Analysis and Applications, Vol. 4, No. 4 (2006) 1–12

ON SOME NONLOCAL VARIATIONAL PROBLEMS

MICHEL CHIPOT

Institut für Mathematik, Angewandte Mathematik, Universität Zürich
Winterthurerstrasse 190, CH–8057 Zürich, Switzerland
m.m.chipot@math.unizh.ch

WILFRID GANGBO

School of Mathematics, Georgia Institute of Technology
Atlanta, GA 30332, USA
gangbo@math.gatech.edu

BERND KAWOHL

Mathematisches Institut, Universität zu Köln
D 50923 Köln, Germany
kawohl@mi.uni-koeln.de

Received 7 March 2006

We study uniqueness and non uniqueness of minimizers of functionals involving nonlocal quantities. We give also conditions which lead to a lack of minimizers and we show how minimization on an infinite dimensional space reduces here to a minimization on $\mathbb{R}$. Among other things, we prove that uniqueness of minimizers of functionals of the form $\int_\Omega a(\int_\Omega gu\,dx)|\nabla u|^2\,dx - 2\int_\Omega fu\,dx$ is ensured if $a > 0$ and $1/a$ is stricly concave in the sense that $(1/a)'' < 0$ on $(0, \infty)$.

Keywords:

Mathematics Subject Classification 2000:

1. Introduction

Throughout this note, Ω is a bounded domain of $\mathbb{R}^N$ with boundary Γ. Let $\tilde{A} : H_0^1(\Omega) \to M_+^{N\times N}$ be a map whose range is contained in the set $M_+^{N\times N}$ of $N \times N$ positive definite matrices. We are interested in the case where $\tilde{A}(u)$ has a nonlocal dependence in u. An example could be

$$\tilde{A}(u) = A\left(\int_\Omega gu\,dx, \|\nabla u\|_{L^2(\Omega)}\right)$$

for prescribed functions, say, $g \in L^2(\Omega)$ and $A : \mathbb{R}^2 \to M_+^{N\times N}$. In fact, later, we will relax the assumption on g to $g \in H^{-1}(\Omega)$. In the above, we have denoted by

$\|\nabla u\|_{L^2(\Omega)}$ the norm

$$\|\nabla u\|_{L^2(\Omega)} = \left\{ \int_\Omega |\nabla u|^2 \, dx \right\}^{\frac{1}{2}}.$$

It is well known that solving the boundary value problem

$$\begin{cases} -\mathrm{div}(\tilde{A}(u)\nabla u) = f & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

reduces to solving a nonlinear sytem of equations in $\mathbb{R}^2$, (see [2]). Up to now such a theory was unavailable for the minimization of

$$J[u] := \frac{1}{2} \int_\Omega \tilde{A}(u)\nabla u \cdot \nabla u \, dx - \int_\Omega f u \, dx,$$

say on $H_0^1(\Omega)$ (in the above integral and below the scalar product between vectors will be denoted by a dot). One of the goals of this note is to fill out this gap and to show for instance, that in the case when

$$\tilde{A}(u) = a\left(\int_\Omega g u \, dx \right) I$$

then the minimization of J on a linear space reduces to the minimization of a single function on $\mathbb{R}$, i.e. to a problem in $\mathbb{R}$ and not in an infinite dimensional space (see Sec. 2, I denotes the identity matrix). One should note of course that (1.1) is not the Euler equation corresponding to the minimization of $J[u]$.

From the point of view of the applications and when

$$g = \frac{1}{|\Omega|},$$

with $|\Omega|$ denoting the Lebesgue measure of Ω, the minimization of J on $H_0^1(\Omega)$ corresponds to the search of the displacement of an elastic membrane spanned along the boundary of Ω and submitted to a force f. The elasticity coefficients, i.e. the entries of A, are supposed to depend on the average displacement and on the elastic energy of this membrane.

Equation (1.1) has also its interpretation in population dynamics (see [3, 1] and the references therein). It gives in particular the stationary equilibria of an evolution process.

The experience gained in Sec. 2 in a simple situation allows us to give in Sec. 3 sharp existence and uniqueness results for the minimization of J on a closed convex set of $H_0^1(\Omega)$.

2. The Case $A = aI$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing on $H^{-1}(\Omega) \times H_0^1(\Omega)$ where $H_0^1(\Omega)$ is equipped with the norm

$$\|u\|_1 = \|\nabla u\|_{L^2(\Omega)}.$$

Throughout this section,

$$f, g \in H^{-1}(\Omega), \tag{2.1}$$

and for each $m \in \mathbb{R}$, we define

$$K_m = \{u \in H_0^1(\Omega) : l(u) = m\}, \quad l(u) = \langle g, u\rangle. \tag{2.2}$$

We assume that $a \in C(\mathbb{R}, (0, +\infty])$ and we set $A = aI$ where I is the identity matrix. We define

$$J[u] = \frac{a(l(u))}{2}\int_\Omega |\nabla u|^2\, dx - \langle f, u\rangle \tag{2.3}$$

and set

$$\tilde{J}(m) = \operatorname*{Inf}_{K_m} J[u]. \tag{2.4}$$

As mentioned below in Sec. 3, Proposition 3.2, the existence of a minimizer of J over $H_0^1(\Omega)$ or K_m can be easily obtained by direct methods of the calculus of variations. Uniqueness of a minimizer of J over $H_0^1(\Omega)$ needs to be justified whereas uniqueness of a minimizer u_m on K_m is trivial. Since for all $w \in K_0$, $u_m + tw \in K_m$, as usual, one can simply deduce that

$$\frac{d}{dt} J[u_m + tw]|_{t=0} = 0$$

and obtain the following characterization of u_m:

Lemma 2.1. *For every m in $\mathbb{R}$, the unique minimizer $u_m \in K_m$ of J over K_m is characterized by the equation*

$$\int_\Omega a(m)\nabla u_m \cdot \nabla w\, dx = \langle f, w\rangle \quad \forall\, w \in K_0. \tag{2.5}$$

Theorem 2.2. *Let S be the set of minimizers of J over $H_0^1(\Omega)$ and let S' be the set of minimizers of $\tilde{J}$ over $\mathbb{R}$. Then*

$$l : u \mapsto l(u)$$

is a one-to-one mapping from S onto S'.

Proof. Let u be a minimizer of J on $H_0^1(\Omega)$. Let $m_0 = l(u)$. One has

$$\tilde{J}(m_0) = J[u] = \operatorname*{Inf}_{K_{m_0}} \left\{\frac{a(m_0)}{2}\int_\Omega |\nabla u|^2\, dx - \langle f, u\rangle\right\} \le J[v] \quad \forall\, v \in H_0^1(\Omega). \tag{2.6}$$

In particular, if $m \in \mathbb{R}$ and $u_m \in K_m$ minimizes J over K_m, (2.6) implies that

$$\tilde{J}(m_0) \le J[u_m] = \tilde{J}(m). \tag{2.7}$$

Hence, $m_0 = l(u)$ is a minimizer of $\tilde{J}$. This proves that the range of l is contained in S'.

To show that l is surjective, we choose an arbitrary m_0 minimizer of $\tilde{J}$ and denote by u_{m_0} the unique minimizer of J over K_{m_0}. If $v \in H_0^1(\Omega)$ and $m = l(v)$, we have that

$$J[u_{m_0}] = \tilde{J}(m_0) \leq \tilde{J}(m) \leq J[v].$$

This proves that u_{m_0} is a minimizer of J over $H_0^1(\Omega)$ and $J[u_{m_0}] = \tilde{J}(m_0)$. Thus, l is surjective. If u_1, u_2 are two minimizers with $l(u_1) = l(u_2)$ then (under an obvious abuse of notation) clearly $u_1 = u_2 = u_{l(u_i)}$ and the injectivity is proved. □

Let us define θ_g to be the unique weak solution of

$$\begin{cases} -\Delta\theta_g = g & \text{in } \Omega, \\ \theta_g \in H_0^1(\Omega). \end{cases} \tag{2.8}$$

Lemma 2.3. *Given $m \in \mathbb{R}$ and $g \not\equiv 0$, let u_m be the unique minimizer of J over K_m. Then, u_m satisfies*

$$-a(m)\Delta u_m = f + c_m g \quad \text{in } \mathcal{D}'(\Omega), \tag{2.9}$$

where c_m is the constant given by

$$c_m = \frac{a(m)m - \langle f, \theta_g\rangle}{l(\theta_g)}. \tag{2.10}$$

Proof. Since $g \not\equiv 0$, $\theta_g \neq 0$ and from (2.8), we deduce

$$l(\theta_g) = \langle g, \theta_g\rangle = \int_\Omega |\nabla\theta_g|^2\, dx > 0.$$

Let $\mathcal{D}(\Omega)$ be the set of C^∞ functions whose support is contained in Ω. We may find $\varrho \in \mathcal{D}(\Omega)$ such that

$$l(\varrho) = 1.$$

For each $v \in \mathcal{D}(\Omega)$, $w = v - l(v)\varrho \in K_0$ and so, by (2.5)

$$\begin{aligned} \langle -a(m)\Delta u_m - f, v\rangle &= \int_\Omega a(m)\nabla u_m \cdot \nabla v\, dx - \langle f, v\rangle \\ &= \int_\Omega l(v)a(m)\nabla u_m \cdot \nabla\varrho\, dx - \langle f, l(v)\varrho\rangle \\ &= l(v)\left\{\int_\Omega a(m)\nabla u_m \cdot \nabla\varrho\, dx - \langle f, \varrho\rangle\right\} := c_m l(v), \quad \forall\, v \in \mathcal{D}(\Omega). \end{aligned}$$

Setting $c_m = \int_\Omega a(m)\nabla u_m \cdot \nabla\varrho\, dx - \langle f, \varrho\rangle$, we have proven that

$$\langle -a(m)\Delta u_m - f - c_m g, v\rangle = 0 \tag{2.11}$$

for all $v \in \mathcal{D}(\Omega)$ and so,

$$a(m)\Delta u_m + f + c_m g = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.12}$$

We choose $v = \theta_g$ in (2.11) to obtain that

$$0 = \langle -a(m)\Delta u_m - f - c_m g, \theta_g\rangle = a(m)\langle u_m, -\Delta\theta_g\rangle - \langle f, \theta_g\rangle - c_m l(\theta_g),$$

and thus

$$c_m l(\theta_g) = a(m)l(u_m) - \langle f, \theta_g\rangle = a(m)m - \langle f, \theta_g\rangle.$$

This concludes the proof. □

Remark 2.4. Note that if $\tilde{A}(u) = a(l(u))I$, then by (2.9) the solutions of (1.1) are of the form u_m with $a(m)m = \langle f, \theta_g\rangle$ or $c_m = 0$.

Theorem 2.5. *We have*

$$\tilde{J}(m) = \frac{1}{2\langle g, \theta_g\rangle}\left\{\frac{(a(m)m - \langle f, \theta_g\rangle)^2 - \langle g, \theta_g\rangle\langle f, \theta_f\rangle}{a(m)}\right\}. \tag{2.13}$$

Proof. By (2.9),

$$\Delta(a(m)u_m - \theta_f - c_m\theta_g) = 0.$$

By the uniqueness of the solution of the Dirichlet problem, we conclude that

$$a(m)u_m = \theta_f + c_m\theta_g. \tag{2.14}$$

Testing (2.9) with u_m and recalling (2.2) we obtain

$$a(m)\int_\Omega |\nabla u_m|^2\,dx = \langle f, u_m\rangle + c_m m.$$

Thus

$$\tilde{J}(m) = \frac{a(m)}{2}\int_\Omega |\nabla u_m|^2\,dx - \langle f, u_m\rangle = \frac{1}{2}\{c_m m - \langle f, u_m\rangle\}. \tag{2.15}$$

We apply f to (2.14) to obtain that

$$a(m)\langle f, u_m\rangle = \langle f, \theta_f\rangle + c_m\langle f, \theta_g\rangle. \tag{2.16}$$

We combine (2.10), (2.15) and (2.16) to conclude after easy computations that

$$\tilde{J}(m) = \frac{1}{2\langle g, \theta_g\rangle}\,\frac{a(m)^2m^2 - 2a(m)m\langle f, \theta_g\rangle + \langle f, \theta_g\rangle^2 - \langle g, \theta_g\rangle\langle f, \theta_f\rangle}{a(m)}.$$

This completes the proof. □

Remark 2.6. If we set

$$\langle f, \theta_g\rangle = \alpha, \quad \langle g, \theta_g\rangle\langle f, \theta_f\rangle = \|f\|_{-1}^2\|g\|_{-1}^2 = \beta > 0, \tag{2.17}$$

the minimization of $\tilde{J}$ reduces to the minimization of

$$\mathcal{J}(m) = \frac{(a(m)m - \alpha)^2 - \beta}{a(m)}. \tag{2.18}$$

Since

$$\langle f, \theta_g\rangle = \int_\Omega \nabla\theta_f\cdot\nabla\theta_g\,dx, \quad \langle g, \theta_g\rangle = \|\nabla\theta_g\|_{L^2(\Omega)}^2, \quad \langle f, \theta_f\rangle = \|\nabla\theta_f\|_{L^2(\Omega)}^2,$$

by the Cauchy–Schwarz inequality, $\alpha^2 \leq \beta$. It is clear that $\tilde{J}$ and $\mathcal{J}$ are continuous functions of m if a is continuous. Recall that a is assumed to be positive throughout the paper.

Note that $\mathcal{J}(0) = \{\alpha^2 - \beta\}/a(0) \leq 0$ and so,

$$\tilde{J}(0) \leq 0.$$

We have shown in Theorem 2.2 that J admits minimizers iff $\mathcal{J}$ admits minimizers on $\mathbb{R}$. This leads us to:

Theorem 2.7. *Suppose that $a \in C(\mathbb{R}; (0, \infty])$.*

(i) *If for $|m|$ large enough,*

$$a(m) \geq \frac{\delta}{|m|}, \tag{2.19}$$

where δ is a positive constant such that

$$(\delta - |\alpha|)^2 > \beta, \tag{2.20}$$

then $J[\cdot]$ and $\mathcal{J}$ admit minimizers.

(ii) *If for $|m|$ large enough,*

$$a(m) = \frac{\delta}{|m|}$$

with $(\delta - |\alpha|)^2 < \beta$, then $J[\cdot]$ fails to have minimizers.

Proof. If (2.19) holds for $|m|$ large enough, we use the fact that $\alpha^2 \leq \beta$ to obtain that

$$\begin{aligned}\mathcal{J}(m) &= a(m)m^2 - 2\alpha m + \frac{\alpha^2 - \beta}{a(m)} \geq \frac{\delta}{|m|}m^2 - 2\alpha m + \frac{\alpha^2 - \beta}{\delta}|m| \\ &= \delta|m| - 2\alpha m + \frac{\alpha^2 - \beta}{\delta}|m|.\end{aligned}$$

This, together with (2.20) yields that

$$\begin{aligned}\mathcal{J}(m) &\geq \delta|m| - 2|\alpha||m| + \frac{\alpha^2 - \beta}{\delta}|m| \\ &= |m|\left\{\frac{(\delta - |\alpha|)^2 - \beta}{\delta}\right\} \to +\infty \quad \text{when } |m| \to +\infty.\end{aligned}$$

Thus the minimization of $\mathcal{J}$ reduces to a minimization on a compact set and since $\mathcal{J}$ is a continuous function, a minimizer does exist.

In the case where $a(m) = \frac{\delta}{|m|}$ for $|m|$ large enough, we have

$$\begin{aligned}\mathcal{J}(m) &= \delta|m| - 2\alpha m + \frac{\alpha^2 - \beta}{\delta}|m| \\ &= |m|\left\{\frac{(\delta - |\alpha|)^2 - \beta}{\delta}\right\} \quad \text{for } \operatorname{sign}(m) = \operatorname{sign}(\alpha)\end{aligned}$$

and $\mathcal{J}$ is not bounded below for $(\delta - |\alpha|)^2 < \beta$. This completes the proof of the theorem. □

Remark 2.8. It is clear that (2.19) holds for instance when

$$a(m) \geq \delta > 0$$

or, more generally, when

$$a(m) \geq \delta |m|^{-\gamma} \quad \text{for } |m| \text{ large},$$

γ being a constant such that $0 < \gamma < 1$, δ being here an arbitrary positive constant.

In case where the continuity of a fails, we can show:

Theorem 2.9. *Suppose that*

$$a \geq \delta > 0. \tag{2.21}$$

Then, if a is discontinuous $J[\cdot]$ might fail to have a minimizer.

Proof. Indeed let a be a continuous function satisfying (2.21). Then $\mathcal{J}$ admits minimizers. Let m_0 be one of them. One has

$$\mathcal{J}(m_0) = a(m_0)m_0^2 - 2\alpha m_0 + \frac{\alpha^2 - \beta}{a(m_0)}.$$

If m_0 and $(\alpha^2 - \beta)$ are not both zero, the function

$$a \to am_0^2 - 2\alpha m_0 + \frac{\alpha^2 - \beta}{a}$$

is clearly increasing and one can change the value of $a(m_0)$ in such a way that m_0 is no longer a minimizer. For this new (and discontinuous) a, the functional J has no minimizer since the function $\mathcal{J}$ has none. □

Regarding uniqueness, we have:

Theorem 2.10. *If $\mathcal{J}$ is strictly convex, then $J[\cdot]$ admits a unique minimizer. Otherwise, J can have as many minimizers as we wish — even for a smooth coefficient function a.*

Proof. The first point is clear. Note that

$$\mathcal{J}(m) = a(m)m^2 - 2\alpha m + \frac{\alpha^2 - \beta}{a(m)}$$

and this function is strictly convex, in particular when

$$\mathcal{J}''(m) = a''m^2 + 4a'm + 2a - \frac{(\alpha^2 - \beta)}{a^2}\left\{a'' - 2\frac{a'^2}{a}\right\} > 0.$$

This is in particular the case when

$$a'' > 2\frac{a'^2}{a}, \tag{2.22}$$

i.e. when $\frac{1}{a}$ is strictly concave. Indeed the inequality is clear when $m = 0$ (recall that $\alpha^2 - \beta \leq 0$). For $m \neq 0$, we have

$$\mathcal{J}''(m) > 2\frac{a'^2}{a}m^2 + 4a'm + 2a = \frac{2}{a}\{a'm + a\}^2 \geq 0.$$

Suppose now — this is of course always possible,

$$\alpha^2 - \beta < 0.$$

Then, consider a function $\mathcal{J}$ having as many minimizers as we wish (even a continuum). It is always possible to find a positive a such that

$$2a\mathcal{J}(m) = (am - \alpha)^2 - \beta \Leftrightarrow a^2m^2 - 2a(\alpha m + \mathcal{J}(m)) + \alpha^2 - \beta = 0.$$

Indeed the discriminant of this equation is

$$\Delta = 4\{(\alpha m + \mathcal{J}(m))^2 - m^2(\alpha^2 - \beta)\} \tag{2.23}$$

and it has its roots in $\mathbb{R}$. Moreover, since $\alpha^2 - \beta < 0$, the roots do not have the same signs and one is positive. We call it $a(m)$. It varies of course, continuously with m, and for the corresponding problem of minimizing (2.3) one has as many solutions as $\mathcal{J}$ has of minimizers. We can also have an arbitrary number of minimizers in the case where $\beta = \alpha^2$. Let $j \in C^2(\mathbb{R})$ be a function having the number of minimizers that we wish and which satisfies the following conditions:

$$j(m) > 2\alpha m \quad \forall\, m \neq 0, \quad j(0) = 0, \quad j'(0) = -2\alpha, \quad j''(0) > 0.$$

It is clear that there are infinitely many functions j satisfying these assumptions. We set

$$a(m) = \begin{cases} \dfrac{2\alpha m + j(m)}{m^2} & \text{if } m \neq 0 \\ \dfrac{1}{2}j''(0) & \text{if } m = 0. \end{cases}$$

We have that $a \in C^1(\mathbb{R})$. In fact, the smoothness of a does not matter. Checking that $\mathcal{J} = j$, we conclude the proof. □

Example 2.11. In biological applications it is often *a priori* known that the average population density m is nonnegative. In that case a typical example of a coefficient function a for which $\tilde{J}$ has at most one minimizer is

$$a(m) = \begin{cases} m^{-\gamma} & \text{if } m > 0 \\ +\infty & \text{if } m \leq 0, \end{cases}$$

where $\gamma \in (0, 1)$. Clearly, $\frac{1}{a}$ is strictly concave on $[0, \infty)$ and solutions with nonnegative mean value cannot exist because they are penalized with infinite costs.

3. The General Case

The main issue in this section is not the existence of minimizers for the class of variational problems that we consider. They are given by standard and direct methods of the calculus of variations which we briefly describe. We will instead keep our focus on uniqueness of these minimizers. In the sequel,

$$\emptyset \neq K \subset H_0^1(\Omega) \quad \text{is closed under the weak } H_0^1(\Omega) \text{ topology} \tag{3.1}$$

and

$$f, g \in H^{-1}(\Omega).$$

For each $m \in \mathbb{R}$, we define

$$K_m = \{u \in K : l(u) = m\}, \quad l(u) = \langle g, u\rangle.$$

We set

$$J[u] = \frac{1}{2}\int_\Omega A(l(u))\ \nabla u \cdot \nabla u\, dx\ - \langle f, u\rangle, \tag{3.2}$$

where A is a matrix-valued map

$$A \in C(\mathbb{R}, M_+^{N\times N}) \tag{3.3}$$

such that there exist positive constants λ, δ with

$$A(m)\xi \cdot \xi \geq \min\left\{\lambda, \frac{\delta}{|m|}\right\}|\xi|^2 \tag{3.4}$$

for all $\xi \in \mathbb{R}^N$ and all $m \in \mathbb{R}$.

Remark 3.1. Since A is continuous, if there exists $M > 0$ such that

$$A(m)\xi \cdot \xi \geq \frac{\delta}{|m|}|\xi|^2$$

for all $|m| \geq M$ and all $\xi \in \mathbb{R}^N$, then (3.4) holds.

If (3.4) holds and $\{u_n\}_{n=1}^{+\infty} \subset K$ converges weakly to u then $\{l(u_n)\}_{n=1}^{+\infty}$ converges to $l(u)$ and so $\{A(l(u_n))\}_{n=1}^{+\infty}$ converges to $A(l(u))$. Similarly, $\{\langle f, u_n\rangle\}_{n=1}^{+\infty}$ converges to $\langle f, u\rangle$. Using that $\xi \to |\xi|^2$ is convex and that $A(l(u_n)) > 0$, we conclude that J is weakly lower semicontinous on K. By (3.4), for $u \in K$,

$$J[u] \geq \frac{1}{2}\min\left\{\lambda, \frac{\delta}{|l(u)|}\right\}\|u\|_1^2 - \|f\|_{-1}\|u\|_1. \tag{3.5}$$

Thus, for every constant $C > 0$, we have that

$$\{u \in K : J[u] \leq C\} \subset U_1 \cup U_2, \tag{3.6}$$

where

$$U_1 = \left\{u \in K : \frac{\lambda}{2}\|u\|_1^2 \leq C + \|g\|_{-1}\ \|u\|_1\right\}$$

and

$$U_2 = \left\{ u \in K : \frac{\delta}{2}\|u\|_1^2 \le |C|\|g\|_{-1}\,\|u\|_1 + \|f\|_{-1}\,\|g\|_{-1}\|u\|_1^2 \right\}.$$

Using the fact that J is weakly lower semicontinous on K, we exploit (3.5) and (3.6) to obtain the following proposition (see [4]).

Proposition 3.2. *Assume that* (3.3) *and* (3.4) *hold.*

(i) *If K_m is nonempty, then J admits a unique minimizer over K_m.*
(ii) *If, in addition, $\delta > 2\|f\|_{-1}\,\|g\|_{-1}$, then J admits a minimizer over K.*

Remark 3.3. (i) Uniqueness of the minimizer over K_m results from the fact that the restriction of J over K_m is simply $u \to \int_\Omega A(m)\nabla u \cdot \nabla u\,dx - \langle f, u\rangle$, which is strictly convex.

(ii) To obtain uniqueness of minimizers of J over K, we will need to impose additional assumptions on A.

Suppose now that A is symmetric. Let us denote by A' the matrix whose entries are derivatives of the entries of A. Note that A', A'' and $A'A^{-1}A'$ are symmetric. If $\xi \in \mathbb{R}^N$, then

$$A'A^{-1}A'\xi \cdot \xi = A^{-1}(A'\xi)\cdot(A'\xi) \ \ge 0$$

since A^{-1} is positive definite. Thus, $A'A^{-1}A'$ is nonnegative definite. We denote by M the set of minimizers of J over K. One remarks from (3.5) and (3.6) that M is *a priori* bounded. We have:

Theorem 3.4. *Assume that $E \subset \mathbb{R}$ is an open interval, that $A \in C(\mathbb{R}) \cap C^2(E)$ is symmetric in E, that the range $\ell(K \cap M)$ of $K \cap M$ is contained in E, that* (3.4) *holds and that*

$$A'' > 2A'A^{-1}A' \quad \text{on } E.$$

Then, if K is convex, J has at most one minimizer over K. (*The above inequality means simply that $A'' - 2A'A^{-1}A'$ is positive definite, and it is the matrix version of* (2.22).)

Proof. It suffices to show that if u, v are two distinct elements of $K \cap M$ then $t \to J[u + t(v-u)]$ is strictly convex on $(0,1)$. For that, it suffices to show that

$$t \to I[u_t] = \int_\Omega A(l(u_t))\nabla u_t \cdot \nabla u_t\,dx$$

is strictly convex on $(0,1)$, where $u_t = u + t(v-u)$. Note that $\{l(u_t) : t \in [0,1]\}$ is a compact subset of $\mathbb{R}$ and so, the fact that $A'' > 2A'A^{-1}A'$ implies the existence of some $\lambda_o > 2$ such that $A''(l(u_t)) > \lambda_o A'(l(u_t))A^{-1}(l(u_t))A'(l(u_t))$ for $t \in [0,1]$.

Direct computations give that

$$\frac{d}{dt}I[u_t] = \int_\Omega (A'(l(u_t))\nabla u_t \cdot \nabla u_t \; l(v-u) + 2A(l(u_t))\nabla u_t \cdot \nabla(v-u))\,dx$$

and that

$$\begin{aligned}\frac{d^2}{dt^2}I[u_t] &= \int_\Omega (A''(l(u_t))\nabla u_t \cdot \nabla u_t)(l(v-u))^2\,dx \\ &\quad + 4\int_\Omega A'(l(u_t))\nabla u_t \cdot \nabla(v-u) l(v-u)\,dx \qquad (3.7)\\ &\quad + 2\int_\Omega A(l(u_t))\nabla(v-u)\cdot\nabla(v-u)\,dx. \qquad (3.8)\end{aligned}$$

We apply the Cauchy–Schwarz and the Young inequalities to estimate the term in (3.7) as follows:

$$\begin{aligned}|A'\nabla u_t \cdot \nabla(v-u)\, l(v-u)| &= |A^{-\frac{1}{2}}A'\nabla u_t \cdot A^{\frac{1}{2}}\nabla(v-u) l(v-u)| \\ &\le |A^{-\frac{1}{2}}A'\nabla u_t|\;|A^{\frac{1}{2}}\nabla(v-u)|\;|l(v-u)| \\ &\le \frac{\lambda_o}{4}|A^{-\frac{1}{2}}A'\nabla u_t|^2 l^2(v-u) + \frac{1}{\lambda_o}|A^{\frac{1}{2}}\nabla(v-u)|^2 \qquad (3.9)\\ &= \frac{\lambda_o}{4}A'A^{-1}A'\nabla u_t \cdot \nabla u_t \; l^2(v-u) \qquad (3.10)\\ &\quad + \frac{1}{\lambda_o}A\nabla(v-u)\cdot\nabla(v-u). \qquad (3.11)\end{aligned}$$

To obtain (3.10) from (3.9), we have used the fact that A is symmetric. We next use (3.8), (3.11) and the fact that

$$A''(l(u_t)) > \lambda_o A'l(u_t)A^{-1}l(u_t)A'l(u_t)$$

for $t \in [0,1]$, to conclude that

$$\begin{aligned}\frac{d^2}{dt^2}I[u_t] &\ge \int_\Omega (A''(l(u_t))\nabla u_t \cdot \nabla u_t)(l(v-u))^2\,dx \\ &\quad - \lambda_o\int_\Omega A'A^{-1}A'\nabla u_t \cdot \nabla u_t \; l^2(v-u)\,dx \\ &\quad + \left(2 - \frac{4}{\lambda_o}\right)\int_\Omega A(l(u_t))\nabla(v-u)\cdot\nabla(v-u)\,dx \\ &\ge \int_\Omega \left(A''(l(u_t)) - \lambda_o A'A^{-1}A'\right)\nabla u_t \cdot \nabla u_t (l(v-u))^2\,dx \\ &\quad + \left(2 - \frac{4}{\lambda_o}\right)\int_\Omega A(l(u_t))\nabla(v-u)\cdot\nabla(v-u)\,dx > 0, \qquad (3.12)\end{aligned}$$

if $\nabla(v-u) \not\equiv 0$. □

Acknowledgment

MC has been supported by the Swiss National Foundation under the contract number 20-105155/1, WG gratefully acknowledges the support provided by NSF grants DMS-00-74037, DMS-02-00267 and DMS-03-54729. BK gratefully acknowledges the support provided by the German Research Foundation DFG.

References

[1] À. Calsina and M. Sanchón, Stability and instability of equilibria of an equation of size structured population dynamics, *J. Math. Anal. Appl.* **286** (2003) 435–452.

[2] M. Chipot and B. Lovat, Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **8**(1) (2001) 35–51.

[3] M. Chipot and S. Zheng, Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms, *Asymptot. Anal.* **45** (2005) 301–312.

[4] E. Giusti, *Direct Methods in the Calculus of Variations* (World Scientific, 2003).