OPTIMAL MAPS FOR THE MULTIDIMENSIONAL MONGE-KANTOROVICH PROBLEM

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Abstract

Let $\mu_1, \cdots, \mu_N$ be Borel, probability measures on $\mathbb{R}^d$. Denote by $\Gamma(\mu_1, \cdots, \mu_N)$ the set of all N-tuples $T = (T_1, \cdots, T_N)$ such that $T_i : \mathbb{R}^d \to \mathbb{R}^d$ $(i = 1 \cdots, N)$ are Borel measurable and satisfy $\mu_1[T_i^{-1}(V)] = \mu_i[V]$, for all Borel $V \subset \mathbb{R}^d$. The Multidimensional Monge-Kantorovich Problem investigated in this paper consists of finding $S = (S_1, \cdots, S_N) \in \Gamma(\mu_1, \cdots, \mu_N)$ minimizing

$$I[T] = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{\mathbb{R}^d} \frac{|T_i(\mathbf{x}) - T_j(\mathbf{x})|^2}{2} d\mu_1(x)$$

over the set $\Gamma(\mu_1, \cdots, \mu_N)$. We study the case where the μ_i's have finite second moments and vanish on $(d-1)$-rectifiable sets. We prove existence and uniqueness of optimal maps S when we impose that $S_1(\mathbf{x}) \equiv \mathbf{x}$, and give an explicit form of the maps S_i. The result is obtained by variational methods and to the best of our knowledge is the first available in the literature in this generality. As a consequence we obtain uniqueness and characterization of an optimal measure for the Multidimensional Kantorovich Problem.

1 Introduction

Mass transportation problems have attracted a lot of attention in recent years and have found applications in many fields of mathematics such as statistics and fluid mechanics (see [17] and [3]). In [10] and [5] the existence of optimal maps for the transportation problem was used as a tool for solving PDE's, and in [14] the problem was applied to the study of attracting gases. ODE and PDE methods were introduced in [6] to obtain a constructive solution of the Monge-Kantorovich Problem. Applications of marginal problems in probability and statistics can be found in [17], [16], [19]

There are very few results available when the problem involves more than two marginals. In this paper we begin the investigation of the existence and uniqueness of optimal maps for the *Multidimensional Monge-Kantorovich Problem*. Given Borel, probability measures $\mu_1, \cdots, \mu_N$ on $\mathbb{R}^d$, the problem consists of finding an *optimal* way of successively rearranging μ_1 onto μ_i against a certain cost function $c : \mathbb{R}^{Nd} \to [0, +\infty)$. If $T_i : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth one-to-one mapping, and $\mu_i = \rho_i d\mathcal{L}^d$ where $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$, the requirement that T_i rearranges μ_1 onto μ_i means

$$\rho_i(T_i(\mathbf{x}))|\text{det} DT_i(\mathbf{x})| = \rho_1(\mathbf{x}) \quad (x \in \mathbb{R}^d). \tag{1}$$

Motivated by the papers of Olkin and Rachev [15] and Knott and Smith [13] we have chosen to work with a cost function

$$c(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{2}. \tag{2}$$

The precise formulation of the Multidimensional Monge-Kantorovich Problem studied in this paper is the following:

Denote by $\Gamma(\mu_1, \cdots, \mu_N)$ the set of all N-tuples $T = (T_1, \cdots, T_N)$ such that $T_i : \mathbb{R}^d \to \mathbb{R}^d$ $(i = 1 \cdots, N)$ are Borel measurable and satisfy

$$\mu_1[T_i^{-1}(V)] = \mu_i[V], \tag{3}$$

for all Borel $V \subset \mathbb{R}^d$. We look for $S = (S_1, \cdots, S_N) \in \Gamma(\mu_1, \cdots, \mu_N)$ such that

$$I[S] = \inf_{T \in \Gamma(\mu_1, \cdots, \mu_N)} I[T], \tag{4}$$

where

$$I[T] = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{\mathbb{R}^d} \frac{|T_i(\mathbf{x}) - T_j(\mathbf{x})|^2}{2} d\mu_1(x). \tag{5}$$

If T_i satisfies (3) we say that T_i pushes μ_1 forward to μ_i and we write $\mu_i = T_i \sharp \mu_1$. If in addition T_i is defined μ_1-almost everywhere, we also say that T_i is *measure-preserving* between μ_1 and μ_i.

There are other problems that are related to (4) and that will be of interest to us. The first consists of finding $\mathbf{u} = (u_1, \cdots, u_N) \in \mathcal{K}$ such that

$$J[\mathbf{u}] = \sup_{\mathbf{w} \in \mathcal{K}} J[\mathbf{w}], \tag{6}$$

where

$$J[\mathbf{w}] = \sum_{i=1}^{N} \int_{\mathbb{R}^d} w_i(\mathbf{x}) d\mu_i(\mathbf{x}) \quad (\mathbf{w} = (w_1, \cdots, w_N) \in \mathcal{K}), \tag{7}$$

and $\mathcal{K}$ is the set of all N-tuples $\mathbf{w} = (w_1, \cdots, w_N)$ such that $w_i \in L^1(\mathbb{R}^d, \mu_i)$, w_i is upper semicontinuous and

$$\sum_{i=1}^{N} w_i(\mathbf{t}_i) \leq \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{|\mathbf{t}_i - \mathbf{t}_j|^2}{2}$$

for all $\mathbf{t}_1, \cdots, \mathbf{t}_N \in \mathbb{R}^d$. Under general assumptions on the μ_i's, it is well known that problem (6) admits a maximizer $\mathbf{u}$ (see [17], [12]). Moreover, (6) has been long-known to be dual to the problem of finding

$$\inf\{L(\nu) \ \mid \ \nu \in \mathcal{P}(\mu_1, \cdots, \mu_N)\}, \tag{8}$$

where $\mathcal{P}(\mu_1, \cdots, \mu_N)$ is the set of all Borel probability measures ν on $\mathbb{R}^{Nd}$ with fixed marginals $\mu_1, \cdots, \mu_N$ and

$$L(\nu) = \int_{\mathbb{R}^{Nd}} [\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{|\mathbf{t}_i - \mathbf{t}_j|^2}{2}] d\nu(\mathbf{t}_1, \cdots, \mathbf{t}_N) \tag{9}$$

(see [11]). The minimization (8) is known as the *Multidimensional Kantorovich Problem.*

The cost function in our multidimensional marginal problem belongs to the class discussed by Rachev in [16], Section 5.2, and [17], Section 3, where $U = \mathbb{R}^d$ equipped with the usual metric, the seminorm $\|\cdot\|$ is the euclidean norm, and $H(r) = r^2$. The functional $L(\nu)$ arises in probability and statistics. Generally speaking it measures a natural distance among probability measures $\mu_1, ..., \mu_N$ or equivalently among random variables $\mathbf{x}_1, ..., \mathbf{x}_N$ with laws $\mu_1, ..., \mu_N$. In this setup it is easy to see that (8) is equivalent to maximizing

$$E\left(\sum_{i=1}^{N} \sum_{j=i+1}^{N} < \mathbf{x}_i ; \mathbf{x}_j >\right)$$

over all $\mathbf{x}_i \sim \mu_i, i = 1, ..., N$, where E denotes the expectation. Therefore, if the random variables $\mathbf{x}_i, i = 1, ..., N$ have mean zero, minimizing (9) corresponds to finding a joint distribution of $(\mathbf{x}_1, ..., \mathbf{x}_N)$ that maximizes the traces of all covariance matrices of $\mathbf{x}_1, ..., \mathbf{x}_N$. If $L(\nu)$ is small the random variables are "close" in the sense that they are far from being mutually independent. For $N = 3$ and normal random variables this problem was studied by Olkin and Rachev in [15] and Knott and Smith in [13]. Applications of Multidimensional Kantorovich Problems of the type considered by us to convergence of probability metrics (merging of sequences of vectors of probability measures) and minimal distances are discussed in [16], Sections 7.4 and 7.5 (see also [15] where this is discussed for $N = 3$ and $L(\nu)$).

In the paper we prove the existence and uniqueness of optimal maps for the Multidimensional Monge-Kantorovich Problem. This extends the results of [2], [4], [8], and [9] to the case $N > 2$. Partial results for the case $N = 3$ were obtained by Knott and Smith [13], and Olkin and Rachev [15]. Recently we have learned that the three margin problem was also studied by Rüschendorf and Uckelmann [20]. They obtained a characterization of optimal maps. Our primary contribution is the observation that when the measures $\mu_1, \cdots, \mu_N$ vanish on $(d-1)$-rectifiable sets and have finite second moments, i.e.

$$\int_{\mathbb{R}^d} |\mathbf{x}|^2 d\mu_i(\mathbf{x}) < +\infty \quad (i = 1, \cdots, N) \tag{10}$$

then the infimum in (8) is attained by a measure μ defined by some $S \in \Gamma(\mu_1, \cdots, \mu_N)$ so that

$$\int_{\mathbb{R}^{Nd}} F(\mathbf{t}_1, \cdots, \mathbf{t}_N) d\mu(\mathbf{t}_1, \cdots, \mathbf{t}_N) = \int_{\mathbb{R}^d} F(S_1(\mathbf{x}), \cdots, S_N(\mathbf{x})) d\mu_1(\mathbf{x})$$

holds for all $F \in C(\mathbb{R}^{Nd})$. This means that the support of the measure μ is the graph of a mapping from $\mathbb{R}^d$ into $\mathbb{R}^{Nd}$.

In the present work we do not use the duality between (8) and (6) as a known fact but we rather recover this result (as in [4] and [8]) by writing the Euler-Lagrange equation of (6) to discover that the points where

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{|\mathbf{t}_i - \mathbf{t}_{i+1}|^2}{2} = \sum_{i=1}^{N} u_i(\mathbf{t}_i)$$

lie in the graph of a map from $\mathbb{R}^d$ into $\mathbb{R}^{Nd}$ when $\mathbf{u} = (u_1, \cdots, u_N)$ is a maximizer for (6). Furthermore, for μ_1-almost every $\mathbf{t}_1$, the points $\mathbf{t}_2, \cdots, \mathbf{t}_N$ are uniquely determined.

Notice that the duality relationship between (4) and (6) can be expressed as

$$\inf_T \sup_{\mathbf{w}} \mathcal{F}(T, \mathbf{w}) = \sup_{\mathbf{w}} \inf_T \mathcal{F}(T, \mathbf{w}),$$

where the infimum is performed over the set of all $T = (T_1, \cdots, T_N)$ such that $T_1, \cdots, T_N : \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable, $T_1(\mathbf{x}) = \mathbf{x}$, and the supremum is performed over the set of all $w_2, \cdots, w_N$ which are upper semicontinuous. Here the functional $\mathcal{F}$ is defined by

$$\begin{aligned} \mathcal{F}(T, w_2, \cdots, w_N) \;&=\; \sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{\mathbb{R}^d} \frac{|T_i(\mathbf{x}) - T_j(\mathbf{x})|^2}{2} d\mu_1(\mathbf{x}) \\ &- \sum_{i=2}^{N} \int_{\mathbb{R}^d} w_i(T_i(\mathbf{x})) d\mu_1(\mathbf{x}) + \sum_{i=2}^{N} \int_{\mathbb{R}^d} w_i(\mathbf{x}) d\mu_i(\mathbf{x}). \end{aligned}$$

For the reader's convenience we recall two definitions needed in the sequel.

Definition 1.1 *Let X be a metric space and let μ be a positive, finite Borel measure on X. The support of μ is the smallest closed set spt$(\mu) \subset X$ such that $\mu(spt(\mu)) = \mu(X)$.*

Definition 1.2 *We that $M \subset \mathbb{R}^d$ is a $(d-1)$-rectifiable set if M is a countable union of C^1 $(d-1)$-hypersurfaces and sets of zero $(d-1)$-dimensional Hausdorff measure.*

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2 Existence of optimal maps.

The main theorem of the paper (Theorem 2.1) yields the existence of optimal maps for the Multidimensional Monge-Kantorovich Problem when the cost function is given by (2), the measures vanish on all $(d-1)$-rectifiable sets and have finite second moments. The dual problem (6) plays the crucial role in the proof of the theorem. Claim (i) in Theorem 2.1 stating the existence of a maximizer $\mathbf{u}$ for (6) is well-known in the literature (see [17] or [12]).

Theorem 2.1 *Assume that $\mu_1, \cdots, \mu_N$ are nonnegative Borel probability measures vanishing on $(d-1)$-rectifiable sets and having finite second moments. Set $X_i := spt(\mu_i)$ for $i = 1, \cdots, N$. Then:*

(i) Problem (6) admits a maximizer $\mathbf{u} = (u_1, \cdots, u_N) \in \mathcal{K}$.

(ii) There is a minimizer $S = (S_1, \cdots, S_N) \in \Gamma(\mu_1, \cdots, \mu_N)$ for (4) satisfying $S_1(\mathbf{x}) = \mathbf{x}$ $(\mathbf{x} \in \mathbb{R}^d)$. The S_i are one-to-one μ_i-almost everywhere, are uniquely determined, and have the form $S_i(\mathbf{x}) = Df_i^*(Df_1(\mathbf{x}))$ $(\mathbf{x} \in \mathbb{R}^d)$, where

$$f_i(\mathbf{x}) = \frac{|\mathbf{x}|^2}{2} + \phi_i(\mathbf{x}), \tag{11}$$

the ϕ_i are convex functions, and $f_i^* \in C^1(\mathbb{R}^d)$.

(iii) Duality holds: the optimal values in Problems (4) and (6) coincide.

(iv) If $\bar{\mathbf{u}} = (\bar{u}_1, \cdots, \bar{u}_N) \in \mathcal{K}$ is another maximizer for Problem (6) we can modify the $\bar{u}_i$'s on sets of zero μ_i measure to obtain a maximizer, still denoted $\bar{\mathbf{u}}$, such that $\bar{u}_i$ is differentiable μ_i-almost everywhere. Furthermore,

$$Du_i = D\bar{u}_i \quad \mu_i - \text{almost everywhere}. \tag{12}$$

Before proving Theorem 2.1 we state a corollary giving the existence and uniqueness of an optimal measure for the Multidimensional Kantorovich Problem. Theorem 2.1 (ii) provides a geometrical characterization of the optimal measure.

Corollary 2.2 *Assume that $\mu_1, \cdots, \mu_N$ are nonnegative Borel probability measures vanishing on $(d-1)$-rectifiable sets and having finite second moments. Then problem (8) admits a unique minimizer $\mu \in \mathcal{P}(\mu_1, \cdots, \mu_N)$.*

Remark 2.3 *If the measures μ_i's are absolutely continuous with respect to the Lebesgue measure and $\mu_i = \rho_i d\mathcal{L}^d$ then formally the f_i's satisfy the Monge-Ampere equations*

$$\rho_i(Df_i^*(\mathbf{x}))\mathrm{det}D^2 f_i^*(\mathbf{x}) = \rho_1(Df_1^*(\mathbf{x}))\mathrm{det}D^2 f_1^*(\mathbf{x}) = \rho(\mathbf{x}).$$

The function ρ is not known.

We now give the proof of Theorem 2.1. To keep the focus on the main ideas of the proof we defer the technical details to Proposition 3.1 in the Appendix.

Proof of Theorem 2.1.

Step 1. For the proof of (i) we refer the reader to [12]. We also claim that we can assume without loss of generality that the u_i satisfy

$$u_i(\mathbf{t}_i) = \inf\{\sum_{k=1}^{N}\sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2} - \sum_{j=1, j\neq i}^{N} u_j(\mathbf{t}_j) \mid \mathbf{t}_1, \cdots, \mathbf{t}_{i-1}, \mathbf{t}_{i+1}, \cdots, \mathbf{t}_N \in \mathbb{R}^d\}, \tag{13}$$

for all $\mathbf{t}_i \in \mathbb{R}^d$.

Indeed, let $\tilde{\mathbf{v}} = (\tilde{v}_1, \cdots, \tilde{v}_N) \in \mathcal{K}$ be a maximizer for Problem (6). As in [17] we define

$$v_1(\mathbf{t}_1) = \inf\{\sum_{k=1}^{N}\sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2} - \sum_{j=2}^{N} \tilde{v}_j(\mathbf{t}_j) \mid \mathbf{t}_2, \cdots, \mathbf{t}_N \in \mathbb{R}^d\}, \quad (\mathbf{t}_1 \in \mathbb{R}^d),$$

and

$$v_i(\mathbf{t}_i) = \inf\{\sum_{k=1}^{N}\sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2} - \sum_{j=1}^{i-1} v_j(\mathbf{t}_j) - \sum_{j=i+1}^{N} \tilde{v}_j(\mathbf{t}_j) \mid \mathbf{t}_1, \cdots, \mathbf{t}_{i-1}, \mathbf{t}_{i+1}, \cdots, \mathbf{t}_N \in \mathbb{R}^d\}, \tag{14}$$

for $i = 2, \cdots, N$ and $\mathbf{t}_i \in \mathbb{R}^d$. Since $(v_1, \cdots, v_i, \tilde{v}_{i+1}, \cdots, \tilde{v}_N) \in \mathcal{K}$, we iteratively obtain that

$$\tilde{v}_i \leq v_i \quad (i = 1, \cdots, N). \tag{15}$$

Note also that

$$\mathbf{v} = (v_1, \cdots, v_N) \in \mathcal{K}, \tag{16}$$

and so, in light of (15) and (16) we deduce that $\mathbf{v}$ is a maximizer for Problem (6).

We finally introduce the functions

$$u_1(\mathbf{t}_1) = \inf\{\sum_{k=1}^{N}\sum_{j=k+1}^{N}\frac{|\mathbf{t}_k-\mathbf{t}_j|^2}{2} - \sum_{j=2}^{N} v_j(\mathbf{t}_j) \mid \mathbf{t}_2,\cdots,\mathbf{t}_N \in \mathbb{R}^d\}, \quad (\mathbf{t}_1 \in \mathbb{R}^d),$$

and

$$u_i(\mathbf{t}_i) = \inf\{\sum_{k=1}^{N}\sum_{j=k+1}^{N}\frac{|\mathbf{t}_k-\mathbf{t}_j|^2}{2} - \sum_{j=1}^{i-1} u_j(\mathbf{t}_j) - \sum_{j=i+1}^{N} v_j(\mathbf{t}_j) \mid \mathbf{t}_1,\cdots,\mathbf{t}_{i-1},\mathbf{t}_{i+1},\cdots,\mathbf{t}_N \in \mathbb{R}^d\}, \tag{17}$$

for $i = 2,\cdots,N$ and $\mathbf{t}_i \in \mathbb{R}^d$. Using arguments similar to those yielding (15) and using (17), we obtain that

$$\tilde{v}_i \le v_i \le u_i \quad (i = 2,\cdots,N) \tag{18}$$

and

$$\mathbf{u} = (u_1,\cdots,u_N) \in \mathcal{K}. \tag{19}$$

We claim that

$$v_i = u_i \quad (i = 1,\cdots,N). \tag{20}$$

Indeed, (18) yields

$$\sum_{k=1}^{N}\sum_{j=k+1}^{N}\frac{|\mathbf{t}_k-\mathbf{t}_j|^2}{2} - \sum_{j=1}^{i-1} u_j(\mathbf{t}_j) - \sum_{j=i+1}^{N} v_j(\mathbf{t}_j) \le \sum_{k=1}^{N}\sum_{j=k+1}^{N}\frac{|\mathbf{t}_k-\mathbf{t}_j|^2}{2} - \sum_{j=1}^{i-1} v_j(\mathbf{t}_j) - \sum_{j=i+1}^{N} \tilde{v}_j(\mathbf{t}_j),$$

which, together with (14) and (17), implies

$$u_i \le v_i \quad (i = 1,\cdots,N). \tag{21}$$

This concludes the proof of (20) and consequently we may assume in the sequel that (13) holds. Moreover we notice that since $\mathbf{u}$ and $\tilde{\mathbf{v}}$ are maximizers for (6) we have

$$u_i = \tilde{v}_i \quad \mu_i - \text{almost everywhere}, \tag{22}$$

for all $i = 1,\cdots,N$.

Define

$$\phi_i(\mathbf{x}) = \frac{N-1}{2}|\mathbf{x}|^2 - u_i(\mathbf{x}), \quad (\mathbf{x} \in \mathbb{R}^d), \tag{23}$$

and their domains of definition

$$dom(\phi_i) = \{\mathbf{x} \in \mathbb{R}^d \mid -\infty < \phi_i(\mathbf{x}) < +\infty\}.$$

We also denote

$$dom(D\phi_i) = \{\mathbf{x} \in \mathbb{R}^d \mid \ \phi_i \text{ is differentiable at } \mathbf{x}\}.$$

Step 2. We now study the properties of the ϕ_i. By (13) and (23) we have

$$\phi_i(\mathbf{t}_i) = \sup\{\sum_{k=2}^{N}\sum_{j=1}^{k-1} < \mathbf{t}_k; \mathbf{t}_j > - \sum_{j=1, j\neq i}^{N} \phi_j(\mathbf{t}_j) | \, \mathbf{t}_1, \cdots, \mathbf{t}_{i-1}, \mathbf{t}_{i+1}, \cdots, \mathbf{t}_N \in \mathbb{R}^d\}, \tag{24}$$

for all $\mathbf{t}_i \in \mathbb{R}^d$, and therefore the ϕ_i are convex and lower semicontinuous as supremums of linear functions.

Since $\mathbf{u} = (u_1, \cdots, u_N) \in \mathcal{K}$ we have $u_i \in L^1(\mathbb{R}^d, \mu_i)$ $i = 1, \cdots, N$. This, together with (10) and (23), implies

$$\phi_i \in L^1(\mathbb{R}^d, \mu_i) \quad (i = 1, \cdots, N) \tag{25}$$

and hence there exist sets $A_i \subset \mathbb{R}^d$ such that

$$dom(\phi_i) = \mathbb{R}^d \setminus A_i \tag{26}$$

and

$$\mu_i(A_i) = 0. \tag{27}$$

Since the ϕ_i are convex, it is well-known that

$$dom(\phi_i) \text{ is convex and } \phi_i \text{ is continuous in the interior of } dom(\phi_i), \tag{28}$$

(see [18]) there exist Borel sets $B_i \subset \mathbb{R}^d$ such that

$$dom(D\phi_i) = dom(\phi_i) \setminus B_i, \tag{29}$$

and

$$B_i \text{ is a } (d-1) - \text{Hausdorff dimensional set} \tag{30}$$

(see [1]) for all $i = 1, \cdots, N$. Also, observe that since $dom(\phi_i)$ is convex,

$$\partial(dom(\phi_i)) \text{ is contained in a } (d-1) - \text{rectifiable set} \tag{31}$$

for all $i = 1, \cdots, N$ (see [18]). Combining (27), (30), (31) and the fact that the μ_i vanish on $(d-1)$-rectifiable sets we deduce that if $C_i = A_i \cup B_i \cup \partial(dom(\phi_i))$ then

$$\mu_i[spt(\mu_i) \setminus C_i] = 1, \quad (i = 1, \cdots, N) \tag{32}$$

and

$$\phi_i \text{ is differentiable in } spt(\mu_i) \setminus C_i. \quad (i = 1, \cdots, N) \tag{33}$$

By (24) and (25) we deduce that

$$\text{the } \phi_i \text{ satisfy assumptions } (60) - (62) \text{ of the Appendix} \tag{34}$$

and hence $f_i^* \in C^1(\mathbb{R}^d)$ for $i = 1, ..., N$ (see Step 4 in the Appendix.)

We can now define the maps S_i by

$$S_i(\mathbf{x}) = Df_i^*(Df_1(\mathbf{x})),$$

for $\mathbf{x} \in spt(\mu_1) \setminus C_1$, for all $i = 1, \cdots, N$. By (32) the S_i's are defined μ_i-almost everywhere and are clearly Borel maps.

Step 3. We claim that S_i pushes μ_1 forward to μ_i $(i = 1, \cdots, N)$.

Proof. The argument is variational. Firstly, invoking Remark 3.2 we notice that if $\mathbf{t}_1 \in dom(Du_1)$ and

$$\sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2} = \sum_{i=1}^{N} u_i(\mathbf{t}_i),$$

then the $\mathbf{t}_i$ are uniquely determined by the equation

$$\mathbf{t}_i = Df_i^*(Df(\mathbf{t}_1)).$$

Secondly, by (32) and (33), there exists a set $N_1 \subset \mathbb{R}^d$ such that $\mu_1(N_1) = 0$ and

$$\mathbb{R}^d \setminus N_1 = dom(Du_1).$$

Since S_1 is the identity mapping we trivially have that S_1 pushes μ_1 forward to μ_1. To prove that S_2 pushes μ_1 forward to μ_2, we choose an arbitrary bounded function $F \in C(\mathbb{R}^d)$ and define for each $r \in (-1, 1)$

$$\phi_i^r(\mathbf{t}_i) = \phi_i(\mathbf{t}_i) \quad (\mathbf{t}_i \in \mathbb{R}^d),\ (i = 3, \cdots, N) \tag{35}$$

$$\phi_2^r(\mathbf{t}_2) = \phi_2(\mathbf{t}_2) + rF(\mathbf{t}_2), \quad (\mathbf{t}_2 \in \mathbb{R}^d) \tag{36}$$

$$\phi_1^r(\mathbf{t}_1) = \sup\{\sum_{i=1}^{N} \sum_{j=i+1}^{N} < \mathbf{t}_i; \mathbf{t}_j > - \sum_{j=2}^{N} \phi_j^r(\mathbf{t}_j) |\ \mathbf{t}_2, \cdots, \mathbf{t}_N \in \mathbb{R}^d\}, \quad (\mathbf{t}_1 \in \mathbb{R}^d) \tag{37}$$

and as in (23) define u_i^r by

$$u_i^r(\mathbf{x}) = \frac{N-1}{2}|\mathbf{x}|^2 - \phi_i^r(\mathbf{x}), \quad (\mathbf{x} \in \mathbb{R}^d). \tag{38}$$

Then $\mathbf{u}^r = (u_1^r, \cdots, u_N^r) \in \mathcal{K}$ and so, using the Lebesgue dominated convergence theorem, (13), (32), (34), (38), and Proposition 3.1 we have

$$\begin{aligned} 0 &= \lim_{r\to 0} \frac{J(\mathbf{u}^r) - J(\mathbf{u})}{r} \\ &= -\int_{\mathbb{R}^d} F(\mathbf{x})d\mu_2(\mathbf{x}) + \lim_{r\to 0}\int_{\mathbb{R}^d} \frac{u_1^r(\mathbf{x}) - u_1(\mathbf{x})}{r} d\mu_1(\mathbf{x}) \\ &= -\int_{\mathbb{R}^d} F(\mathbf{x})d\mu_2(\mathbf{x}) - \lim_{r\to 0}\int_{\mathbb{R}^d} \frac{\phi_1^r(\mathbf{x}) - \phi_1(\mathbf{x})}{r} d\mu_1(\mathbf{x}) \\ &= -\int_{\mathbb{R}^d} F(\mathbf{x})d\mu_2(\mathbf{x}) + \int_{\mathbb{R}^d} F(S_2(\mathbf{x}))d\mu_1(\mathbf{x}). \end{aligned} \tag{39}$$

Thus, by (39), we have

$$\int_{\mathbb{R}^d} F(\mathbf{x})d\mu_2(\mathbf{x}) = \int_{\mathbb{R}^d} F(S_2(\mathbf{x}))d\mu_1(\mathbf{x}),$$

for all $F \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Similarly

$$\int_{\mathbb{R}^d} F(\mathbf{x})d\mu_i(\mathbf{x}) = \int_{\mathbb{R}^d} F(S_i(\mathbf{x}))d\mu_1(\mathbf{x}), \tag{40}$$

for all $F \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and all $i = 2, \cdots, N$. This implies that

$$\mu_i[A] = \mu_1[S_i^{-1}(A)], \tag{41}$$

for all $A \subset \mathbb{R}^d$ Borel.

Step 4. Notice as in Step 3 that there exist $N_1, \cdots, N_N \subset \mathbb{R}^d$ such that

$$\mu_i[N_i] = 0 \quad (i = 1, \cdots, N), \tag{42}$$

$$dom(Du_i) = \mathbb{R}^d \setminus N_i \quad (i = 1, \cdots, N),$$

and

$$S_i(\mathbf{x}) = Df_i^*(Df_1((\mathbf{x}))) \quad (\mathbf{x} \in dom(Du_1))$$

for $i = 1, \cdots, N$. Define

$$L_i(\mathbf{x}) = Df_1^*(Df_i(\mathbf{x})),$$

for $\mathbf{x} \in dom(Du_i)$, and for $i = 1, \cdots, N$. In light of (41) and (42) we have

$$\mu_1[dom(Du_1) \setminus S_i^{-1}(N_i)] = 1, \quad (i = 1, \cdots, N).$$

Clearly

$$L_i(S_i(\mathbf{x})) = \mathbf{x} \quad (\mathbf{x} \in dom(Du_1) \setminus S_i^{-1}(N_i))$$

for $i = 1, \cdots, N$. Consequently, S_i is μ_1-almost everywhere one-to-one. The fact that $S = (\mathbf{x}, S_2, \cdots, S_N)$ is unique will be shown in Step 6.

Step 5. To prove (iii) we first notice that for each $T = (T_1, \cdots, T_N) \in \Gamma(\mu_1, \cdots, \mu_N)$ and for each $\mathbf{w} = (w_1, \cdots, w_N) \in \mathcal{K}$, using (41) we have

$$\sum_{i=1}^{N} \int_{\mathbb{R}^d} w_i(\mathbf{x}) d\mu_i(\mathbf{x}) = \int_{\mathbb{R}^{Nd}} [\sum_{i=1}^{N} w_i(T_i(\mathbf{x}))] d\mu_1(\mathbf{x}) \leq \int_{\mathbb{R}^{Nd}} [\sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|T_k(\mathbf{x}) - T_j(\mathbf{x})|^2}{2}] d\mu_1(\mathbf{x}). \tag{43}$$

Hence

$$\min_{T \in \Gamma(\mu_1, \cdots, \mu_N)} I[T] \geq \sup_{\mathbf{w} \in \mathcal{K}} J[\mathbf{w}]. \tag{44}$$

In light of (ii) and Remark 3.2, we have

$$\sum_{i=1}^{N} \int_{\mathbb{R}^d} u_i(\mathbf{x}) d\mu_i(\mathbf{x}) = \int_{\mathbb{R}^{Nd}} [\sum_{i=1}^{N} u_i(S_i(\mathbf{x}))] d\mu_1(\mathbf{x}) = \int_{\mathbb{R}^{Nd}} [\sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|S_k(\mathbf{x}) - S_j(\mathbf{x})|^2}{2}] d\mu_1(\mathbf{x}).$$

Hence

$$I[S] = J[\mathbf{u}], \tag{45}$$

and therefore (4) and (6) are dual.

Step 6. Assume that $\bar{S}_1(\mathbf{x}) \equiv \mathbf{x}$ and $\bar{S} = (\bar{S}_1, \cdots, \bar{S}_N) \in \Gamma(\mu_1, \cdots, \mu_N)$ is a minimizer for (4). We will prove that $\bar{S}_i = S_i$ μ_1-almost everywhere. Indeed, by (45)

$$I[\bar{S}] = J[\mathbf{u}]$$

and so, since $\mathbf{u} \in \mathcal{K}$ we deduce that

$$\sum_{i=1}^{N} u_i(\bar{S}_i(\mathbf{x})) = \sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|\bar{S}_k(\mathbf{x}) - \bar{S}_j(\mathbf{x})|^2}{2} \quad \mu_1 \ a.e. \tag{46}$$

Using (46) and Remark 3.2 we deduce that

$$\bar{S}_i(\mathbf{x})) = Df_i^*(Df_1(\mathbf{x})) \quad \mu_1 \ a.e.$$

i.e. $\bar{S}_i = S_i$ μ_1 *a.e.*

Step 7. Recall that by (22) we can modify each component $\bar{u}_i$ of any maximizer $\bar{\mathbf{u}}$ of (6) on a set of zero μ_i-measure to obtain a maximizer whose components are differentiable

μ_i-almost everywhere. We next show that $Du_i = D\bar{u}_i$ for $i = 1, ..., N$, where $\mathbf{u}$ is defined in Step 1.

Indeed, as in Step 2 we obtain $\bar{S} = (\bar{S}_1, \cdots, \bar{S}_N) \in \Gamma(\mu_1, \cdots, \mu_N)$ such that $\bar{S}$ is a minimizer for (4),

$$\bar{S}_1(\mathbf{x}) \equiv \mathbf{x},$$

and

$$D\bar{u}_1(\mathbf{x}) = (N-1)\mathbf{x} - \sum_{i=2}^{N} \bar{S}_i(\mathbf{x}). \tag{47}$$

However by Step 6, since we have imposed that $\bar{S}_1(\mathbf{x}) \equiv \mathbf{x}$, we have $\bar{S}_i = S_i$ up to a set of zero μ_1-measure which, together with (47), implies

$$D\bar{u}_1(\mathbf{x}) = (N-1)\mathbf{x} - \sum_{i=2}^{N} S_i(\mathbf{x}) = Du_1(\mathbf{x}), \tag{48}$$

up to a set of zero μ_1-measure. Similarly,

$$D\bar{u}_i(\mathbf{x}) = Du_i(\mathbf{x}), \tag{49}$$

up to a set of zero μ_i-measure for all $i = 1, \cdots, N$. Thus, (12) is proven.

∎

The proof of Corollary 2.2 is a consequence of Theorem 2.1.

Proof of Corollary 2.2.

Step 1. Observe that for each $\nu \in \mathcal{P}(\mu_1, \cdots, \mu_N)$ and each $\mathbf{w} \in \mathcal{K}$ we have

$$\sum_{i=1}^{N} \int_{\mathbb{R}^d} w_i(\mathbf{x}) d\mu_i(\mathbf{x}) = \int_{\mathbb{R}^{Nd}} [\sum_{i=1}^{N} w_i(\mathbf{t}_i)] d\nu(\mathbf{t}_1, \cdots, \mathbf{t}_N) \leq \int_{\mathbb{R}^{Nd}} [\sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2}] d\nu(\mathbf{t}_1, \cdots, \mathbf{t}_N). \tag{50}$$

Hence

$$\sup\{J[\mathbf{w}] \mid \mathbf{w} \in \mathcal{K}\} \leq \inf\{L[\nu] \mid \nu \in \mathcal{P}(\mu_1, \cdots, \mu_N)\}. \tag{51}$$

Define a Borel measure μ on $\mathbb{R}^{Nd}$ by

$$\mu[A_1 \times \cdots \times A_N] = \mu_1[\cap_{i=1}^{N} S_i^{-1}(A_i)] \tag{52}$$

for all Borel sets $A_i \subset \mathbb{R}^d$. In light of Theorem 2.1 and (52) we have

$$\mu[\mathbb{R}^d \times \cdots \times \mathbb{R}^d \times A_i \times \mathbb{R}^d \cdots \times \mathbb{R}^d] = \mu_1[S_i^{-1}(A_i)] = \mu_i[A_i],$$

and therefore

$$\mu \in \mathcal{P}(\mu_1, \cdots, \mu_N).$$

Using Theorem 2.1 (iii) we have

$$J[\mathbf{u}] = I[S] = L[\mu] \tag{53}$$

where $\mathbf{u}$ is the maximizer of J obtained in Theorem 2.1. Combining (51) and (53) we deduce that μ is a minimizer for (8).

Step 2. We now prove that the minimizer for (8) is unique. Indeed, let $\bar{\mu} \in \mathcal{P}(\mu_1, \cdots, \mu_N)$ be another minimizer for (8). Then by (53)

$$J[\mathbf{u}] = L[\bar{\mu}],$$

i.e.

$$\int_{\mathbb{R}^{Nd}} [\sum_{i=1}^{N} u_i(\mathbf{t}_i)] d\bar{\mu}(\mathbf{t}_1, \cdots, \mathbf{t}_N) = \int_{\mathbb{R}^{Nd}} [\sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2}] d\bar{\mu}(\mathbf{t}_1, \cdots, \mathbf{t}_N). \tag{54}$$

Since $\mathbf{u} \in \mathcal{K}$, (54) implies that there exists a set $N \subset \mathbb{R}^{Nd}$ such that

$$\bar{\mu}[N] = 0 \tag{55}$$

and

$$\sum_{i=1}^{N} u_i(\mathbf{t}_i) = \sum_{k=1}^{N} \sum_{j=k+1}^{N} \frac{|\mathbf{t}_k - \mathbf{t}_j|^2}{2}, \quad ((\mathbf{t}_1, \cdots, \mathbf{t}_N) \in \mathbb{R}^{Nd} \setminus N).$$

Let $N_1 \subset \mathbb{R}^d$ be the set where u_1 is not differentiable. By (32) and (33) we have $\mu_1(N_1) = 0$ and since furthermore $\bar{\mu} \in \mathcal{P}(\mu_1, \cdots, \mu_N)$ we have

$$\bar{\mu}(N_1 \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d) = 0.$$

Hence, we may assume without loss of generality that

$$N_1 \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \subset N. \tag{56}$$

Let B_1 be the first projection of the complement of N, namely,

$$B_1 = \{\mathbf{t}_1 \in \mathbb{R}^d \mid \exists \mathbf{t}_2, \cdots, \mathbf{t}_N, (\mathbf{t}_1, \cdots, \mathbf{t}_N) \in \mathbb{R}^{Nd} \setminus N\}.$$

We have

$$\mu_1[\mathbb{R}^d \setminus B_1] = 0, \tag{57}$$

and by (56)

$$N_1 \subset \mathbb{R}^d \setminus B_1.$$

Let $(\mathbf{t}_1, \cdots, \mathbf{t}_N) \in \mathbb{R}^{Nd} \setminus N$. In light of Remark 3.2, (34), and (56) it follows that

$$\mathbf{t}_i = S_i(\mathbf{t}_1), \quad (i = 1, \cdots, N). \tag{58}$$

In order to prove that $\bar{\mu} = \mu$ we fix Borel sets $A_1, \cdots, A_N \subset \mathbb{R}^d$. Since $\bar{\mu} \in \mathcal{P}(\mu_1, \cdots, \mu_N)$, by (52), (55), and (58) we have

$$\begin{aligned} \mu[A_1 \times \cdots \times A_N] &= \mu_1[\cap_{i=1}^N S_i^{-1}(A_i)] \\ &= \bar{\mu}[\cap_{i=1}^N S_i^{-1}(A_i) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus N] \\ &= \bar{\mu}[A_1 \times \cdots \times A_N]. \end{aligned}$$

Consequently,

$$\bar{\mu} = \mu.$$

∎

Remark 2.4 *A straightforward consequence of Theorem 2.1 (iii) is the following necessary and sufficient condition for optimality in the Multidimensional Monge-Kantorovich Problem. The maps* $T_1, \cdots, T_N : \mathbb{R}^d \to \mathbb{R}^d$ *such that* $\mu_i = T_i \sharp \mu_1$ *for* $i = 1, ..., N$ *are opimal for (4) if and only if there exist lower semicontinuous functions* $\phi_1, \cdots, \phi_N : \mathbb{R}^d \to \mathbb{R} \cap \{+\infty\}, \phi_i \in L^1(\mathbb{R}^d, \mu_i)$ *for* $i = 1, ..., N$*, satisfying* $\sum_{i=1}^d \phi_i(x_i) \geq \sum_{i<j} < x_i; x_j >$ *for all* $x_i \in \mathbb{R}^d$*,* $i = 1, ..., N$*, such that*

$$\sum_{i=1}^d \phi_i(T_i(x)) = \sum_{i<j} < T_i(x); T_j(x) > \quad \mu_1 - a.e. \tag{59}$$

We refer the reader to [20] for another necessary and sufficient condition for optimality in the case $N = 3$*..*

3 Appendix

Throughout the appendix we assume that

$$\phi_1, \cdots, \phi_N : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \text{ are convex, lower semicontinuous,} \tag{60}$$

and

$$\phi_1(\mathbf{t}_1) = \sup\{\sum_{k=1}^{N}\sum_{j=k+1}^{N} < \mathbf{t}_k; \mathbf{t}_j > - \sum_{j=2}^{N} \phi_j(\mathbf{t}_j) |\ \mathbf{t}_2, \cdots, \mathbf{t}_N \in \mathbb{R}^d\} \quad (\mathbf{t}_1 \in \mathbb{R}^d). \tag{61}$$

To ensure that each ϕ_i is bounded below by some linear function we assume in addition that

$$dom(\phi_i) \neq \emptyset \quad (i = 1, \cdots, N). \tag{62}$$

Observe that if $\mathbf{a}_i \in dom(\phi_i)$ $(i = 1, \cdots, N)$ then (61) gives

$$\phi_i(\mathbf{x}) \geq < \mathbf{p}_i; \mathbf{x} - \mathbf{a}_i > + l_i \quad (\mathbf{x} \in \mathbb{R}^d), \quad i = 1, \cdots, N, \tag{63}$$

where

$$\mathbf{p}_i = \sum_{j \neq i} \mathbf{a}_j$$

and l_i is a polynomial of the $\mathbf{a}_j$'s and the $\phi_j(\mathbf{a}_j)$'s.

Let $F : \mathbb{R}^d \to \mathbb{R}$ be a bounded, continuous function. For each $r \in (-1, 1)$ we define

$$\phi_i^r(\mathbf{x}) = \phi_i(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^d), \quad (i = 3, \cdots, N) \tag{64}$$

$$\phi_2^r(\mathbf{x}) = \phi_2(\mathbf{x}) + rF(\mathbf{x}), \quad (\mathbf{x} \in \mathbb{R}^d) \tag{65}$$

$$\phi_1^r(\mathbf{t}_1) = \sup\{\sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_i; \mathbf{t}_j > - \sum_{j=2}^{N} \phi_j^r(\mathbf{t}_j) |\ \mathbf{t}_2, \cdots, \mathbf{t}_N \in \mathbb{R}^d\}, \quad (\mathbf{t}_1 \in \mathbb{R}^d). \tag{66}$$

The following proposition proves the existence of a maximizer in (66).

Proposition 3.1 *Assume that* $\mathbf{t}_1^0 \in dom(D\phi_1)$. *Then:*

(i) For each $r \in (-1, 1)$ *there exist* $\mathbf{t}_{i,r} \in dom(\phi_i)$ $(i = 2, \cdots, N)$ *such that*

$$\sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_{i,r}; \mathbf{t}_{j,r} > = \sum_{i=1}^{N} \phi_i^r(\mathbf{t}_{i,r}), \tag{67}$$

where we have set $\mathbf{t}_{1,r} = \mathbf{t}_1^0$.

(ii) The points $\mathbf{t}_{2,0}, \cdots, \mathbf{t}_{N,0}$ *are uniquely determined and*

$$\lim_{r \to 0} \mathbf{t}_{i,r} = \mathbf{t}_{i,0} = Df_i^*(Df_1(\mathbf{t}_1^0)) \quad (i = 2, \cdots, N), \tag{68}$$

where

$$f_i(\mathbf{x}) = \frac{|\mathbf{x}|^2}{2} + \phi_i(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^d), \quad (i = 1, \cdots, N), \tag{69}$$

and f_i^* denotes the Legendre transform of f_i (see [18]).

(iii) We have

$$\lim_{r\to 0}\frac{\phi_1^r(\mathbf{t}_{1,0})-\phi_1(\mathbf{t}_{1,0})}{r}=-F(\mathbf{t}_{2,0}).$$

Proof.

Step 1. Let $\mathbf{t}_1^0\in dom(D\phi_1)$. Then

$$\phi_1 \text{ is bounded in a neighborhood } U \text{ of } \mathbf{t}_1^0, \tag{70}$$

and clearly

$$||\phi_i^r-\phi_i||_\infty\le r||F||_\infty,\quad (i=1,\cdots,N). \tag{71}$$

Combining (70), (71) and Theorem 1, page 236 of [7] we deduce that there exists a closed ball B independent of r such that

$$\partial\phi_1^r(U)\subset\frac{1}{2}B. \tag{72}$$

Now, let $\{\mathbf{t}_i^n\}_n\subset dom(\phi_i)$ be sequences depending on r such that

$$\phi_1^r(\mathbf{t}_1^0)=\lim_{n\to\infty}\sum_{i=1}^N\sum_{j=i+1}^N<\mathbf{t}_i^n;\mathbf{t}_j^n>-\sum_{i=2}^N\phi_i^r(\mathbf{t}_i^n), \tag{73}$$

where we have set $\mathbf{t}_1^n=\mathbf{t}_1^0$. Clearly, $\mathbf{t}_2^n+\cdots+\mathbf{t}_N^n$ converges to some point $\mathbf{c}\in\partial\phi_1^r(\mathbf{t}_1^0)$ and thus, by (72), we can assume without loss of generality that

$$\mathbf{c}_n:=\mathbf{t}_2^n+\cdots+\mathbf{t}_N^n\in B. \tag{74}$$

Step 2. We claim that the $\{\mathbf{t}_i^n\}_n$ are bounded by a constant that does not depend on n and r.

Proof. We use (73) to obtain

$$\begin{aligned}\phi_1^r(\mathbf{t}_1^0) &= \lim_{n\to\infty}(<\mathbf{t}_1^0;\mathbf{c}_n>+<\mathbf{t}_2^n;\mathbf{c}_n-\mathbf{t}_2^n>+\cdots+<\mathbf{t}_{N-1}^n;\mathbf{c}_n-\mathbf{t}_2^n-\cdots-\mathbf{t}_{N-1}^n>\\ &- [\phi_2^r(\mathbf{t}_2^n)+\cdots+\phi_{N-1}^r(\mathbf{t}_{N-1}^n)-\phi_N^r(\mathbf{c}_n-\mathbf{t}_2^n-\cdots-\mathbf{t}_{N-1}^n)])\end{aligned} \tag{75}$$

Combining (63) and (75) we deduce that

$$\begin{aligned}\phi_1^r(\mathbf{t}_1^0) &\le \limsup_{n\to\infty}-([|\mathbf{t}_2^n|^2+\cdots|\mathbf{t}_{N-1}^n|^2+<\mathbf{t}_1^0;\mathbf{c}_n>\\ &+ \mathbf{t}_2^n(\mathbf{t}_3^n+\cdots+\mathbf{t}_{N-1}^n)+\cdots+\mathbf{t}_{N-2}^n\mathbf{t}_{N-1}^n]+Q_n(\mathbf{t}_2^n,\cdots,\mathbf{t}_{N-1}^n))\\ &\le \limsup_{n\to\infty}(-\frac{1}{2}[|\mathbf{t}_2^n|^2+\cdots|\mathbf{t}_{N-1}^n|^2]+Q_n(\mathbf{t}_2^n,\cdots,\mathbf{t}_{N-1}^n)),\end{aligned} \tag{76}$$

where

$$Q_n(\mathbf{t}_2^n, \cdots, \mathbf{t}_{N-1}^n) = \sum_{i=2}^{N-1} < \mathbf{t}_i; \mathbf{c}_n - \mathbf{p}_N + \mathbf{p}_i > - \sum_{i=2}^{N} (< \mathbf{p}_i; \mathbf{a}_i > + l_i) + < \mathbf{p}_N + \mathbf{t}_1^0; \mathbf{c}_n > + ||F||_\infty.$$

It follows from (74) that

the coefficients of the polynomial Q_n are bounded by a constant which does not depend on r, n. (77)

In light of (76) and (77) it therefore follows that the sequences $\{\mathbf{t}_i^n\}_n$, $i = 2, \cdots, N-1$, are bounded by a constant which does not depend on r, n. This, together with (74), implies that $\{\mathbf{t}_N^n\}_n$ is bounded by a constant that does not depend on r, n. Hence for each r, up to a subsequence if necessary, we have that

$$\lim_{n\to\infty} \mathbf{t}_i^n = \mathbf{t}_{i,r} \text{ exists}, \quad (i = 1, \cdots, N) \tag{78}$$

where we have set $\mathbf{t}_{1,r} = \mathbf{t}_1^0$. This concludes the proof of the claim.

Step 3. We claim that $\sum_{i=1}^N \sum_{j=i+1}^N < \mathbf{t}_{i,r}; \mathbf{t}_{j,r} >= \sum_{i=1}^N \phi_i^r(\mathbf{t}_{i,r})$.

Proof. Using the subsequence from (78) in (73) we have

$$\lim_{n\to\infty} \sum_{i=1}^{N} \sum_{j=i+1}^{N} < \mathbf{t}_i^n; \mathbf{t}_j^n >= \lim_{n\to\infty} \sum_{i=1}^{N} \phi_i^r(\mathbf{t}_i^n). \tag{79}$$

Since by (60) the ϕ_i are lower semicontinuous, (79) yields

$$\sum_{i=1}^{N} \phi_i^r(\mathbf{t}_{i,r}) \leq \sum_{i=1}^{N} \sum_{j=i+1}^{N} < \mathbf{t}_{i,r}; \mathbf{t}_{j,r} >$$

which, together with (66), concludes the proof.

Step 4. We observe that since the ϕ_i are convex and the $dom(\phi_i)$ are nonempty, the f_i defined in (69) are strictly convex, have quadratic growth, and hence

$$f_i^* \in C^1(\mathbb{R}^d). \tag{80}$$

The task ahead is to prove that $\mathbf{t}_i^0 := \mathbf{t}_{i,0}$ are uniquely determined.

Step 5. We claim that $\mathbf{t}_{i,0} = Df_i^*(Df_1(\mathbf{t}_{1,0}))$, $(i = 2, \cdots, N)$.

Proof. In light of (61) and Step 3 (when $r=0$) we observe that

$$\begin{aligned}\phi_1(\mathbf{t}_1) &\geq\ < \mathbf{t}_1;\mathbf{t}_{2,0}+\cdots+\mathbf{t}_{N,0}> + \sum_{i=2}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_{i,0};\mathbf{t}_{j,0}> - \sum_{i=2}^{N}\phi_i(\mathbf{t}_{i,0}) \\ &=\ < \mathbf{t}_1-\mathbf{t}_{1,0};\mathbf{t}_{2,0}+\cdots+\mathbf{t}_{N,0}> + \phi_1(\mathbf{t}_{1,0}) \qquad (81)\end{aligned}$$

for all $\mathbf{t}_1\in\mathbb{R}^d$ which yields

$$\mathbf{t}_{2,0}+\cdots+\mathbf{t}_{N,0}\in\partial\phi_1(\mathbf{t}_{1,0}),$$

and which, by (69), is equivalent to

$$\mathbf{t}_{1,0}+\cdots+\mathbf{t}_{N,0}\in\partial f_1(\mathbf{t}_{1,0}).$$

Similarly, using again (61) and Step 3 (when $r=0$), we obtain

$$\mathbf{t}_{1,0}+\cdots+\mathbf{t}_{N,0}\in\partial f_i(\mathbf{t}_{i,0}),\quad (i=1,\cdots,N). \qquad (82)$$

Since by assumption f_1 is differentiable at $\mathbf{t}_{1,0}$ (82) gives

$$\mathbf{t}_{1,0}+\cdots+\mathbf{t}_{N,0}=Df_1(\mathbf{t}_{1,0}). \qquad (83)$$

Therefore, combining (80), (82), and (83) we obtain

$$\mathbf{t}_{i,0}=Df_i^*(Df_1(\mathbf{t}_{1,0})).\quad (i=1,\cdots,N)$$

Step 6. We claim that $\lim_{r\to 0}\mathbf{t}_{i,r}=\mathbf{t}_{i,0}$ $(i=2,\cdots,N)$.

Proof. By (78) and by Step 1 the families $\{\mathbf{t}_{i,r}\}_r$ are bounded by a constant which does not depend on r and so we may extract sequences $\{\mathbf{t}_{i,r_k}\}_k\subset\{\mathbf{t}_{i,r}\}_r$ ($r_k\to 0$ when k goes to infinity) and find points $\mathbf{b}_i\in B$ in the closed ball B found in (72) such that

$$\mathbf{t}_{i,r_k}\to\mathbf{b}_i,\quad (i=2,\cdots,N). \qquad (84)$$

By (71) the $\phi_i^{r_k}$ converge uniformly to the ϕ_i. Furthermore, by Step 3, we have

$$\sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_{i,r_k};\mathbf{t}_{j,r_k}> = \sum_{i=1}^{N}\phi_i^{r_k}(\mathbf{t}_{i,r_k}),$$

and therefore, thanks to (84) and the lower semicontinuity of the ϕ_i, we conclude that

$$\sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{b}_i;\mathbf{b}_j> \geq \sum_{i=1}^{N}\phi_i(\mathbf{b}_i), \qquad (85)$$

where we have set $\mathbf{b}_1 := \mathbf{t}_{1,0}$. Invoking (61) and (85) we thus obtain

$$\sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{b}_i; \mathbf{b}_j >= \sum_{i=1}^{N} \phi_i(\mathbf{b}_i). \tag{86}$$

Since $\mathbf{b}_1 = \mathbf{t}_{1,0}$, in light of Step 5 and (86) we deduce that

$$\mathbf{b}_i = Df_i^*(Df_1(\mathbf{t}_{1,0})), \quad (i = 2, \cdots, N).$$

Finally, since the subsequence $\{r_k\}_k \subset (-1, 1)$ was chosen arbitrarily, we conclude that

$$\lim_{r\to 0} \mathbf{t}_{i,r} = \mathbf{t}_{i,0}, \quad (i = 2, \cdots, N).$$

Step 7. We claim that $\lim_{r\to 0} \frac{\phi_1^r(\mathbf{t}_{1,0})-\phi_1(\mathbf{t}_{1,0})}{r} = -F(\mathbf{t}_{2,0})$.
Proof. Using the definition of the ϕ_i given in (64), (65), (66), and Step 3 we obtain

$$\begin{aligned}\phi_1^r(\mathbf{t}_{1,0}) &\geq \sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_{i,0}; \mathbf{t}_{j,0} > - \sum_{i=2}^{N} \phi_i(\mathbf{t}_{i,0}) - rF(\mathbf{t}_{2,0}) \\ &= \phi_1(\mathbf{t}_{1,0}) - rF(\mathbf{t}_{2,0}).\end{aligned} \tag{87}$$

Employing (61), and again the definition of the ϕ_i and Step 3, we have

$$\begin{aligned}\phi_1(\mathbf{t}_{1,0}) &\geq \sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_{i,r}; \mathbf{t}_{j,r} > - \sum_{i=2}^{N} \phi_i(\mathbf{t}_{i,r}) \\ &= \phi_1^r(\mathbf{t}_{1,0}) + rF(\mathbf{t}_{2,r}).\end{aligned} \tag{88}$$

Combining Step 6, (87), and (88) we therefore deduce

$$\lim_{r\to 0} \frac{\phi_1^r(\mathbf{t}_{1,0}) - \phi_1(\mathbf{t}_{1,0})}{r} = -F(\mathbf{t}_{2,0}),$$

and the claim follows. ∎

Remark 3.2 *We record the following useful fact. In Step 5 in the proof of Proposition 3.1 we have proven that if*

$$\sum_{i=1}^{N} \phi_i(\mathbf{t}_i) = \sum_{i=1}^{N}\sum_{j=i+1}^{N} < \mathbf{t}_i; \mathbf{t}_j > \tag{89}$$

or equivalently,

$$\sum_{i=1}^{N} u_i(\mathbf{t}_i) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{|\mathbf{t}_i - \mathbf{t}_j|^2}{2}, \tag{90}$$

and $\mathbf{t}_1 \in dom(D\phi_1) = dom(Du_1)$ *then* $\mathbf{t}_2, \cdots, \mathbf{t}_N$ *are uniquely determined and satisfy*

$$\mathbf{t}_i = Df_i^*(Df_1(\mathbf{t}_1)),$$

where

$$u_i(\mathbf{x}) = \frac{N-1}{2}|\mathbf{x}|^2 - \phi_i(\mathbf{x}),$$

for $i = 1, \cdots, N$*. Conversely, if* $\mathbf{t}_i = Df_i^*(Df_1(\mathbf{t}_1))$*, and* $\mathbf{t}_1 \in dom(D\phi_1) = dom(Du_1)$ *then the* $\mathbf{t}_i$*'s satisfy (89) and (90).*

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