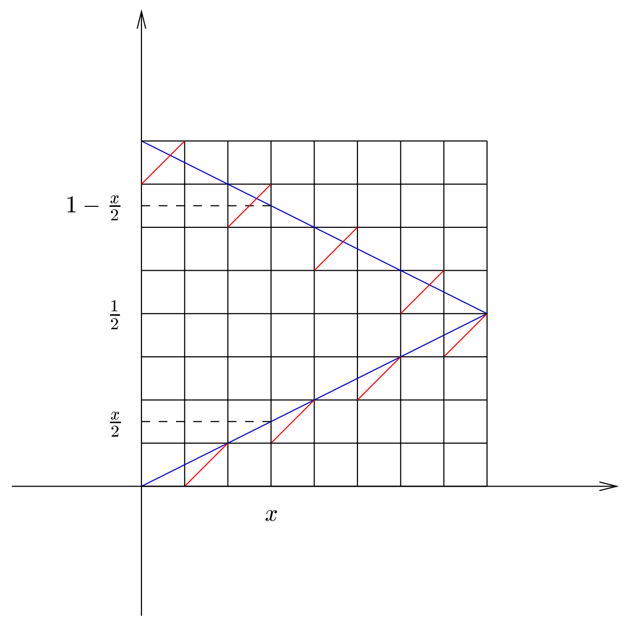

Fig 3: The blue graph represents the measure γ_o which splits masses at each point of $[-1/2, 1/2]$ into two equal masses. The red graph represents a permutation map $\mathbf{p}_o$ such that $\mu_{\mathbf{p}_o}$ approximates γ_o.

Corollary 1.5 *Suppose that $Q \subset \mathbf{R}^d$ is an open cube and that $1 \leq p < +\infty$. Then for every measure-preserving map $\mathbf{s} \in S_Q$ and every integer $n > 0$, there exists a map $\tilde{\mathbf{s}} \in G_Q$ such that $||\mathbf{s} - \tilde{\mathbf{s}}||_{L^p(Q)} \leq 1/n$. In other words we have that $\bar{G}_Q^{L^p} = S_Q$.*

Proof: Let us denote by $\bar{P}^{L^p}$ the closure of P in the L^p-norm. In light of lemma 1.3 and theorem 1.4 we have that $P \subset \bar{G}_Q^{L^p}$ and that $S_Q \subset \bar{P}^{L^p}$. This proves that $S_Q \subset \bar{G}_Q^{L^p}$. The reverse inequality is a direct consequence of the fact that $G_Q \subset S_Q$ and that S_Q is closed in the L^p-norm. QED.

2 Approximation of L^∞ maps by diffeomorphisms

Throughout this section we assume that $\Omega \subset \mathbf{R}^d$ is an open, bounded set.

Lemma 2.1 (Approximation of convex functions) *Suppose that $K \subset B_R \subset \mathbf{R}^d$ is the closure of a convex, bounded and open set. Suppose that $\psi : \mathbf{R}^d \to \mathbf{R}$ is convex and that $1 \leq p < +\infty$. Then there exists a family $\{\psi_\epsilon\}_{\epsilon>0}$ of convex functions such that*
(i) ψ_ϵ, $\psi_\epsilon^* \in C^\infty(\mathbf{R}^d)$.
(ii) ψ_ϵ converges to ψ in $C(K)$ and in $W^{1,p}(int(K))$.
(iii) $\partial\psi_\epsilon(K) \subset [conv\partial\psi(K_\epsilon)]_{R\epsilon}$.

Proof: We first observe that since ψ is convex and assumes only finite values on $\mathbf{R}^d$ then ψ is continuous on $\mathbf{R}^d$. Define

$$\rho_\epsilon(x) := \frac{1}{\epsilon^d}\rho(\frac{x}{\epsilon}) \tag{9}$$

where $\rho \in C_o^\infty(B_1(O))$ is a nonnegative, radial function such that $\operatorname{spt}(\rho) = \bar{B}_1(O)$ and $\int_{\mathbf{R}^d} \rho dx = 1$. Set

$$\psi_\epsilon(x) := \rho_\epsilon * \psi(x) + \epsilon||x||^2/2 \qquad (x \in \mathbf{R}^d).$$

1. Note that since $\rho_\epsilon * \psi \in C^\infty(\mathbf{R}^d)$ is convex then $\psi_\epsilon \in C^\infty(\mathbf{R}^d)$ is strictly convex. Because the eigenvalues of $D^2(\rho_\epsilon * \psi)$ are nonnegative, we readily deduce that the eigenvalues of the matrix $D^2\psi_\epsilon$ are greater than or equal to ϵ. This proves that $D\psi_\epsilon$ is one-to-one on $\mathbf{R}^d$ and

$$\epsilon^d \leq \det(\mathrm{D}^2\psi_\epsilon). \tag{10}$$

It is easy to check that $\mathrm{dom}\psi_\epsilon^* = \mathbf{R}^\mathrm{d}$ and so, using the fact that ψ_ϵ is strictly convex, we deduce that $\mathrm{domD}\psi_\epsilon^* = \mathbf{R}^\mathrm{d}$ and that

$$D\psi_\epsilon^* \circ D\psi_\epsilon = \mathbf{id}. \tag{11}$$

If $y \in \mathbf{R}^d$, setting $x := D\psi_\epsilon^*(y)$ we have that $y \in \partial\psi_\epsilon(x)$, which together with the fact that $\psi_\epsilon \in C^1(\mathbf{R}^d)$ implies $y = D\psi_\epsilon(x)$. This proves that $D\psi_\epsilon$ is surjective. So, we conclude that $D\psi_\epsilon$ and $D\psi_\epsilon^*$ are two homeomorphisms of $\mathbf{R}^d$ onto $\mathbf{R}^d$ that are inverse of each other. We now use the fact that $\psi_\epsilon \in C^\infty(\mathbf{R}^d)$, that $D\psi_\epsilon^* \in C^o(\mathbf{R}^d)$ and (10) to obtain that the function

$$y \to \mathbf{A}(y) := \frac{cof D^2\psi_\epsilon(D\psi_\epsilon^*(y))}{\det(\mathrm{D}^2\psi_\epsilon)(\mathrm{D}\psi_\epsilon^*(\mathrm{y}))}$$

is continuous on $\mathbf{R}^d$. In addition

$$D^2\psi_\epsilon^*(y) = \mathbf{A}^T(y). \tag{12}$$

(See [1], or [11] theorem 6.1). Consequently, $\psi_\epsilon^* \in C^2(\mathbf{R}^d)$. So, using (10) and (12), we inductively deduce that $\psi_\epsilon^* \in C^\infty(\mathbf{R}^d)$. This concludes the proof of (i).

2. Because ψ is continuous on $\mathbf{R}^d$ we have that

$$||\psi||_{L^1(B_{2R})} < +\infty$$

and that $\{\rho_\epsilon * \psi\}_{\epsilon>0}$ converges uniformly to ψ on compact subsets of $\mathbf{R}^d$. Consequently, $\{\psi_\epsilon\}_{\epsilon>0}$ converges uniformly to ψ on compact subsets of $\mathbf{R}^d$, and the constant $k > 0$ defined by

$$k := sup_{\epsilon\in(0,1)}||\psi_\epsilon||_{L^1(B_{2R})}$$

is finite. We deduce that there exists a constant c_d depending only on the dimension d such that

$$\sup_{\epsilon>0}\{||\psi_\epsilon||_{L^\infty(B_R)}\} \leq c_d\frac{||\psi_\epsilon||_{L^1(B_{2R})}}{\mathcal{H}^d(B_{2R})} \leq \frac{kc_d}{\mathcal{H}^d(B_{2R})}. \tag{13}$$

(See [10] pp 236). Similarly,

$$\sup_{\epsilon>0}\{||D\psi_\epsilon||_{L^\infty(B_R)}\} \leq c_d\frac{||\psi_\epsilon||_{L^1(B_{2R})}}{2R\mathcal{H}^d(B_{2R})} \leq \frac{kc_d}{2R\mathcal{H}^d(B_{2R})}. \tag{14}$$

Because ψ_ϵ is convex we have that $D^2\psi_\epsilon$ is a nonnegative definite matrix, and so, $\triangle\psi_\epsilon \geq 0$. We combine (14) and the divergence theorem to deduce the following: there exists a constant k_{rd} independant of ϵ such that

$$\int_{B_{2R}} |\triangle\psi_\epsilon(x)|dx = \int_{B_{2R}} \triangle\psi_\epsilon(x)dx = \int_{\partial B_{2R}} \tau \cdot D\psi_\epsilon(\tau)d\tau \leq k_{rd}, \tag{15}$$

for all $\epsilon \in (0,1)$. This yields that

$$||D^2\psi_\epsilon||_{L^1(B_{2R})} \leq k_{rd}, \tag{16}$$

for all $\epsilon \in (0,1)$. Combining (13), (14) and (16) with the Sobolev imbedding theorems, we deduce the proof of (ii).

3. Let $x_o \in K$. Note that for each $\eta > 0$ we have that

$$\partial\psi(B_\epsilon(x_o)) \subset \partial\psi(K_\epsilon) \subset \mathcal{C}_\eta, \tag{17}$$

where

$$\mathcal{C}_\eta := [conv(\partial\psi(K_\epsilon))]_\eta.$$

The interior of $\mathcal{C}_\eta$ is not empty, and so, translating coordinates if necessary, we may assume that 0 is contained in the interior of $\mathcal{C}_\eta$. Define the *gauge*

$$g_{\mathcal{C}_\eta}(z) = \inf\{\lambda \mid \lambda > 0, z \in \lambda\mathcal{C}_\eta\}, \qquad (z \in \mathbf{R}^d).$$

Because $\mathcal{C}_\eta$ is a convex set, the function $g_{\mathcal{C}_\eta}$ is convex, homogeneous of degree one. Since K_ϵ is a compact set, so are $\psi(K_\epsilon)$ and $\mathcal{C}_\eta$. As a consequence $\mathcal{C}_\eta$ is the unit closed ball of $\mathbf{R}^d$ with respect to the norm $g_{\mathcal{C}_\eta}$, i.e.,

$$\mathcal{C}_\eta = \{z \in \mathbf{R}^d \mid g_{\mathcal{C}_\eta}(z) \leq 1\}.$$

It suffices to show that $g_{\mathcal{C}_\eta}(D(\rho_\epsilon * \psi)(x_o)) \leq 1$ and let η go to 0 to conclude the proof of (iii). We use (17) and Jensen's inequality. We use the fact that $\text{spt}\,(\rho_\epsilon) = B_\epsilon$ and that $g_{\mathcal{C}_\eta}$ is homogeneous of degree one to obtain that

$$g_{\mathcal{C}_\eta}(D(\rho_\epsilon * \psi)(x_o)) \leq \int_{B_\epsilon(x_o)} \rho_\epsilon(x_o - y) g_{\mathcal{C}_\eta}(D\psi(y))dy \leq 1.$$

This is the needed inequality that enable us to conclude the proof of (iii). QED.

We now recall an elementary result of measure theory and skip its proof which can be found in [25].

Lemma 2.2 *Suppose that $\phi : \Omega \to \mathbf{R}^d$ is a Borel map and that $M := ||\phi||_{L^\infty(\Omega)} < +\infty$. Let $\epsilon > 0$. Then there exists a Borel map $\phi_\epsilon \in L^\infty(\Omega)^d$ such that the cardinality of $\phi_\epsilon(\Omega)$ is finite, $||\phi_\epsilon - \phi||_{L^\infty(\Omega)} \leq \epsilon$, and $\phi_\epsilon(\Omega) \subset \phi(\Omega)$.*

Lemma 2.3 *Suppose that $\phi : \Omega \to \mathbf{R}^d$ is a Borel map and that $\phi(\Omega) = \{\mathbf{a}_1, \cdots, \mathbf{a}_k\}$. Set $M := \sup_{x\in\Omega}\{||x||\}$. Then there exists a positive real number ϵ_o depending only on $\phi(\Omega)$, and $diam(\Omega)$, there exists $\phi_\epsilon \in L^\infty(\Omega)^d$ such that*
(i) $||\phi_\epsilon - \phi||_{L^\infty(\Omega)} \leq \epsilon diam(\Omega)$, for every $\epsilon \in (0, \epsilon_o)$.
(ii) $\phi_\epsilon(\Omega) \subset [\phi(\Omega)]_{\epsilon M}$, for every $\epsilon \in (0, \epsilon_o)$.
(iii) ϕ_ϵ is Borel measurable, nondegenerate and one-to-one.

Proof: The proof is simple and is done as follows. Define the real number

$$\epsilon_o := \frac{\min_{i\neq j} ||\mathbf{a}_i - \mathbf{a}_j||}{4diam(\Omega)},$$

the sets

$$A_i := \phi^{-1}\{\mathbf{a}_i\} \qquad (i = 1, \cdots, k)$$

and the maps

$$\phi_\epsilon := \phi + \epsilon\mathbf{id}.$$

It is straightforward to check that ϕ_ϵ satisfies (i) and (ii). Also, observe that ϕ_ϵ is one-to-one on A_i and

$$||\phi_\epsilon(x) - \phi_\epsilon(y)|| \geq \epsilon_o diam(\Omega)$$

whenever $i \neq j$, $x \in A_i$ and $y \in A_j$. This proves that ϕ_ϵ is one-to-one on Ω.

We next claim that ϕ_ϵ is nondegenerate. Indeed, let $B \subset \mathbf{R}^d$ be Borel measurable. Using the fact that ϕ_ϵ is one-to-one on Ω and that

$$\phi_\epsilon^{-1}(B) = \cup_{i=1}^k \phi_\epsilon^{-1}(B \cap (\mathbf{a}_i + \epsilon A_i)),$$

we deduce that

$$\mathcal{H}^d[\phi_\epsilon^{-1}(B)] = \sum_{i=1}^k \mathcal{H}^d[\phi_\epsilon^{-1}(B \cap (\mathbf{a}_i + \epsilon A_i))] = \frac{1}{\epsilon^d}\sum_{i=1}^k \mathcal{H}^d[B \cap (\mathbf{a}_i + \epsilon A_i)].$$

Note that from the above calculations if $\mathcal{H}^d[B] = 0$ then $\mathcal{H}^d[\phi_\epsilon^{-1}(B)] = 0$. Hence, ϕ_ϵ is nondegenerate. QED.

Lemma 2.4 *Suppose that $\phi \in L^\infty(\Omega)^d$ is a Borel, nondegenerate map and that $1 \leq p < +\infty$. Then for each $\epsilon > 0$ there exists a diffeomorphism $\phi_\epsilon : \bar{\Omega} \to \phi_\epsilon(\bar{\Omega})$ such that*
(i) $||\phi_\epsilon - \phi||_{L^p(\Omega)} \leq \epsilon$
(ii) $\phi_\epsilon(\bar{\Omega}) \subset [conv\overline{\phi(\Omega)}]_\epsilon$.

Proof: Since $\phi \in L^\infty(\Omega)^d$ is nondegenerate applying polar factorization of maps result in [4] to ϕ we deduce that there exists a measure preserving map $\bar{\mathbf{s}} : \Omega \to \Omega$ and a convex function $\psi : \mathbf{R}^d \to \mathbf{R}$ such that

$$\partial\psi(\mathbf{R}^d) \subset conv(\overline{\phi(\Omega)}) \tag{18}$$

and

$$\phi(x) = D\psi(\bar{\mathbf{s}}(x)),$$

for a.e. $x \in \Omega$. Let Q be an open cube large enough such that $\bar{\Omega}, \overline{\phi(\Omega)} \subset Q$. Define $\mathbf{s}$ on $\bar{Q}$ by

$$\mathbf{s}(x) = \begin{cases} \bar{\mathbf{s}}(x) & \text{if} \quad x \in \Omega \\ x & \text{if} \quad x \in \bar{Q} \setminus \Omega. \end{cases}$$

Note that $\mathbf{s} : \bar{Q} \to \bar{Q}$ is measure preserving and so, in light of corollary 1.5 there exist a set $N_1 \subset \mathbf{R}^d$ of null Lebesgue measure and a sequence $(\mathbf{s}_n)$ of measure-preserving diffeomorphisms of $\bar{Q}$ onto $\bar{Q}$ such that $\mathrm{det}\mathrm{D}\mathbf{s}_\mathrm{n} \equiv 1$ and

$$\mathbf{s}_n(x) \to \mathbf{s}(x) \tag{19}$$

for every $x \in Q \setminus N_1$, as n tends to $+\infty$. Thanks to lemma 2.1 and (18) we deduce that there exists a sequence (ψ_n) of convex function such that $\psi_n \in C^\infty(\mathbf{R}^d)$, $D\psi_n$ is a diffeomorphism of $\mathbf{R}^d$ onto $\mathbf{R}^d$. Furthermore,

$$\psi_n \to \psi \text{ in } C(\bar{Q}) \text{ and in } W^{1,p}(Q) \tag{20}$$

and

$$\partial\psi_n(\bar{Q}) \subset [\psi(\bar{Q}_{1/n})]_{1/n}.$$

This, together with (18) implies that

$$\partial\psi_n(\bar{Q}) \subset [conv\phi(\bar{\Omega})]_{1/n} \subset Q. \tag{21}$$

Set

$$\bar{\phi}_n(x) := D\psi_n(\mathbf{s}_n(x)) \qquad (x \in \bar{\Omega}).$$

We have that $\bar{\phi}_n$ is a diffeomorphism of $\bar{\Omega}$ onto $\phi_n(\bar{\Omega})$. In light of (21) we conclude that

$$\phi_n(\bar{\Omega}) \subset [conv\overline{\phi(\Omega)}]_{1/n} \subset Q. \tag{22}$$

Using the fact that

$$||\bar{\phi}_n - \phi||_{L^p(\Omega)} \leq ||D\psi_n \circ \mathbf{s}_n - D\psi \circ \mathbf{s}_n||_{L^p(\Omega)} + ||D\psi \circ \mathbf{s}_n - D\psi \circ \mathbf{s}||_{L^p(\Omega)}$$

and that $\mathbf{s}_n$ is measure-preserving, we obtain that

$$||\bar{\phi}_n - \phi||_{L^p(\Omega)} \leq ||D\psi_n - D\psi||_{L^p(\Omega)} + ||D\psi \circ \mathbf{s}_n - D\psi \circ \mathbf{s}||_{L^p(\Omega)}. \tag{23}$$

Now, define

$$N := \cup_{n=1}^{\infty} \mathbf{s}_n^{-1}[\mathbf{R}^d \setminus \mathrm{domD}\psi] \cup \mathbf{s}^{-1}[\mathbf{R}^{\mathrm{d}} \setminus \mathrm{domD}\psi] \cup \mathrm{N}_1.$$

Observe that N has null Lebesgue measure and by (19) we have

$$\lim_{n\to+\infty} D\psi(\mathbf{s}_n(x)) = D\psi(\mathbf{s}(x)), \tag{24}$$

for all $x \in \Omega \setminus N$. By (18) $\{D\psi \circ \mathbf{s}_n\}_{n=1}^{\infty}$ is bounded in $L^\infty(\Omega)^d$. So, using (24) we conclude that

$$\lim_{n\to+\infty} ||D\psi \circ \mathbf{s}_n - D\psi \circ \mathbf{s}||_{L^p(\Omega)} = 0. \tag{25}$$

Combining (20), (23) and (25) we obtain that

$$\bar{\phi}_n \to \phi \text{ in } L^p(\Omega), \tag{26}$$

as n tends to ∞. Given $\epsilon > 0$ we choose n large enough in (22) and (26) to conclude the proof of theorem 2.4. QED.

Theorem 2.5 (Main results) *Suppose that $\phi \in L^p(\Omega)^d$ is a Borel map, where $1 \leq p < +\infty$. Then for each $\epsilon > 0$ there exists a diffeomorphism $\phi_\epsilon : \bar{\Omega} \to \phi_\epsilon(\bar{\Omega})$ such that*
(i) $||\phi_\epsilon - \phi||_{L^p(\Omega)} \leq \epsilon$
(ii) If in addition $\phi \in L^\infty(\Omega)^d$, then ϕ_ϵ can be chosen so that $\phi_\epsilon(\bar{\Omega}) \subset [\overline{conv\phi(\Omega)}]_\epsilon$.

Proof: Note that since every map in $L^p(\Omega)$ map can be approximated by a sequence of maps in L^∞ map, it suffices to prove theorem 2.5 when $\phi \in L^\infty(\Omega)^d$. Assume then that $\phi \in L^\infty(\Omega)^d$ and that $1 \leq p < +\infty$. Combining lemmas 2.2–2.4, we deduce that there exists a diffeomorphism $\phi_\epsilon : \bar{\Omega} \to \phi_\epsilon(\bar{\Omega})$ such that $||\phi_\epsilon - \phi||_{L^p(\Omega)} \leq \epsilon/2$ and (ii) hold. QED.

References

[1] J. Ball. Global invertibility of Sobolev functions and the interpenetration of the matter, *Proceedings of the Royal Society of Edinburgh*, **88A,** 315–328, 1981.

[2] G. Birkhoff. Tres observaciones sobre el algebra lineal, *Univ. Nac. Tucuman Rev.*, Ser. **A 5,** 147–150, 1946.

[3] Y. Brenier. Unpublished Lecture Notes (ENS Ulm, 1992).

[4] Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C.R. Acad. Sci. Paris Sér. I Math.* **305**, 805–808, 1987.

[5] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44:375–417, 1991.

[6] G.R. Burton, R.J. Douglas. Rearrangements and polar factorisation of countably degenerate functions. *Proceedings of the Royal Society of Edinburgh*, **128 A,** 671–681, 1998.

[7] L.A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.*, 5:99–104, 1992.

[8] L. Caffarelli. Boundary regularity of maps with convex potentials. *Comm. Pure Appl. Math.*, 45:1141–1151, 1992.

[9] L.A. Caffarelli. Boundary regularity of maps with convex potentials — II. *Ann. of Math. (2)*, 144: 453–496, 1996.

[10] L.C. Evans and R. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, 1992.

[11] I. Fonseca and W. Gangbo. Degree Theory in Analyis and Applications. *Clarendon Press Oxford* 1995.

[12] I. Fonseca and L. Tartar. The displacement problem for elastic crystals. *Proc. Royal Soc. Edinburgh* **113 A**, 159–180, 1989.

[13] W. Gangbo. An elementary proof of the polar factorization of vector-valued functions. *Arch. Rational Mech. Anal.*, 128:381–399, 1994.

[14] W. Gangbo. The Monge mass transfer problem and its applications. NSF-CBMS Conference on the Monge-Ampere equation: applications to geometry and optimization, July 09–13 1997. *Contemporary Mathematics*, 128: 381–399, 1999.

[15] W. Gangbo and R.J. McCann. The geometry of optimal transportation. *Acta Math.*, 177:113–161, 1996.

[16] W. Gangbo and R. Van der Putten. Uniqueness of Equilibrium Configurations in Solid Crystals (To appear in SIAM Mathematical Analysis).

[17] P.R. Halmos. Approximation theories for measure preserving transformations. *Trans. Amer. Math. Soc.*, 55:1–18, 1944.

[18] W. Huebsch and M. Morse. Schoenflies extensions without interior differential singularities. *Ann. of Math.* (2), 76, 18–54, 1962.

[19] W. Huebsch and M. Morse. Analytic diffeomorphims approximating C^m–diffeomorphisms. *Rend. Circ. Mat. Palermo* (2), 11, 25–46, 1962.

[20] A.B. Katok, A.M. Stepin. Metric properties of homeomorphisms that preserve measure. *Uspehi Mat. Nauk*, 25:193–220, 1970.

[21] P.D. Lax. Approximation of measure preserving transformations. *Comm. Pure Appl. Math.*, 24:133–135, 1971.

[22] R.J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.*, 80: 309–323, 1995.

[23] H. Minc. Nonnegative Matrices. *Wiley-Interscience Series in Discrete Mathematics and Optimization* 1988.

[24] M. Roesch. Thèse de Doctorat de l'Université Pierre et Marie Curie, Paris VI, 1995.

[25] W. Rudin. Real and Complex Analysis. third edition, *McGraw Hill,* New York 1987.

[26] A.I. Shnirelman. The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid. *Mat. Sb. (N.S.)*, 128(170), no. 1, 82–109, 1985.

[27] A.I. Shnirelman. Generalized fluid flows, their approximation and applications. *Geom. Funct. Anal.*, 4, no. 5, 586–620, 1994.

[28] H.E. White. The approximation of one-one measurable transformations by diffeomorphism. *Trans. Amer. Math. Soc.*, 141, 305–322, 1969.