An elementary proof of the polar factorization of vector-valued functions.

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Abstract: We present an elementary proof of an important result of Y. Brenier ([Br1], [Br2]), namely that vector fields in $\mathbb{R}^d$ satisfying a nondegeneracy condition admit the polar factorization

$$(*) \qquad \mathbf{u}(x) = \nabla\psi(\mathbf{s}(x)),$$

where ψ is a convex function and $\mathbf{s}$ is a measure-preserving mapping. Brenier solves a minimization problem using Monge-Kantorovich theory, whereas we turn our attention to a dual problem, whose Euler-Lagrange equation turns out to be $(*)$.

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1 Introduction.

Given a vector-valued mapping $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ $(1 \leq p \leq \infty)$, we cannot expect in general that $\mathbf{u}$ is the gradient of a mapping $\psi : \mathbb{R}^d \to \mathbb{R}$. However Y. Brenier proved in [Br1] and in [Br2] that there exists a convex function whose gradient is equal to $\mathbf{u}$ up to a rearrangement, provided that $\mathbf{u}$ satisfies a nondegeneracy condition, so-called N^{-1}-property (see below). Precisely, he proved that there exist a convex function $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and a measure-preserving mapping $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ such that $\mathbf{u} = \nabla\psi \circ \mathbf{s}$. He termed this decomposition formula the polar factorization of $\mathbf{u}$.

This paper provides a new and elementary proof of the polar factorization of vector-valued mappings $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ $(1 \leq p \leq \infty)$ satisfying the $N^{-1}-$ property. The proof does not use any tools of Monge-Kantorovich theory as in [Br1] and in [Br2] and relies instead on convex analysis, namely on Lemma 2.4 which says under some technical assumptions, that for every $h \in C(\bar{\Omega})$ we have

$$\lim_{r \to 0} \frac{\phi_r(y) - \psi^*(y)}{r} = -h(\nabla\psi^*(y)) \tag{1}$$

for almost every $y \in B_R$, where

$$\phi_r(y) = \sup\{yz - \psi(z) - rh(z) \mid z \in \Omega\} \text{ and } \psi^* = \phi_0.$$

We establish first the polar factorization formula for every mappings $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^d)$. More precisely in Theorem 2.3, we fix $\Omega \subset \mathbb{R}^d$ an open, bounded set such that $meas(\partial\Omega) = 0$, and $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^d)$, a mapping that satisfies the $N^{-1}-$ property. We assume that the closure of $\mathbf{u}(\Omega)$ is contained in the ball B_R of center zero and radius R. As in [Br1] we prove that the following variational problem admits a minimizer

$$i_\infty = \inf\{I(\phi, \psi) \mid (\phi, \psi) \in E_R\}, \tag{2}$$

where

$$I(\phi, \psi) = \int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx,$$

$$E_R = \{(\phi, \psi) \mid \phi \in C(B_R) \cap L^\infty(B_R),\, \psi \in C(\Omega) \cap L^\infty(\Omega),\, \phi(y) + \psi(z) \geq yz \;\forall (y, z) \in B_R \times \Omega\}$$

and

$$S = \{\mathbf{s} : \bar{\Omega} \to \bar{\Omega} \mid \mathbf{s} \text{ is a measure-preserving mapping}\}.$$

We also prove that a dual problem of the problem in (2) is

$$\sup\{\int_\Omega \mathbf{u}(x)\mathbf{s}(x)dx \mid \mathbf{s} \in S\}. \tag{3}$$

or, equivalently, the projection problem

$$\frac{1}{2}\inf\{\int_\Omega ||\mathbf{u}(x) - \mathbf{s}(x)||^2 dx \mid \mathbf{s} \in S\}. \tag{4}$$

We prove that i_∞ is attained by convex functions $(\psi^*, \psi) \in E_R$ where ψ^* denotes the Legendre-Fenchel transform of ψ (see Definition 4.1). As (ψ^*, ψ) is a minimizer of (2) and as $((\psi + rh)^*, \psi + rh) \in E_R$, one can readily check, as we do in the next section, that

$$\lim_{r \to 0} \frac{I((\psi + rh)^*, \psi + rh) - I(\psi^*, \psi)}{r} = 0. \tag{5}$$

As $\mathbf{u}$ satisfies the $N^{-1}-$property, (1) and (5) yield

$$\int_\Omega h(x)dx = \int_\Omega h(\nabla\psi^*(\mathbf{u}(x))dx.$$

Since $h \in C(\bar{\Omega})$ is arbitrary, we deduce that $\mathbf{s} := \nabla\phi^* \circ \mathbf{u}$ is a measure-preserving mapping from $\bar{\Omega}$ into $\bar{\Omega}$; hence, the polar factorization formula is established for L^∞ mappings. Using an approximation argument and Proposition 4.4, it is straightforward to check that the polar factorization formula holds for L^p mappings (see Corollary 3.1).

By a well-known property of convex analysis relating ψ^* to ψ, the equation $\mathbf{s}(x) = \nabla\psi^* \circ \mathbf{u}(x)$ implies

$$\psi^*(\mathbf{u}(x)) + \psi(\mathbf{s}(x)) = \mathbf{u}(x)\mathbf{s}(x) \tag{6}$$

for almost every $x \in \Omega$. As a consequence, a dual to the problem in (2) is the problem in (3) or equivalently the problem in (4).

This paper is organized as follows. In the second section we recall first some well-known definitions and then we state the main result of this paper for L^∞ functions, namely, the factorization formula for L^∞ mappings. We deduce that a dual of the problem in (2) is the problem in (3). In the third section, using the factorization formula for L^∞ mappings and an approximation argument, we deduce the factorization formula for L^p mappings and prove a duality result. For completeness, we end up this paper with an appendix, where we review some well-known definitions and some properties of convex functions.

After this work was completed, I looked in L. Caffarelli's paper, "Boundary regularity of maps with convex potentials", Comm. Pure Appl. Math XLV (1992), p. 1141-1151, which notes in passing that somewhat similar approach to Brenier's theory has apparently been observed by S. Varadhan. I am not aware of any detailed proofs in the literature, however.

2 The polar factorization for L^∞ mappings.

In this section we state the main result of this paper that is the polar factorization formula for L^∞ mappings (Theorem 2.3). Throughout this section $\gamma > 0$ is a constant, $\Omega \subset \mathbb{R}^d$ is an open bounded set such that $meas(\partial\Omega) = 0$ and $\bar{\Omega} \subset B_\gamma$. For each $R > 0$ we define E_R to be the set

$$E_R = \{(\phi, \psi) \,|\, \phi \in C(B_R) \cap L^\infty(B_R), \psi \in C(\Omega) \cap L^\infty(\Omega),\ \phi(y) + \psi(z) \geq yz\, \forall (y, z) \in B_R \times \Omega\},$$

and we recall the following well-known definitions.

Definition 2.1 *Assume that $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$. We say that $\mathbf{s}$ is a measure-preserving mapping if $\mathbf{s}$ satisfies the following equivalent properties:*

(i) $$\mathbf{s}^{-1}(A) \textit{ is measurable and } meas(\mathbf{s}^{-1}(A)) = meas(A),$$

for every $A \subset \bar{\Omega}$ measurable set.

(ii) $$f \circ \mathbf{s} \in L^1(\Omega) \textit{ and } \int_\Omega f \circ \mathbf{s}(x)dx = \int_\Omega f(x)dx,$$

for every $f \in L^1(\Omega)$.

Definition 2.2 Let $E \subset \mathbb{R}^d$ be a measurable set. A mapping $\mathbf{u} : E \to \mathbb{R}^d$ is said to satisfy the $N^{-1}-$property if $meas(\mathbf{u}^{-1}(N)) = 0$ whenever $N \subset \mathbb{R}^d$ and $meas(N) = 0$.

Theorem 2.3 *Let $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^d)$ be a vector-valued mapping satisfying the $N^{-1}-$property and let $R > 0$ be such that $\overline{\mathbf{u}(\Omega)} \subset B_R$. Consider the following variational problem*

$$i_\infty = \inf\{I(\phi, \psi) := \int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx \mid (\phi, \psi) \in E_R\}. \tag{7}$$

Then
(i) (Brenier) The minimum is attained for some pair $(\phi_0, \psi_0) \in E_R$.
(ii) There exist two convex, Lipschitz functions $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$ *that extend* ϕ_0, ψ_0 *respectively.*
(iii) The Euler-Lagrange equation corresponding to (7) is:

$$\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x)), \tag{8}$$

for almost every $x \in \Omega$, *where* $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ *is a measure-preserving mapping.*
(iv) A dual problem of the problem in (7) is

$$j_\infty = \inf\{\int_\Omega \mathbf{u}(x)\mathbf{s}(x)dx \mid \mathbf{s} \text{ is a measure-preserving mapping on } \bar{\Omega}\}, \tag{9}$$

in the sense that $i_\infty = j_\infty$.

We prove in the next section that there exist a unique pair of functions $(\psi, \mathbf{s})$ satisfying the assumptions of Theorem 2.3 and satisfying $\int_\Omega \psi(z)dz = 0$.

We prove first the following lemma, which is the main key in proof of Theorem 2.3.

Lemma 2.4 *Let* $\psi : \mathbb{R}^d \to \mathbb{R}$ *be a convex function such that*

$$\psi^*(y) = \sup\{yz - \psi(z) \mid z \in \bar{\Omega}\} \tag{10}$$

for every $y \in \mathbb{R}^d$. *For* $h \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and* $r \in [0, 1]$ *define*

$$f_r(y) = \sup\{yz - \psi(z) - rh(z) \mid z \in \bar{\Omega}\}.$$

Then
(i)

$$||f_r(y) - f_0(y)|| \leq r||h||_\infty,$$

for every $y \in \mathbb{R}^d$ *and*
(ii)

$$\lim_{r\to 0} \frac{f_r(y) - f_0(y)}{r} = -h(\nabla f_0(y)) = -h(\nabla\psi^*(y))$$

for almost every $y \in \mathbb{R}^d$.

Proof. Assertion (i) is trivial. To prove (ii) we first define

$$N = \{y \in \mathbb{R}^d \mid \psi^* \text{ is not differentiable at } y\}.$$

By Proposition 4.5 in the Appendix, $dom(\psi^*) := \{x \in \mathbb{R}^d \mid \psi^*(x) < \infty\} = \mathbb{R}^d$, and so by Proposition 4.3

$$meas(N) = 0. \tag{11}$$

Let $y \in \mathbb{R}^d \setminus N$ and let $\{a_r, b_r\}_{r\in(-1,1)} \subset \bar{\Omega}$ be such that

$$f_r(y) \leq yb_r - \psi(b_r) - rh(b_r) + r^2 \leq f_0(y) - rh(b_r) + r^2 \tag{12}$$

and

$$f_0(y) \leq ya_r - \psi(a_r) + r^2 \leq f_r(y) + rh(a_r) + r^2. \tag{13}$$

We observe

$$-h(a_{r_i}) - r_i \leq \frac{f_{r_i}(y) - f_0(y)}{r_i} \leq -h(b_{r_i}) + r_i \tag{14}$$

for any sequence $\{r_i\}_{i=1}^{+\infty} \subset (0,1)$ that converges to 0 as i goes to $+\infty$. Up to a subsequence of $\{r_i\}_{i=1}^{+\infty}$ we still denote by $\{r_i\}_{i=1}^{+\infty}$ there exist $a, b \in \bar{\Omega}$ such that

$$a = \lim_{i \to +\infty} a_{r_i} \text{ and } b = \lim_{i \to +\infty} b_{r_i} \tag{15}$$

Letting i go to $+\infty$ in (12) and (13) and noting by (i) that f_r converges to f_0 uniformly in $\mathbb{R}^d$, we deduce that

$$\psi^*(y) = f_0(y) = ya - \psi(a) = yb - \psi(b). \tag{16}$$

Consequently,

$$a, b \in \partial\psi^*(y) = \{\nabla\psi^*(y)\},$$

where $\partial\psi^*(y)$ is the subdifferential of f at the point $y \in \mathbb{R}^d$ (see [Ro] for the definition of the subdifferential). Letting i go to $+\infty$ in (14) we obtain

$$\lim_{i \to +\infty} \frac{f_{r_i}(y) - f_0(y)}{r_i} = -h(\nabla\psi^*(y)).$$

As $\{r_i\}_{i=1}^{+\infty}$ is an arbitrary sequence of $(0,1)$ we deduce that

$$\lim_{r \to 0^+} \frac{f_r(y) - f_0(y)}{r} = -h(\nabla\psi^*(y)).$$

By a similar argument we obtain

$$\lim_{r \to 0^-} \frac{f_r(y) - f_0(y)}{r} = -h(\nabla\psi^*(y)).$$

As $y \in \mathbb{R}^d \setminus N$ is arbitrary, we deduce with the help of (11) that

$$\lim_{r \to 0} \frac{f_r(y) - f_0(y)}{r} = -h(\nabla\psi^*(y))$$

for almost every $y \in \mathbb{R}^d$. ∎

Proof of Theorem 2.3 We divide the proof into two steps.
Step1. Following Brenier (see [Br1] p.414) we prove first that there exists a pair $(\phi, \psi) \in E_R$ such that $i_\infty = I(\phi, \psi)$. Let $\{(f_n, g_n)\} \subset E_R$ be a minimizing sequence. We observe that

$$\{(\tilde{\phi}_n, \tilde{\psi}_n)\} := \{(f_n - \inf_{y \in B_R} f_n(y), g_n + \inf_{y \in B_R} f_n(y)\} \subset E_R$$

and is still a minimizing sequence such that

$$\inf_{y \in B_R} \tilde{\phi}_n(y) = 0. \tag{17}$$

Set

$$\psi_n(z) = \sup\{yz - \tilde{\phi}_n(y) \mid y \in B_R\}, \;\; \forall y \in \mathbb{R}^d \tag{18}$$

and

$$\phi_n(z) = \sup\{yz - \psi_n(z) \mid z \in \Omega\}, \;\; \forall z \in \mathbb{R}^d. \tag{19}$$

By Proposition 4.5, (17), (18) and (19) we have

$$\psi_n, \phi_n \in C(\mathbb{R}^d), \tag{20}$$

$$Lip(\phi_n) \leq d\gamma$$

$$\psi_n(0) = 0,$$

$$Lip(\psi_n) \leq dR$$

$$0 \leq \phi_n(y) \leq \gamma(||y|| + R) \;\; \forall y \in \mathbb{R}^d$$

and

$$||\psi_n(z)|| \leq R||z|| \;\; \forall z \in \mathbb{R}^d.$$

By Ascoli's theorem there exist two functions Lipschitz $f, g : \mathbb{R}^d \to \mathbb{R}$ such that ψ_n converges to g uniformly in every compact set of $\mathbb{R}^d$, ϕ_n converges to f uniformly in every compact set of $\mathbb{R}^d$,

$$g(0) = 0, \;\; Lip(g) \leq d\gamma, \tag{21}$$

$$Lip(f) \leq dR \tag{22}$$

$$0 \leq f(y) \leq \gamma(||y|| + R) \;\; \forall y \in \mathbb{R}^d \tag{23}$$

and

$$||g(z)|| \leq R||z|| \;\; \forall z \in \mathbb{R}^d. \tag{24}$$

Claim1: $\{(\phi_n, \psi_n)\} \subset E_R$ and is still a minimizing sequence.
By (19) we obtain that

$$\psi_n(z) + \phi_n(y) \geq yz \;\; \forall (y, z) \in B_R \times \Omega$$

and so by (20) we deduce that $\{(\phi_n, \psi_n)\} \subset E_R$. As $(\tilde{\phi}_n, \tilde{\psi}_n) \in E_R$, (18) implies that

$$\psi_n(z) \leq \tilde{\psi}_n(z)$$

for all $z \in \Omega$. It is also straightforward to check that

$$\phi_n(y) \leq \tilde{\phi}_n(y)$$

for all $y \in B_R$ and so

$$\int_\Omega [\phi_n(\mathbf{u}(x)) + \psi_n(x)]dx \leq \int_\Omega [\tilde{\phi}_n(\mathbf{u}(x)) + \tilde{\psi}_n(x)]dx.$$

Hence, $\{(\phi_n, \psi_n)\}$ is still a minimizing sequence. As ψ_n converges to g uniformly in every compact set of $\mathbb{R}^d$ and ϕ_n converges to f uniformly in every compact set of $\mathbb{R}^d$, we obtain

$$i_\infty = I(f, g).$$

It is straightforward to check that f and g are convex functions and so (i) and (ii) are proved.

Step2. To prove (iii), we first define

$$\tilde{f}(y) = \sup\{yz - g(z) \mid z \in \bar{\Omega}\} \;\; \forall y \in \mathbb{R}^d.$$

and

$$\psi = (\tilde{f})^*.$$

It is obvious that $\tilde{f}$ and ψ are two convex functions satisfying

$$\tilde{f}(y) \leq f(y)$$

for every $y \in B_R$,

$$\psi(z) \leq g(z)$$

for every $z \in \mathbb{R}^d$ and

$$(\tilde{f}, \psi) \in E_R.$$

Hence

$$i_\infty = I(\tilde{f}, \psi). \tag{25}$$

By Proposition 4.5, (25) and by the definition of ψ, setting $\phi = \tilde{f}$ we obtain that

$$\phi^* = \psi, \ \ \psi^* = \phi, \tag{26}$$

$$\phi|_{\mathbf{u}(\Omega)} = f|_{\mathbf{u}(\Omega)}, \ \ \psi|_\Omega = g|_\Omega \tag{27}$$

and

$$\phi(y) = \sup\{yz - \psi(z), \ | \, z \in \bar{\Omega}\} \ \ \forall y \in \mathbb{R}^d. \tag{28}$$

For each $h \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and each $r \in (0,1)$ we define

$$f_r(y) = \sup\{yz - \psi(z) - rh(z) \, | \, z \in \bar{\Omega}\} \ \ \forall y \in \mathbb{R}^d$$

and

$$g_r(z) = \psi(z) + rh(z) \ \ \forall z \in \mathbb{R}^d.$$

Lemma 2.4, (26) and (28) imply that

$$\lim_{r \to 0} \frac{f_r(y) - f_0(y)}{r} = -h(\nabla f_0(y)) = -h(\nabla \psi^*(y))$$

and

$$|\frac{f_r(y) - f_0(y)}{r}| \leq ||h||_\infty + r$$

for almost every $y \in \mathbb{R}^d$ and so, since $\mathbf{u}$ satisfies the $N^{-1}-$ property and since $meas(\partial\Omega) = 0$ we deduce that

$$\lim_{r \to 0} \frac{f_r(\mathbf{u}(x)) - f_0(\mathbf{u}(x))}{r} = -h(\nabla f_0(\mathbf{u}(x))) = -h(\nabla \psi^*(\mathbf{u}(x))) \tag{29}$$

and

$$|\frac{f_r(\mathbf{u}(x)) - f_0(\mathbf{u}(x))}{r}| \leq ||h||_\infty + r \tag{30}$$

for almost every $x \in \Omega$. One can readily check that $\{(f_r, g_r)\} \subset E_R$ and will the help of (29) and (30) check that

$$\lim_{r \to 0} \frac{I(f_r, g_r) - I(f_0, g_0)}{r} = \int_\Omega [h(x) - h(\nabla \psi^*(\mathbf{u}(x)))] dx$$

and so using the definition of f_0, (26), (28), as (f_0, g_0) is a minimizer of I on E_R and as $\{(f_r, g_r)\} \subset E_R$ we deduce that

$$0 = \int_\Omega [h(x) - h(\nabla \psi^*(\mathbf{u}(x)))].$$

Since $h \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is arbitrary, if we define

$$\mathbf{s}(x) = \nabla \psi^*(\mathbf{u}(x)) \tag{31}$$

for almost every $x \in \Omega$, as $meas(\partial\Omega) = 0$ we may extend $\mathbf{s}$ to $\bar{\Omega}$ and we obtain that $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ is a measure-preserving mapping. By (26) and (31) we deduce that

$$\mathbf{u}(x) = \nabla \psi^{**}(\mathbf{s}(x)) = \nabla \psi(\mathbf{s}(x)) \tag{32}$$

for almost every $x \in \bar{\Omega}$.

Step3. We prove (iv). For every $(f, g) \in E_R$ and every measure-preserving mapping $\mathbf{t}$ on Ω, we have

$$\int_\Omega [f(\mathbf{u}(x)) + g(x)] dx = \int_\Omega [f(\mathbf{u}(x)) + g(\mathbf{t}(x))] dx \geq \int_\Omega \mathbf{u}(x) \mathbf{t}(x) dx$$

and so

$$i_\infty \geq j_\infty. \tag{33}$$

Let $(\phi, \psi) \in E_R$ be the pair of convex functions of step2 and let $\mathbf{s}$ be the measure-preserving mapping on Ω of step2, such that (32) holds. We observe that

$$\phi(\mathbf{u}(x)) + \psi(\mathbf{s}(x)) = \mathbf{u}(x)\mathbf{s}(x)$$

for almost every $x \in \bar{\Omega}$ and so

$$j_\infty \geq \int_\Omega \mathbf{u}(x)\mathbf{s}(x)dx = \int_\Omega [\phi(\mathbf{u}(x)) + \psi(\mathbf{s}(x))]dx \geq i_\infty. \tag{34}$$

By (33) and (34) we deduce that $i_\infty = j_\infty$, i.e. the problem in (7) is dual to the problem in (9). ∎

3 The polar factorization for L^p mappings.

The main result of this section is the polar factorization for L^p mappings and a duality result (Corollary 3.1). In Corollary 3.2 we prove that given a mapping $\mathbf{u}$, the polar factors ψ and $\mathbf{s}$ such that $\mathbf{u} = \nabla\psi \circ \mathbf{s}$ are uniquely determine in a sense to be precised and the mapping $\mathbf{u} \to (\psi, \mathbf{s})$ is a continuous mapping.

Throughout this section $\gamma > 0$ is a constant, $\Omega \subset \mathbb{R}^d$ is an open, bounded set such that $\bar{\Omega} \subset B_\gamma$.

Corollary 3.1 *Let $1 \leq p \leq +\infty$ and let $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ be a vector-valued mapping satisfying the $N^{-1}-$property. Then*
(i) the problem

$$i_p = \inf\{\int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx \mid (\phi, \psi) \in E\} \tag{35}$$

is dual to the problem

$$j_p = \sup\{\int_\Omega \mathbf{s}(x)\mathbf{u}(x) \mid \mathbf{s} \in S\}, \tag{36}$$

in the sense that $i_p = j_p$, where

$$E = \{(\phi, \psi) \,|\, \phi \in C(\mathbb{R}^d) \cap L^1_{\mathbf{u}}(\mathbb{R}^d), \psi \in C(\Omega) \cap L^1(\Omega),\ \phi(y) + \psi(z) \geq yz \,\forall (y, z) \in \mathbb{R}^d \times \Omega\},$$

$$L^1_{\mathbf{u}}(\mathbb{R}^d) = \{\phi : \mathbb{R}^d \to \mathbb{R} \,|\, \phi \circ \mathbf{u} \in L^1(\Omega)\}$$

and as in the previous section

$$S = \{\mathbf{s} : \bar{\Omega} \to \bar{\Omega} \text{ measure-preserving mapping}\}.$$

(ii) In addition $\mathbf{u}$ can be factored into

$$\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x)), \tag{37}$$

for almost every $x \in \Omega$, where $\mathbf{s} \in S$, $\psi \in W^{1,p}(\Omega)$ admits a convex extension on $\mathbb{R}^d$ and satisfies $\int_\Omega \psi(x)dx = 0$.

Corollary 3.2 *With the assumptions of Corollary 3.1 the following properties hold:*
(i) The decomposition in (37) is unique
(ii) The mapping $\mathbf{u} \to (\nabla\psi, \mathbf{s})$ is a continuous mapping from $L^p(\Omega, \mathbb{R}^d) \setminus P$ into $L^p(\Omega, \mathbb{R}^d) \times L^q(\Omega, \mathbb{R}^d)$, for all $q \in [1, +\infty[$, where $P = \{\mathbf{u} \in L^p(\Omega, \mathbb{R}^d) \,|\, \mathbf{u}$ does not satisfies the $N^{-1}-$ property$\}$.

Before given the proof of Corollaries 3.1, 3.2 we make some remarks.

Remarks 3.3

(a) As we already mentioned in the introduction, the polar factorization theorem was first proved by Brenier in [Br1] and [Br2] (see also the references of [Br2] for earlier related results). His proof consisted of studying a Monge-Kantorovich problem which is a minimization problem $\inf\{I(p) \mid p \in \mathcal{P}\}$, where

$$\mathcal{P} = \{p \text{ } probability \text{ } measure \text{ } on \text{ } \bar\Omega\times\bar\Omega \mid \int_{\bar\Omega\times\bar\Omega} f(x)p(dx,dy) = \int_{\bar\Omega\times\bar\Omega} f(y)p(dx,dy)\} = \int_{\bar\Omega} f(x)dx \;\forall f \in C(\bar\Omega)\}.$$

and then he deduced the polar factorization theorem.

(b) For every $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$ satisfying the $N^{-1}-$property and for every Lebesgue measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ one can readily check that $\phi \circ \mathbf{u}$ is measurable.

(c) If $\mathbf{u} = \nabla\psi \circ \mathbf{s}$ then for every $c \in \mathbb{R}$ setting $\psi_c = \psi + c$ we also have $\mathbf{u} = \nabla\psi_c \circ \mathbf{s}$. Therefore we need the condition $\int_\Omega \psi(x)dx = 0$ to ensure the uniqueness of the factorization in Corollary 3.1.

(d) If $\mathbf{u} = \nabla\psi \circ \mathbf{s}$ are as in Theorem 2.3 we observe that

$$\int_\Omega ||\psi(y)||^p dy = \int_\Omega ||\psi(\mathbf{s}(x))||^p dx = \int_\Omega ||\mathbf{u}(x)||^p dx.$$

(e) Assume that $1 \leq p \leq +\infty$, $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$ satisfies the $N^{-1}-$ property, $(\phi,\psi) \in E$ and $\mathbf{s} \in S$. Then

$$\int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx = \int_\Omega [\phi(\mathbf{u}(x)) + \psi(\mathbf{s}(x))]dx \geq \int_\Omega \mathbf{s}(x)\mathbf{u}(x).$$

Hence if

$$\int_\Omega [\phi_0(\mathbf{u}(x)) + \psi_0(x)]dx = \int_\Omega \mathbf{s}_0(x)\mathbf{u}(x)dx$$

for some pair $(\phi_0,\psi_0) \in E$ *and for some* $\mathbf{s}_0 \in S$ *then*

$$i_p = \int_\Omega [\phi_0(\mathbf{u}(x)) + \psi_0(x)]dx = \int_\Omega \mathbf{s}_0(x)\mathbf{u}(x)dx = j_p.$$

(f) We notice by (21), (26), (27) and (28) that there exists a pair $(\phi,\psi) \in E_R$ such that

$$i_\infty = I(\phi,\psi),$$

$$\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x)), \tag{38}$$

$$Lip(\phi) \leq d\gamma, \; Lip(\psi) \leq dR, \tag{39}$$

$$\phi^* = \psi, \; \psi^* = \phi, \tag{40}$$

and

$$\phi(y) = \sup\{yz - \psi(z) \mid z \in \bar\Omega\}, \;\; \forall y \in \mathbb{R}^d. \tag{41}$$

∎

Proof of Corollary 3.1. If $p = +\infty$ then Theorem 2.3 implies Corollary 3.1. Assume in the sequel that $1 \leq p < +\infty$. We split the proof into two steps.
Step1. We prove (i). Define $\mathbf{u}_n : \Omega \to \mathbb{R}^d$ by

$$\mathbf{u}_n(x) = \mathbf{r}_n(\mathbf{u}(x)),$$

where $\mathbf{r}_n$ is a diffeomorphism from $\mathbb{R}^d$ onto B_n such that

$$||\mathbf{r}_n(y)|| \leq ||y||, \tag{42}$$

for all $y \in \mathbb{R}^d$ and

$$\mathbf{r}_n(y) \to y \text{ uniformly on any compact subset of } \mathbb{R}^d. \tag{43}$$

We observe that for each $n \in \mathbb{N}$, $\mathbf{u}_n \in L^\infty(\Omega, \mathbb{R}^d)$ and $\mathbf{u}_n$ satisfies the $N^{-1}-$ property. As

$$\sup\{||\mathbf{u}_n(x)|| \mid x \in \Omega\} \leq n$$

we obtain by Theorem 2.3 that there exist a measure preserving mapping $\mathbf{s}_n$ and a pair of Lipschitz, convex functions $\phi_n, \psi_n : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\mathbf{u}_n(x) = \nabla\psi_n(\mathbf{s}_n(x)), \tag{44}$$

for almost every $x \in \Omega$,

$$\phi_n(y) = \sup\{yz - \psi_n(z) \mid z \in \Omega\},$$

for every $y \in \mathbb{R}^d$. By Remark 3.3 (d), (38), (39), (42) and the Hölder's inequality we obtain that

$$Lip(\phi_n) \leq d\gamma,\ Lip(\psi_n) \leq dn, \tag{45}$$

$$\int_\Omega ||\nabla\psi_n(x)||dx = \int_\Omega ||\mathbf{u}_n(x)||dx \leq \int_\Omega ||\mathbf{u}(x)||dx \leq (1 + meas(\Omega))(\int_\Omega ||\mathbf{u}(x)||^p dx)^{\frac{1}{p}}. \tag{46}$$

We may assume without loss of generality that

$$\int_\Omega \psi_n(z)dz = 0, \tag{47}$$

for every $n \in \mathbb{N}$ and so, with the help of (45) we have

$$\psi_n \in K_0,$$

for every $n \in \mathbb{N}$, where

$$K_0 = \{h \in W^{1,1}(\Omega) \cap C(\Omega) \mid \int_\Omega h(z)dz = 0, \exists h_1 : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}, \text{ convex, l.s.c. such that } h_1|_\Omega = h\}.$$

By Proposition 4.4, (40), (41) and (45) there exist a subsequence still labelled by n and a pair of convex functions (ϕ, ψ) such that

$$\psi \in L^1(\Omega) \cap C(\Omega),\ \phi \in C(\mathbb{R}^d) \text{ and } Lip(\phi) \leq d\gamma \tag{48}$$

$$\phi(y) + \psi(z) \geq yz \text{ for all } (y, z) \in \mathbb{R}^d \times \Omega \tag{49}$$

$$\psi_n \to \psi \text{ in } L^1(\Omega) \text{ and uniformly on any compact subset of } \Omega$$

$$\phi_n \to \phi \text{ uniformly in any compact subset of } \mathbb{R}^d \tag{50}$$

$$||\phi(y)|| \leq \gamma(||y|| + 2M) \text{ for all } y \in \mathbb{R}^d, \tag{51}$$

where $M = (1+meas(\Omega))(\int_\Omega ||\nabla \mathbf{u}(x)||^p dx)^{\frac{1}{p}}$. By Proposition 4.2 and (50) $\nabla\phi_n(y)$ converges to $\nabla\phi(y)$ as n goes to infinity, for almost every $y \in \mathbb{R}^d$. As $\mathbf{u}$ satisfies the $N^{-1}-$ property we deduce that $\{\nabla\phi_n(\mathbf{u}(x))\}$ converges to $\nabla\phi(\mathbf{u}(x))$ as n goes to infinity, for almost every $x \in \Omega$, and so with the help of (45) we obtain

$$\nabla\phi_n(\mathbf{u}_n(x)) \to \nabla\phi(\mathbf{u}(x)) \ \text{ in } \ L^q(\Omega, \mathbb{R}^d), \tag{52}$$

for every $1 \le q < +\infty$. We deduce that for every $h \in C(\mathbb{R}^d) \cap L^\infty(\Omega)$

$$\int_\Omega h(\nabla\phi(\mathbf{u}(x)))dx = \lim_{n\to+\infty}\int_\Omega h(\nabla\phi_n(\mathbf{u}_n(x))) = \lim_{n\to+\infty}\int_\Omega h(\mathbf{s}_n(x)) = \int_\Omega h(x). \tag{53}$$

If we define $\mathbf{s} : \bar\Omega \to \mathbb{R}^d$ by

$$\mathbf{s}(x) = \nabla\phi(\mathbf{u}(x)),$$

(53) implies that $\mathbf{s}$ is a measure preserving mapping from $\bar\Omega$ into $\bar\Omega$ and by (52), $\mathbf{s}_n$ converges to $\mathbf{s}$ in $L^q(\Omega, \mathbb{R}^d)$ for every $1 \le q < +\infty$. Theorem 2.3 implies

$$\int_\Omega [\phi_n(\mathbf{u}_n(x)) + \psi_n(x)]dx = \int_\Omega \mathbf{u}_n(x)\mathbf{s}_n(x)dx$$

and so, letting n go to infinity we deduce that

$$\int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx = \int_\Omega \mathbf{u}(x)\mathbf{s}(x)dx. \tag{54}$$

As $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d) \subset L^1(\Omega, \mathbb{R}^d)$, (48) and (51) imply that $(\phi, \psi) \in E$ and so by Remark 3.3 (e) and (54) we have

$$i_p = \int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx = \int_\Omega \mathbf{s}(x)\mathbf{u}(x) = j_p. \tag{55}$$

step2. We prove (ii).
Claim1. $\phi^*(z) = \psi(z)$ for all $z \in \Omega$.
As $(\phi, \psi) \in E$ we observe that $\phi^* \le \psi$, $(\phi, \phi^*) \in E$ and so by (55)

$$\int_\Omega [\phi(\mathbf{u}(x)) + \phi^*(x)]dx \le \int_\Omega [\phi(\mathbf{u}(x)) + \psi(x)]dx = i_p.$$

Hence,

$$\phi^*|_\Omega = \psi|_\Omega. \tag{56}$$

Claim2. $\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x))$ for almost every $x \in \Omega$.
By (48) and (56) $\Omega \subset dom(\phi^*)$ and so by Proposition 4.3, ϕ^* is differentiable at almost every point $z \in \Omega$. Since $\mathbf{s} : \bar\Omega \to \bar\Omega$ satisfies the $N^{-1}-$ property and $meas(\partial\Omega) = 0$ we deduce that ϕ^* is differentiable at $\mathbf{s}(x)$ for almost every $x \in \Omega$. By (55) and claim1 as $(\phi, \psi) \in E$ we have

$$\int_\Omega [\phi(\mathbf{u}(x)) + \phi^*(\mathbf{s}(x))]dx = \int_\Omega \mathbf{s}(x)\mathbf{u}(x)dx$$

and so

$$\phi(\mathbf{u}(x)) + \phi^*(\mathbf{s}(x)) = \mathbf{s}(x)\mathbf{u}(x),$$

for almost every $x \in \Omega$. Hence,

$$\mathbf{u}(x) = \nabla\phi^*(\mathbf{s}(x)), \tag{57}$$

for almost every $x \in \Omega$. Using again that $\mathbf{s} : \bar\Omega \to \bar\Omega$ satisfies the $N^{-1}-$ property and $meas(\partial\Omega) = 0$, by (56) and (57) we obtain

$$\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x)),$$

for almost every $x \in \Omega$. ∎

We make first the following remark we will use often in the proof of Corollary 3.2.

Remarks 3.4 Assume that $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ is a mapping satisfying the $N^{-1}-$ property such that

$$\mathbf{u}(x) = \nabla\psi(\mathbf{s}(x)), \tag{58}$$

for almost every $x \in \Omega$, where $\psi \in C(\Omega) \cap L^1(\Omega)$, is a function satisfying, $\int_\Omega \psi(x)dx = 0$ *and* $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ is a measure-preserving mapping. Then by Remark 3.3 (d) we observe that in fact, $\psi \in W^{1,p}(\Omega) \cap C(\Omega)$. If we define $\tilde{\psi} : \mathbb{R}^d \to \mathbb{R}$ by

$$\tilde{\psi}(y) = \sup\{yz - \psi(z) \mid z \in \Omega\}$$

then Proposition 4.4 (i), (vi) (with $\psi_n = \psi$) we deduce that

$$Lip(\tilde{\psi}) \leq \gamma d, \quad ||\tilde{\psi}(y)|| \leq \gamma(||y|| + 2M)$$

for all $y \in \mathbb{R}^d$, where $M = \int_\Omega ||\nabla\psi(x)||dx$. By Proposition 4.5 and by a technique we already used in the claim1 of the proof of Corollary 3.1 we deduce that ψ admits a convex, lower semicontinuous extension on $\mathbb{R}^d$ we still denote by ψ, so that

$$\psi^* = \tilde{\psi}, \quad \tilde{\psi}^* = \psi.$$

It follows from (58) that

$$\psi^*(\mathbf{u}(x)) + \psi(\mathbf{s}(x)) = \mathbf{s}(x)\mathbf{u}(x),$$

for almost every $x \in \Omega$.

Proof of Corollary 3.2 We divide the proof into two parts.
Part1. We prove the uniqueness of the factors. One can notice that it suffices to prove the uniqueness of the factors in the case $p = 1$ and the case $1 \leq p \leq +\infty$ readily follows.
Assume that

$$\mathbf{u}(x) = \nabla\psi_1(\mathbf{s}_1(x)) = \nabla\psi_2(\mathbf{s}_2(x)), \tag{59}$$

for almost every $x \in \Omega$, where $\mathbf{s}_1, \mathbf{s}_2 : \bar{\Omega} \to \bar{\Omega}$ are measure-preserving mappings,

$$\psi_1, \psi_2 \in C(\Omega) \cap W^{1,1}(\Omega), \quad i = 1, 2 \tag{60}$$

and

$$\int_\Omega \psi_1(z)dz = \int_\Omega \psi_2(z)dz = 0. \tag{61}$$

Assume moreover that ψ_1 and ψ_2 admit a convex lower semicontinuous extension on $\mathbb{R}^d$ we still denote by ψ_1 and ψ_2. By (59) we have

$$\psi_1^*(\mathbf{u}(x)) + \psi_1(\mathbf{s}_1(x)) = \mathbf{s}_1(x)\mathbf{u}(x), \quad \psi_2^*(\mathbf{u}(x)) + \psi_2(\mathbf{s}_2(x)) = \mathbf{s}_2(x)\mathbf{u}(x), \tag{62}$$

for almost every $x \in \Omega$.

Claim1. $\mathbf{s}_1(x) = \mathbf{s}_2(x)$ for almost every $x \in \Omega$.
Indeed, by Remark 3.3 (e) and (62) as $\mathbf{s}_1$ and $\mathbf{s}_2$ are measure-preserving mappings, we have

$$\int_\Omega \mathbf{s}_1(x)\mathbf{u}(x)dx = \int_\Omega \mathbf{s}_2(x)\mathbf{u}(x)dx = \int_\Omega [\psi_2^*(\mathbf{u}(x)) + \psi_2(\mathbf{s}_2(x))] = \int_\Omega [\psi_2^*(\mathbf{u}(x)) + \psi_2(\mathbf{s}_1(x))]$$

and so, as

$$\psi_2^*(\mathbf{u}(x)) + \psi_2(\mathbf{s}_1(x)) \geq \mathbf{s}_1(x)\mathbf{u}(x),$$

for almost every $x \in \Omega$, we deduce that

$$\psi_2^*(\mathbf{u}(x)) + \psi_2(\mathbf{s}_1(x)) = \mathbf{s}_1(x)\mathbf{u}(x),$$

for almost every $x \in \Omega$ and so

$$\mathbf{s}_1(x) = \nabla\psi_2^*(\mathbf{u}(x)),$$

for almost every $x \in \Omega$. The equation above combined with (59) yields

$$\mathbf{s}_1(x) = \nabla\psi_2^*(\mathbf{u}(x)) = \mathbf{s}_2(x),$$

for almost every $x \in \Omega$.

In the sequel we set

$$\mathbf{s}(x) = \mathbf{s}_1(x) = \mathbf{s}_2(x), \tag{63}$$

for almost every $x \in \Omega$.

Claim2. For every $N \subset \bar{\Omega}$ subset of zero measure we have $meas(\mathbf{s}(\Omega \setminus N)) = meas(\Omega)$.
Indeed, as $meas$ is a Borel regular measure, there exists a measurable set B such that

$$\mathbf{s}(\Omega \setminus N) \subset B \subset \bar{\Omega} \tag{64}$$

and

$$meas(\mathbf{s}(\Omega \setminus N)) = meas(B). \tag{65}$$

As $\mathbf{s}$ is a measure-preserving mapping and as $meas(\partial\Omega) = 0$, (64) and (65) imply that

$$meas(B) = meas(\mathbf{s}^{-1}(B)) \geq meas(\mathbf{s}^{-1}(\mathbf{s}(\Omega \setminus N))) \geq meas(\Omega \setminus N) = meas(\Omega) \geq meas(B)$$

and so

$$meas(\mathbf{s}(\Omega \setminus N)) = meas(\Omega).$$

Claim3 $\psi_1(z) = \psi_2(z)$ for almost every $z \in \Omega$.
Define

$$N_i = \{z \in \Omega \mid \psi_i \text{ is not differentiable at } z\},\ i = 1, 2$$

and

$$M = \{x \in \Omega \mid \mathbf{s}_1(x) = \mathbf{s}_2(x)\}.$$

By Proposition 4.3 and (63)

$$meas(N_1 \cup N_2) = meas(M) = 0. \tag{66}$$

By (59) and (63) we observe that

$$\nabla\psi_1(z) = \nabla\psi_2(z) \tag{67}$$

for every $z \in A := \mathbf{s}(\Omega \setminus M) \setminus (N_1 \cup N_2 \cup \partial\Omega)$. As $meas(\partial\Omega) = 0$, we obtain by Claim2 and (66) that

$$meas(A) = meas(\Omega). \tag{68}$$

Let $z \in \overline{A \setminus (N_1 \cup N_2)}$ and let $\{z_n\} \subset A \setminus (N_1 \cup N_2)$ be a sequence such that

$$z = \lim_{n\to+\infty} z_n.$$

By Proposition 4.3 $\nabla\psi_1$ and $\nabla\psi_2$ are respectively continuous on $\Omega \setminus N_1$ and $\Omega \setminus N_2$ and so with the help of (67) we observe that

$$\nabla\psi_1(z) = \lim_{n\to+\infty} \nabla\psi_1(z_n) = \lim_{n\to+\infty} \nabla\psi_2(z_n) = \nabla\psi_2(z). \tag{69}$$

Using (68) one can readily check that

$$meas(\overline{A \setminus (N_1 \cup N_2)}) = meas(\Omega),$$

hence, (69) holds for almost every $z \in \Omega$. Set

$$\psi = \psi_1 - \psi_2.$$

By (60), (61) and (69) we have

$$\psi \in W^{1,1}(\Omega),$$

$$\int_\Omega \psi(z)dz = 0$$

and

$$\nabla\psi(z) = 0,$$

for almost every $z \in \Omega$. Hence by Poincare's inequality we deduce that

$$\psi(z) = 0,$$

for almost every $z \in \Omega$. As by (60) $\psi \in C(\Omega)$ we deduce that

$$\psi(z) = 0,$$

for every $z \in \Omega$.

Part2. Let $q \in [1, +\infty[$. We prove that for each $1 \leq p \leq +\infty$ the mapping $H_p : \mathbf{u} \to (\nabla\psi, \mathbf{s})$ a continuous mapping from $L^p(\Omega, \mathbb{R}^d) \setminus P$ into $L^p(\Omega, \mathbb{R}^d) \times L^q(\Omega, \mathbb{R}^d)$. One can notice that it suffices to prove that $H := H_1$ is continuous. Let $\mathbf{u} \in L^1(\Omega, \mathbb{R}^d) \setminus P$ and let $\{\mathbf{u}_n\} \subset L^1(\Omega, \mathbb{R}^d) \setminus P$ be an arbitrary sequence converging to $\mathbf{u}$ in $L^1(\Omega, \mathbb{R}^d)$. As $L^1(\Omega, \mathbb{R}^d) \setminus P$ is a normed space, if we prove that we may extract a subsequence $\{\mathbf{u}_{n_i}\} \subset L^p(\Omega, \mathbb{R}^d) \setminus P$ such that $H(\mathbf{u}_{n_i})$ converges to $H(\mathbf{u})$ we are done. Assume as in Corollary 3.1 that $\mathbf{u}_n$ and $\mathbf{u}$ are factored

$$\mathbf{u}_n = \nabla\psi_n \circ \mathbf{s}_n, \quad \mathbf{u} = \nabla\psi \circ \mathbf{s}, \tag{70}$$

where $\psi_n, \psi \in K_0$ (see Appendix for the definition of K_0) and $\mathbf{s}_n, \mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ are measure-preserving mappings. Define

$$\phi(y) = \sup\{yz - \psi(z) \mid z \in \Omega\}, \ \forall y \in \mathbb{R}^d,$$

$$\phi_n(y) = \sup\{yz - \psi_n(z) \mid z \in \Omega\}, \ \forall y \in \mathbb{R}^d$$

and

$$M = \max\{\sup\{\int_\Omega ||\nabla\psi_n(z)||dz \ n \in \mathbb{N}\}, \int_\Omega ||\nabla\psi(z)||dz\}.$$

Since

$$\int_\Omega ||\nabla\psi_n(z)||dz = \int_\Omega ||\mathbf{u}_n(z)||dz, \quad \int_\Omega ||\nabla\psi(z)||dz = \int_\Omega ||\mathbf{u}(z)||dz$$

and since $\{\mathbf{u}_n\}$ converges to $\mathbf{u}$ in $L^1(\Omega, \mathbb{R}^d)$ we observe that

$$M < +\infty$$

and so, by Proposition 4.4 there exists a subsequence still labelled by n and a pair (f, g) such that

$$\begin{array}{ll} (i) & f \in C(\mathbb{R}^d) \text{ and } Lip(f) \leq d\gamma \\ (ii) & g \in L^1(\Omega) \cap C(\Omega), \\ (iii) & f(y) + g(z) \geq yz \text{ for all } (y, z) \in \mathbb{R}^d \times \Omega \\ (iv) & \psi_n \to g \text{ in } L^1(\Omega) \text{ and uniformly on any compact subset of } \Omega \\ (v) & \phi_n \to f \text{ uniformly in any compact subset of } \mathbb{R}^d \\ (vi) & ||\phi_n(y)|| \leq \gamma(||y|| + 2M) \text{ for all } y \in \mathbb{R}^d, \end{array} \tag{71}$$

By Remark 3.4 and (70) we obtain that

$$\mathbf{s}_n = \nabla\phi_n \circ \mathbf{u}_n \text{ and } \mathbf{s} = \nabla\phi \circ \mathbf{u}. \tag{72}$$

As $\{\mathbf{u}_n\}$ converges to $\mathbf{u}$ in $L^1(\Omega, \mathbb{R}^d)$, up to a subsequence we still labelled by n, $\{\mathbf{u}_n(x)\}$ converges to $\mathbf{u}(x)$ for almost every $x \in \Omega$ and so as $\mathbf{u}_n$ and $\mathbf{u}$ satisfy the $N^{-1}-$ property we obtain by Proposition 4.2 and (71) that $\mathbf{s}_n = \nabla \phi_n \circ \mathbf{u}_n(x)$ converges to $\nabla f \circ \mathbf{u}(x)$ for almost every $x \in \Omega$. As $\{\mathbf{s}_n\}$ is bounded in $L^\infty(\Omega, \mathbb{R}^d)$, using Egorov's theorem we deduce that

$$\mathbf{s}_n \equiv \nabla \phi_n \circ \mathbf{u}_n \to \nabla f \circ \mathbf{u} \;\text{ in }\; L^q(\Omega, \mathbb{R}^d).$$

One can readily check that $\nabla f \circ \mathbf{u}$ is a measure-preserving mapping. Using an argument similar to the one above we may deduce that

$$\mathbf{u}_n = \nabla \psi_n \circ \mathbf{s}_n \to \nabla g \circ \nabla f \circ \mathbf{u} \;\text{ in }\; L^1(\Omega, \mathbb{R}^d)$$

and so as by assumption $\{\mathbf{u}_n\}$ converges to $\mathbf{u}$ in $L^1(\Omega, \mathbb{R}^d)$, we deduce that

$$\mathbf{u} = \nabla g \circ \nabla f \circ \mathbf{u}.$$

By the uniqueness property of the polar decomposition we obtain that

$$g = \psi \;\text{ and }\; \mathbf{s} = \nabla f \circ \mathbf{u}.$$

Therefore

$$\mathbf{s}_n \to \mathbf{s} \;\text{ in }\; L^q(\Omega, \mathbb{R}^d)$$

and

$$\psi_n \to \psi \;\text{ in }\; L^1(\Omega, \mathbb{R}^d)$$

hence H is continuous. $\blacksquare$

Warning If $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ is a measure-preserving mapping then, $meas(\mathbf{s}(\Omega)) = meas(\Omega)$, but, as we don't know whether or not $\mathbf{s}(\Omega)$ is measurable we cannot deduce that $meas(\Omega \setminus \mathbf{s}(\Omega)) = 0$.

4 Appendix.

Throughout this section $\Omega \subset \mathbb{R}^d$ is an open, bounded of $\mathbb{R}^d$. We recall some definitions and review some results of the convex analysis needed in this paper. We complete the section by making a remark on measure-preserving mappings.

Definition 4.1 *If $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ we define ψ^* the Legendre-Fenchel transform of ψ by*

$$\psi^*(y) = \sup\{yz - \psi(z) \mid z \in \mathbb{R}^d\}$$

and we define ψ^{**} to be $(\psi^*)^*$.

Proposition 4.2 *Let $C \subset \mathbb{R}^d$ be a convex, open set let $f_i : C \to \mathbb{R}$ $i = 0, 1, 2, \cdots,$ be a sequence of convex functions converging pointwise to $f : C \to \mathbb{R}$ on C. Let*

$$N = \{x \in C \,|\, \exists i \in N, \nabla f_i \text{ is not differentiable at } x\} \cup \{x \in C \,|\, \nabla f \text{ is not differentiable at } x\}.$$

Then,

$$meas(N) = 0$$

and

$$\nabla f_i(x_i) \to \nabla f(x)$$

for every $x \in C \setminus N$ and for every sequence $\{x_i\} \subset C \setminus N$ converging to x.

Proof. The Proposition is a consequence of Theorem 24.5 and 25.5 in [Ro]. ∎

Proposition 4.3 *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex function and let $D \subset \mathbb{R}^d$ be the set of the points where f is differentiable. Then D is a dense subset of $int(dom(f))$ and its complement in $int(dom(f))$ is a set of zero measure. Furthermore, the gradient mapping $\nabla f : x \to \nabla f(x)$ is continuous on D.*

Proof. We refer the reader to Theorem 25.5 in [Ro].

Following [Br1] we define the set K_0.

$$K_0 = \{\psi \in W^{1,1}(\Omega) \cap C(\Omega) \mid \int_\Omega \psi(z)dz = 0, \exists \psi_1 : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}, \text{ convex, l.s.c. such that } \psi_1|_\Omega = \psi\},$$

where l.s.c. stands for the abreviation of the expression lower semicontinuous.

Proposition 4.4 *Let $M > 0$ be a constant and $\{\psi_n\}_{n=1}^\infty \subset K_0$ be a sequence such that*

$$M := \sup_{n \in \mathbb{N}} \int_\Omega ||\psi_n(z)||dz < +\infty.$$

Then there exists a subsequence, still labelled by n, and a pair (ϕ, ψ) such that

$$\begin{array}{ll} (i) & \phi \in C(\mathbb{R}^d) \text{ and } Lip(\phi) \le d\gamma \\ (ii) & \psi \in L^1(\Omega) \cap C(\Omega), \\ (iii) & \phi(y) + \psi(z) \ge yz \text{ for all } (y,z) \in \mathbb{R}^d \times \Omega \\ (iv) & \psi_n \to \psi \text{ in } L^1(\Omega) \text{ and uniformly on any compact subset of } \Omega \\ (v) & \tilde{\psi}_n \to \phi \text{ uniformly in any compact subset of } \mathbb{R}^d \\ (vi) & ||\tilde{\psi}_n(y)|| \le \gamma(||y|| + 2M) \text{ for all } y \in \mathbb{R}^d, \end{array}$$

where $\tilde{\psi}_n : \mathbb{R}^d \to \mathbb{R}$ is the convex function defined by

$$\tilde{\psi}_n(y) = \sup\{yz - \psi_n(z) \mid z \in \Omega\}.$$

Proof. We refer the reader to Proposition 3.3 in [Br1]. ∎

Proposition 4.5 *Assume that $\bar{\Omega} \subset B_R$ and $f : \mathbb{R}^d \to \mathbb{R}$. Define $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ by*

$$\tilde{f}(y) = \sup\{yz - f(z) \mid z \in \Omega\}.$$

(a) If $f \in L^\infty(\Omega)$ then $\tilde{f}$ is a convex, Lipschitz function and

$$|\tilde{f}(y_1) - \tilde{f}(y_2)| \le dR||y_1 - y_2||$$

for every $y_1, y_2 \in \mathbb{R}^d$.
(b)

$$\tilde{f}(y) = \sup\{yz - (\tilde{f})^*(z) \mid z \in \Omega\} = \sup\{yz - (\tilde{f})^*(z) \mid z \in \mathbb{R}^d\}$$

for every $y \in \mathbb{R}^d$.

Proof

Proof of (a). Since $\tilde{f}$ is a supremum of convex functions it is straightforward to check that $\tilde{f}$ is still a convex function and as Ω is bounded and $f \in L^\infty(\Omega)$ one can readily check that $dom(\tilde{f}) = \mathbb{R}^d$. Hence (see Proposition 4.3) $\tilde{f}$ is differentiable at almost every point of $\mathbb{R}^d$. Let $y \in \mathbb{R}^d$ be a point such that $\tilde{f}$ is differentiable at y and let $z_h, \bar{z}_h \in \Omega$ be such that

$$\tilde{f}(y+h) \leq (y+h)z_h - f(z_h) + ||h||^2 \leq \tilde{f}(y) + hz_h + ||h||^2$$

and

$$\tilde{f}(y) \leq y\bar{z}_h - f(\bar{z}_h) + ||h||^2 \leq \tilde{f}(y+h) - h\bar{z}_h + ||h||^2.$$

We obtain that

$$-||h|| + \frac{h}{||h||}\bar{z}_h \leq \frac{\tilde{f}(y+h) - \tilde{f}(y)}{||h||} \leq ||h|| + \frac{h}{||h||}z_h$$

and so

$$|\frac{\partial \tilde{f}(y)}{\partial y_i}| \leq R$$

for every $i = 1, \cdots, d$ which completes the proof of (a).

Proof of (b). We define $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(z) = \begin{cases} f(z) & \text{if } z \in \Omega \\ +\infty & \text{if } z \notin \Omega \end{cases}$$

We have (see [Da]) $\phi^{***} = \phi^* = \tilde{f}$ and $\phi^{**} \leq \phi$. Hence

$$\begin{aligned} \phi^*(y) = \phi^{***}(y) &= \sup\{yz - \phi^{**}(z) \mid z \in \mathbb{R}^d\} \\ &\geq \sup\{yz - \phi^{**}(z) \mid z \in \Omega\} \\ &\geq \sup\{yz - \phi(z) \mid z \in \Omega\} \\ &= \sup\{yz - f(z) \mid z \in \Omega\} = \phi^*(y) \end{aligned} \tag{73}$$

and so

$$\phi^*(y) = \sup\{yz - \phi^{**}(z) \mid z \in \mathbb{R}^d\} = \sup\{yz - \phi^{**}(z) \mid z \in \Omega\}$$

for every $y \in \mathbb{R}^d$. ∎

We end this section by making a remark on measure-preserving mappings.

Remarks 4.6 It is well known that a measure-preserving mapping $\mathbf{s}$ is not necessarily one-to-one. Indeed, if $\Omega = [0,1]^d$ and if $\mathbf{s} : \bar{\Omega} \to \bar{\Omega}$ is defined by $\mathbf{s}(x_1, \cdots, x_d) = (\min(2x_1, 2 - 2x_1), x_2, \cdots, x_d)$ then $\mathbf{s}$ is a measure-preserving mapping such that

$$\mathbf{s} \in W^{1,\infty}(\Omega, \mathbb{R}^d),$$

$$|det(\nabla \mathbf{s}(x)| = 2,$$

for almost every $x \in \Omega$ *and*

$$\mathbf{s}(x_1, x_2 \cdots, x_d) = \mathbf{s}(1 - x_1, x_2 \cdots, x_d),$$

for every $x \in \Omega$.

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