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Some examples of rank one convex functions in dimension two

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Synopsis

We study the rank one convexity of some functions $f(\xi)$ where ξ is a 2×2 matrix. Examples such as $|\xi|^{2\alpha}+h(\det\xi)$ and $|\xi|^{2\alpha}(|\xi|^2-\gamma\det\xi)$ are investigated. Numerical computations are done on the example of Dacorogna and Marcellini, $|\xi|^4-\frac{4}{\sqrt{3}}|\xi|^2\det\xi$, indicating that this function is quasiconvex.

Introduction

In this paper we consider

$$I(u)=\int_\Omega f(\nabla u(x))\,dx, \tag{0.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded open set, $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ and therefore $\nabla u\in\mathbb{R}^{2\times 2}$ (the set of 2×2 matrices which will often be identified with $\mathbb{R}^4$) and $f:\mathbb{R}^4\to\mathbb{R}$ is a continuous function.

One of the fundamental problems is then to minimise I with prescribed Dirichlet conditions on the boundary $\partial\Omega$. To do this, usually the only available tool is to prove that I is *sequentially weakly lower semicontinuous* (abbreviated w.l.s.c.) in a certain Sobolev space $W^{1,p}$. Therefore it is of primary importance to know conditions on f which ensure this property of I. Surprisingly, this problem turns out to be difficult and it has not yet received a fully satisfactory answer. This problem was first formulated by Bliss in 1937 in his seminar on the calculus of variations and has received considerable attention, in particular by Albert [1], Reid [17]. MacShane [12], Hestenes and MacShane [9], Terpstra [19], Van Hove [20], Serre [18] and Marcellini [13] for the quadratic case and in a more general context by Morrey [14, 15] (for a survey of this problem, see [4, 5, 7]).

In order to describe our results we need the following definitions.

Definitions 0.1. (i) f is said to be *quasiconvex* if

$$\int_\Omega f(\xi+\nabla\phi(x))\,dx\geqq f(\xi)\operatorname{meas}\Omega \tag{0.2}$$

for every $\xi\in\mathbb{R}^4$ and for every $\phi\in W_0^{1,\infty}(\Omega;\mathbb{R}^2)$ (the set of Lipschitz functions vanishing on $\partial\Omega$).

(ii) f is said to be *rank one convex* if

$$f(\lambda\xi+(1-\lambda)\eta)\leqq\lambda f(\xi)+(1-\lambda)f(\eta) \tag{0.3}$$

for every $\lambda \in [0, 1]$, ξ, $\eta \in \mathbb{R}^4$ with $\det(\xi - \eta) = 0$ (if $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$, then $\det \xi = \xi_1\xi_4 - \xi_2\xi_3$).

(iii) f is said to be *polyconvex* if there exists ϕ: $\mathbb{R}^5 \to \mathbb{R}$ convex such that

$$f(\xi) = \phi(\xi, \det \xi) \tag{0.4}$$

for every $\xi \in \mathbb{R}^4$.

Remarks 0.2. (i) It can be proved (cf. [7] for example) that if (0.2) holds for one domain Ω, it holds for any domain.

(ii) In the second definition it is easy to see that if f is C^2, then the rank one convexity of f is equivalent to the classical *Legendre–Hadamard* condition

$$\sum_{i,j=1}^{2} \sum_{\alpha,\beta=1}^{2} \frac{\partial^2 f(\xi)}{\partial \xi_{i\alpha}\, \partial \xi_{j\beta}} \lambda_i \lambda_j \mu_\alpha \mu_\beta \geqq 0 \tag{0.5}$$

for every $\xi \in \mathbb{R}^{2\times 2}$ and every λ, $\mu \in \mathbb{R}^2$.

In general, one has the following diagram:

$$f \text{ convex} \underset{\Rightarrow}{\not\Leftarrow} f \text{ polyconvex} \underset{\Rightarrow}{\not\Leftarrow} f \text{ quasiconvex} \underset{\Rightarrow}{\overset{?}{\Leftarrow}} f \text{ rank one convex}$$

$$\Updownarrow$$

$$I \text{ is w.l.s.c.}$$

Morrey's conjecture 0.3. It was conjectured by Morrey [14], that in fact f rank one convex $\not\Rightarrow f$ quasiconvex.

It is the aim of this article to study the rank one convexity of some functions f and marginally their polyconvexity and quasiconvexity. In the first section, we study functions of the type

$$f(\xi) = g(|\xi|^2, \det \xi), \tag{0.6}$$

where g: $\mathbb{R}^2 \to \mathbb{R}$ and $|\xi|^2 = \sum_{i=1}^4 \xi_i^2$, $\det \xi = \xi_1\xi_4 - \xi_2\xi_3$ whenever $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$. These functions are important in elasticity (cf. for example [4, 7]) and in optimal design (cf. [11]).

Usually when one studies functions as in (0.6), it is more convenient to write them as $h(\lambda, \mu)$, where (λ, μ) are the eigenvalues of $(\xi\xi^t)^{1/2}$. The rank one convexity of f being then expressed in terms of some convexity of h (cf. [2–4, 10]). However in some cases, (cf. [8]) it may be easier to study the Legendre–Hadamard condition (0.5) directly.

In particular, we shall concentrate on the following two examples:

$$f(\xi) = |\xi|^{2\alpha} + h(\det \xi) \tag{0.7}$$

$$f(\xi) = |\xi|^{2\alpha}(|\xi|^2 - a \det \xi). \tag{0.8}$$

The first one is particularly important for applications and has been studied in the case $\alpha < 2$ by Ball and Murat [6]. The second one is interesting, since it gives the first example (in the case $\alpha = 1$) of a rank one convex function (for $a = 4/\sqrt{3}$) which is not polyconvex (cf. [2, 8]).

In Section 2 we return to the example of Dacorogna and Marcellini [8]

$$f(\xi) = |\xi|^4 - \frac{4}{\sqrt{3}} |\xi|^2 \det \xi. \tag{0.9}$$

Because of the above observation, and in view of the above diagram, f is a candidate for answering Morrey's conjecture. However, analytical computations seem to be a very hard method for deciding whether or not f is quasiconvex. We present some numerical computations which tend to indicate that f is indeed quasiconvex, leaving, therefore, Morrey's conjecture unanswered.

We conclude by a remark on the proofs of Section 1. The proofs require only elementary calculus. In this sense they are simple, though a bit tricky.

1. Examples of rank one convex functions

We start with some properties of functions of the type

$$f(\xi)=g(|\xi|^2, \det \xi), \tag{1.1}$$

where for $\xi=(\xi_1, \xi_2, \xi_3, \xi_4)\in \mathbb{R}^4$, $|\xi|^2=\sum_{i=1}^4 \xi_i^2$ and $\det \xi = \xi_1\xi_4-\xi_2\xi_3$.

Proposition 1.1. (i) *If f is rank one convex, $f(0)=0$ and $g(x,-y)=g(x,y)$, for every $x, y\geqq 0$, then $f\geqq 0$.*

(ii) *f is rank one convex if and only if f is convex with respect to the first variable.*

(iii) *Let f and g be C^2. The following properties are then equivalent:*

(a) *f is rank one convex;*

(b) *for every $\xi\in\mathbb{R}^4$, the following holds:*

$$4g_{xx}(|\xi|^2, \det \xi)\xi_1^2+4g_{xy}(|\xi|^2, \det \xi)\xi_1\xi_4 + g_{yy}(|\xi|^2, \det \xi)\xi_4^2+2g_x(|\xi|^2, \det \xi)\geqq 0, \tag{1.2}$$

where $g_x=\dfrac{\partial}{\partial x}g$ and similarly for g_{xx}, g_{xy}, g_{yy};

(c) *the following holds*

$$4g_{xx}(x,y)u^2+4g_{xy}(x,y)uv+g_{yy}(x,y)v^2+2g_x(x,y)\geqq 0, \tag{1.3}$$

for every $(x,y,u,v)\in\mathbb{R}^4$ satisfying

$$\begin{cases} u^2+v^2\leqq x, \\ (u+v)^2-x\leqq 2y\leqq x-(u-v)^2. \end{cases} \tag{1.4}$$

Remarks 1.2. (i) The first part rules out examples of the type $|\xi|^{2\alpha}(|\xi|^4-\gamma(\det \xi)^2)$ when $\gamma>2$. It also trivially implies that f is quasiconvex at 0, i.e., $\int_\Omega f(\nabla\phi(x))\,dx\geqq 0$ for every $\phi\in W_0^{1,\infty}(\Omega;\mathbb{R}^2)$.

(ii) In many instances, as for example the two cases discussed below, it is easier to show (1.3), than to verify the Legendre–Hadamard condition (i.e. (1.2)) or the equivalent properties for the function $h(\lambda,\mu)$, where (λ,μ) are the eigenvalues of $(\xi\xi^t)^{1/2}$ and $f(\xi)=g(|\xi|^2, \det\xi)=h(\lambda,\mu)$ (see [3, 10] for more details on this last approach).

Before proceeding with the proof, we need to introduce some notations and to establish an elementary lemma.

Notation 1.3. Let $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4$; we denote by $\tilde{\xi}\in\mathbb{R}^4$, the vector $\tilde{\xi}=(\xi_4,-\xi_3,-\xi_2,\xi_1)$. For $\xi,\lambda\in\mathbb{R}^4$ we let $(\xi/\lambda)=\sum_{i=1}^4 \xi_i\lambda_i$.

Remark 1.4. Observe that $2\det\xi=(\hat{\xi}/\xi)$.

LEMMA 1.5. *Let* $\xi,\ \lambda\in\mathbb{R}^4$ *with* $|\lambda|^2=1$ *and* $\det\lambda=0$. *Then there exists* $\eta=(\eta_1,\eta_2,\eta_3,\eta_4)\in\mathbb{R}^4$ *with*

$$\begin{cases}|\eta|^2=|\xi|^2, & \det\eta=\det\xi\\ \eta_1=(\xi/\lambda), & \eta_4=(\hat{\xi}/\lambda).\end{cases}$$

Proof. It only remains to find η_2 and η_3, which are given by

$$\begin{cases}\eta_2\eta_3=-\det\xi+(\hat{\xi}/\lambda)(\xi/\lambda)\\ \eta_2^2+\eta_3^2=|\xi|^2-(\xi/\lambda)^2-(\hat{\xi}/\lambda)^2.\end{cases}\tag{1.5}$$

The system is solvable provided

$$(\eta_2-\eta_3)^2=|\xi|^2+2\det\xi-[(\xi/\lambda)+(\hat{\xi}/\lambda)]^2\geqq 0.\tag{1.6}$$

$$(\eta_2+\eta_3)^2=|\xi|^2-2\det\xi-[(\xi/\lambda)-(\hat{\xi}/\lambda)]^2\geqq 0.\tag{1.7}$$

We show (1.6) ((1.7) being shown similarly), which is equivalent to showing that

$$\min_{|\xi|=1}\{|\xi|^2+2\det\xi-2(\xi/\lambda)(\hat{\xi}/\lambda)-(\xi/\lambda)^2-(\hat{\xi}/\lambda)^2\}\geqq 0.\tag{1.8}$$

Let α be the Lagrange multiplier and $\phi(\xi,\lambda)=|\xi|^2+2\det\xi-2(\xi/\lambda)(\hat{\xi}/\lambda)-(\xi/\lambda)^2-(\hat{\xi}/\lambda)^2-\alpha(|\xi|^2-1)$. We find that ϕ attains its minimum on $|\xi|=1$ and where $\nabla\phi=0$, i.e.

$$\xi+\hat{\xi}-(\hat{\xi}/\lambda)\lambda-(\xi/\lambda)\hat{\lambda}-(\xi/\lambda)\lambda-(\hat{\xi}/\lambda)\hat{\lambda}=\alpha\xi.\tag{1.9}$$

Multiplying (1.9) first by ξ, then by λ and $\hat{\lambda}$ (bearing in mind that $|\lambda|^2=1$, $|\xi|^2=1$ and $\det\lambda=0$) we find that $\alpha=\phi(\xi,\lambda)$, $\alpha(\xi/\lambda)=\alpha(\hat{\xi}/\lambda)=0$. We therefore deduce that either $\alpha=0$ or $(\xi/\lambda)=(\hat{\xi}/\lambda)=0$, which, in any case, imply (1.8). Proceeding in the same way with (1.7), we have indeed established the lemma. □

We now turn to the proof of Proposition 1.1:

Proof of Proposition 1.1. (i) Assume for contradiction that there exists $\xi^1=(\xi_1,\xi_2,\xi_3,\xi_4)$ such that $f(\xi^1)<0$. Let $\xi^2=(\xi_1,\xi_2,-\xi_3,-\xi_4)$, $\xi^3=(-\xi_1,-\xi_2,\xi_3,\xi_4)$ and $\xi^4=-\xi^1$. Since

$$\begin{cases}\det\xi^1=\det\xi^4, & \det\xi^2=\det\xi^3=-\det\xi^1\\ \det(\xi^1-\xi^2)=\det(\xi^3-\xi^4)=0,\end{cases}$$

we deduce from the facts that f is rank one convex and that $g(x,y)=g(x,-y)$, that

$$f(\tfrac12\xi^1+\tfrac12\xi^2)=f(\xi_1,\xi_2,0,0)\leqq\tfrac12 f(\xi^1)+\tfrac12 f(\xi^2)<0,$$

$$f(\tfrac12\xi^3+\tfrac12\xi^4)=f(-\xi_1,-\xi_2,0,0)\leqq\tfrac12 f(\xi^3)+\tfrac12 f(\xi^4)<0.$$

Thus, using the rank one convexity of f again, we deduce that

$$f(0)=f(\tfrac12(-\xi_1,-\xi_2,0,0)+\tfrac12(\xi_1,\xi_2,0,0))<0,$$

which is absurd.

(ii) *Necessity*: this part is obvious.

Sufficiency: we now assume that f is convex with respect to the first variable and we wish to show that f is rank one convex which is equivalent to showing that $f(\xi + t\lambda)$ is convex in $t \in \mathbb{R}$ for every ξ, $\lambda \in \mathbb{R}^4$ with $\det \lambda = 0$. Applying Lemma 1.5 to ξ and to $\frac{\lambda}{|\lambda|}$, we can find $\eta \in \mathbb{R}^4$ with $\eta_1 = \left(\xi \Big/ \frac{\lambda}{|\lambda|}\right)$, $\eta_4 = \left(\xi \Big/ \frac{\lambda}{|\lambda|}\right)$, $\det \eta = \det \xi$ and $|\eta|^2 = |\xi|^2$. We then obtain that

$$\begin{aligned} f(\xi + t\lambda) &= g(|\xi|^2 + 2t(\xi/\lambda) + t^2 |\lambda|^2, \det \xi + t(\hat{\xi}/\lambda)) \\ &= g(|\eta|^2 + 2t\eta_1 |\lambda| + t^2 |\lambda|^2, \det \eta + t\eta_4 |\lambda|) \\ &= f(\eta + t(|\lambda|, 0, 0, 0)). \end{aligned} \tag{1.10}$$

Since f is convex with respect to the first variable, we find that $f(\xi + t\lambda)$ is convex in t for every ξ, $\lambda \in \mathbb{R}^4$ with $\det \lambda = 0$ and thus f is rank one convex.

(iii) The fact that (a) and (b) are equivalent results from (ii) and from a direct computation of $f_{\xi_1\xi_1}$, which is positive. The fact that (b) and (c) are equivalent is easily seen, by setting $u = \xi_1$, $v = \xi_4$, $y = \det \xi$, $x = |\xi|^2$ and we leave out the details. □

We now turn our attention to some examples.

THEOREM 1.6. *For $\alpha \geqq \frac{1}{2}$ and $h \in C^2(\mathbb{R})$, let*

$$f(\xi) = |\xi|^{2\alpha} + h(\det \xi). \tag{1.11}$$

(1) *The following conditions are equivalent*:

(i) *f rank one convex*;

(ii) *for every $z \in \mathbb{R}$*,

$$h''(z) \geqq -\alpha(\alpha - 1)2^{\alpha-1} \qquad b_0 |z|^{\alpha-2}, \tag{1.12}$$

where

$$b_0 = \begin{cases} 0 & \text{if } \frac{1}{2} \leqq \alpha < 2, \\ 1 & \text{if } \alpha = 2, \\ b_1 & 2 < \alpha < 2 + \sqrt{2}, \\ b_2 & \alpha \geqq 2 + \sqrt{2}, \end{cases} \tag{1.13}$$

with

$$b_2 = \frac{\dfrac{2(\alpha - 1)}{2\alpha - 1}}{\left(1 - \dfrac{\alpha}{2(\alpha - 1)^2}\right)^{\alpha - 2}}$$

$$b_1 = \frac{2(\alpha + (\alpha - 1)\bar{x})}{(\alpha - 1)(1 - \bar{x}^2)^{\alpha/2 - 1}(1 - \bar{x})}$$

and $\bar{x}$ is given by

$$\bar{x} = \frac{-(\alpha^2 - 1) + [(\alpha^2 - 1)^2 - 4(\alpha - 1)(\alpha - 2)(2\alpha - 1)]^{1/2}}{2(\alpha - 1)(\alpha - 2)}.$$

(2) *Let* $0 \leqq b \leqq b_0$, *where* b_0 *is as in* (1.13); *then*

$$\bar{f}(\xi) = |\xi|^{2\alpha} - 2^{\alpha-1} b\, |\det \xi|^{\alpha} \tag{1.14}$$

is rank one convex.

Remarks 1.7. (i) The case $\alpha \in [\frac{1}{2}, 2)$ has been established by Ball and Murat [6].

(ii) Functions f as in (1.11) often occur in nonlinear plane elasticity where the strain energy function has the above form (see **[4, 7, 16]** for more details).

(iii) We now point out a curious result which occurs when $\alpha = 2$, i.e. when $\bar{f}(\xi) = |\xi|^4 - 2(\det \xi)^2$. In fact it is easily seen that $\bar{f}$ is not only rank one convex but also *convex*. (To see this, let (λ, μ) be the eigenvalues of $(\xi\xi^t)^{1/2}$ and observe that $\bar{f}(\xi) = (\lambda^2 + \mu^2)^2 - 2\lambda^2\mu^2 = \lambda^4 + \mu^4$. A general result (cf. [4]) immediately implies that $\bar{f}$ is convex.)

(iv) More generally if $\alpha \in [\frac{1}{2}, 2]$, then in view of the theorem and of the above remark $\bar{f}$ is convex if and only if it is rank one convex. However, one should be careful not to conclude from this remark that if f is as in (1.11), then f is convex if and only if f is rank one convex. Indeed $f(\xi) = |\xi|^2 - \gamma \det \xi$ is rank one convex for every γ while it is not convex if $\gamma > 2$.

Before proceeding with the proof we give our next example.

THEOREM 1.8. *For* $\alpha \geqq 1$, $\gamma \geqq 0$, *let*

$$f(\xi) = |\xi|^{2\alpha}(|\xi|^2 - \gamma \det \xi). \tag{1.15}$$

Then, f is rank one convex if and only if $0 \leqq \gamma \leqq \gamma_0$, *where*

$$\gamma_0 = \begin{cases} \gamma_1 & \textit{if} \quad \alpha \in \left[1, \dfrac{9+5\sqrt{5}}{4}\right), \\ \gamma_2 & \textit{if} \quad \alpha \in \left[\dfrac{9+5\sqrt{5}}{4}, +\infty\right), \end{cases} \tag{1.16}$$

where

$$\gamma_2 = 2 - \frac{1 + \dfrac{1}{\alpha}}{2\alpha + \sqrt{4\alpha^2 - 2\alpha - 2}} = 1 + \sqrt{1 - \frac{1}{2\alpha} - \frac{1}{2\alpha^2}},$$

$$\gamma_1 = \left(1 + \frac{1}{\alpha}\right) m(\alpha) \equiv \left(1 + \frac{1}{\alpha}\right) \min_{t>0} \left\{\frac{t^4 + 2(\alpha+1)t^2 + 2\alpha + 1}{3t^3 + (2\alpha+1)t}\right\}.$$

Remark 1.9. The case $\alpha = 1$, which gives $\gamma_0 = \gamma_1 = \dfrac{4}{\sqrt{3}}$, has been studied by Dacorogna and Marcellini **[8]** (see also **[2]**). This example is interesting in the sense that, with $\gamma = \dfrac{4}{\sqrt{3}}$, the function f is rank one convex but not polyconvex, thus answering an open question.

To see that f is not polyconvex, it suffices to observe that if $\xi = (1, 0, 0, 1)$ then $f(t\xi) = 2\left(2 - \dfrac{4}{\sqrt{3}}\right)t^4$, and then to use the Hahn–Banach theorem to show that f

cannot be polyconvex (see [7, 8] for details). We therefore use strongly the fact that $\lim_{t\to\infty} f(t\xi)/t^2 = -\infty$. However, in view of (1.16), except for α close to 1, $f(\xi) \geqq 0$ for every $\xi \in \mathbb{R}^4$. Therefore if α is not close to 1, one cannot use the above argument, thus leaving the question of the polyconvexity of f unsettled.

We now proceed with the proofs of the theorems.

Proof of Theorem 1.6

We consider only the case $\alpha \geqq 2$. Ball and Murat [6] have obtained the result when $\frac{1}{2} \leqq \alpha < 2$. We start by proving part (2).

Proof of part (2). To show that $\bar{f}$ is rank one convex, we proceed in three steps.

Step 1. By Proposition 1.1, the rank one convexity of $\bar{f}$ is equivalent to

$$4(\alpha(\alpha-1)x^{\alpha-2})u^2 - 2^{\alpha-1}b\alpha(\alpha-1)\,|y|^{\alpha-2}v^2 + 2\alpha x^{\alpha-1} \geqq 0, \tag{1.17}$$

for every u, v, x, y satisfying $u^2+v^2 \leqq x$, $(u+v)^2 - x \leqq 2y \leqq x-(u-v)^2$. Using the homogeneity of $\bar{f}$, we find that (1.17) is equivalent to

$$2(\alpha-1)u^2 - b(\alpha-1)\,|2y|^{\alpha-2}\,v^2 + 1 \geqq 0, \tag{1.18}$$

whenever $u^2+v^2 \leqq 1$, $(u+v)^2 - 1 \leqq 2y \leqq 1-(u-v)^2$.

Obviously the function $|2y|^{\alpha-2}$ attains its maximum either for $2y = (u+v)^2 - 1$ or for $1-(u-v)^2$. Furthermore, since changing v to $-v$ does not alter (1.18), we find that the rank one convexity of $\bar{f}$ is equivalent to showing that

$$\phi_b(u,v) \equiv 1 + 2(\alpha-1)u^2 - b(\alpha-1)\,|1-(u-v)^2|^{\alpha-2}\,v^2 \geqq 0, \tag{1.19}$$

whenever $u^2+v^2 \leqq 1$.

On letting

$$\sigma(b) = \min\{\phi_b(u,v) : u^2+v^2 \leqq 1\} \tag{1.20}$$

and observing that $\sigma(0)=1$, $\sigma(+\infty) = -\infty$ and that σ is continuous and strictly decreasing, we find that there exists a unique b_0 such that

$$\sigma(b_0) = 0. \tag{1.21}$$

Consequently $\bar{f}$ is rank one convex if and only if $0 \leqq b \leqq b_0$. It is the aim of the next steps to show that b_0 is as stated in the theorem.

Step 2. First observe that the case $\alpha = 2$ is immediately settled, leading to $b_0 = 1$ as claimed. So we may assume from now on that $\alpha > 2$. We next let $B = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\}$, $C_+ = \{(u,v) \in \mathbb{R}^2 : (u-v)^2 < 1\}$ and $C_- = \{(u,v) \in \mathbb{R}^2 : (u-v)^2 > 1\}$. We then claim the following facts, which will be proved in Step 3:

Fact 1: $\phi_{b|_{\partial C_+}} = \phi_{b|_{\partial C_-}} \geqq 1 = \phi(0,1) \geqq \min\{\phi_b(u,v) : (u,v) \in \partial B\}$.

Fact 2: $\operatorname{grad} \phi_b(u,v) \neq 0$ if $(u,v) \in B \cap C_-$.

Fact 3: Let b, $(\bar{u},\bar{v}) \in C_+$, then $\phi_b(\bar{u},\bar{v}) = 0$ and $\operatorname{grad}\phi_b(\bar{u},\bar{v}) = 0$

$$\Leftrightarrow b = b_2,\ \bar{u}^2 = \frac{1}{2\alpha}, \qquad \bar{v} = \frac{2\alpha-1}{\alpha-1}\bar{u}. \tag{1.22}$$

Furthermore, if $\alpha \geqq 2+\sqrt{2}$, then $(\bar{u},\bar{v}) \in \bar{B} \cap C_+$.

Fact 4: $(\bar{u}, \bar{v})$ defined as in (1.22) is the absolute minimum of ϕ_{b_2} over C_+.

Fact 5: The minimum of ϕ_b on ∂B is the same as that on $\partial B \cap C_+$, $\partial B \cap C_-$, $\partial B \cap \{u, v \geqq 0\}$, $\partial B \cap \{u \geqq 0,\ v \leqq 0\}$, $\partial B \cap \{u \leqq 0,\ v \geqq 0\}$ and $\partial B \cap \{u, v \leqq 0\}$.

To conclude the proof of part (2) of Theorem 1.6, we split the discussion into two cases.

Case 1. *let* $\alpha \geqq 2+\sqrt{2}$. From all the above facts we may conclude that $0=\phi_{b_2}(\bar{u}, \bar{v}) = \min\{\phi_{b_2}(u, v)\colon (u, v) \in \bar{B}\}$ and hence by (1.21), we have $b_0 = b_2$ as claimed.

Case 2. *if* $2<\alpha<2+\sqrt{2}$, we have $b_0=b_1$. Indeed since $\bar{B}=\{(u, v)\in \mathbb{R}^2\colon u^2+v^2 \leqq 1\}$ is compact we have by (1.20) and (1.21) that there exists $(\bar{u}, \bar{v}) \in \bar{B}$ such that $\phi_{b_0}(\bar{u}, \bar{v}) = 0 = \sigma(b_0)$, since $(\bar{u}, \bar{v}) \notin \partial C_+ \cup \partial C_-$ and grad $\phi_{b_0}(\bar{u}, \bar{v}) \neq 0$ (by Facts 2, 3 and 4) we must have $(\bar{u}, \bar{v}) \in \partial B$ and thus $\min\{\phi_{b_0}\colon (u, v) \in \bar{B}\} = \min\{\phi_{b_0}\colon (u, v) \in \partial B\} = 0 = \sigma(b_0)$. By Fact 5 we have also that

$$0=\min\{1+2(\alpha-1)u^2-b_0(\alpha-1)(2uv)^{\alpha-2}v^2\colon u^2+v^2=1,\ u,\ v \geqq 0\}. \quad (1.23)$$

Writing $u=\cos\theta$, $v=\sin\theta$, $x=\cos 2\theta$, we find that (1.23) is equivalent to

$$\min\left\{\psi(x) \equiv \alpha+(\alpha-1)x-\frac{b_0}{2}(\alpha-1)(1-x^2)^{\alpha/2-1}(1-x)\colon \quad x \in [-1, 1]\right\}=0. \quad (1.24)$$

Since $\psi(1)=2\alpha-1$ and $\psi(-1)=1$, we find that b_0 is characterised by $\bar{x} \in (-1, 1)$ such that $\psi(\bar{x}) = \psi'(\bar{x}) = 0$. An elementary computation then gives $b_0 = b_1$, as claimed in Theorem 1.6.

Step 3. Therefore, to conclude the proof of part (2), it remains to show Facts 1 to 5 of Step 2.

Fact 1: The first fact is trivial.

Fact 2: If $(u, v) \in C_-$ and grad $\phi_b(u, v) = 0$, we find that

$$\begin{cases} 2u = b(\alpha-2)v^2(u-v)((u-v)^2-1)^{\alpha-3}, \\ (u-v)^2-1=(\alpha-2)v(u-v). \end{cases}$$

By multiplying the first equation by $((u-v)^2-1)$ and then using the second, we find that $2uv = bv^2((u-v)^2-1)^{\alpha-2}$, which implies in particular that $uv \geqq 0$; however, if $(u, v) \in \bar{B} \cap C_-$ we should have $0 \geqq u^2+v^2-1 \geqq 2uv$, which is absurd. Thus there is no stationary point of ϕ_b in $\bar{B} \cap C_-$.

Fact 3: We now let $(\bar{u}, \bar{v}) \in C_+$ such that grad $\phi_b(\bar{u}, \bar{v}) = 0$, this implies that either $\bar{u} = \bar{v} = 0$ (which gives $\phi_b(0, 0) = 1$), or

$$\begin{cases} 2\bar{u} = b(\alpha-2)\bar{v}^2(\bar{v}-\bar{u})(1-(\bar{u}-\bar{v})^2)^{\alpha-3}, & (1.25) \\ 1-(\bar{u}-\bar{v})^2 = (\alpha-2)\bar{v}(\bar{v}-\bar{u}). & (1.26) \end{cases}$$

As above, we then obtain

$$\begin{cases} 2\bar{u}\bar{v} = b\bar{v}^2(1-(\bar{u}-\bar{v})^2)^{\alpha-2}, & (1.27) \\ 1 = \bar{u}^2+(\alpha-1)\bar{v}^2-\alpha\bar{u}\bar{v}. & (1.28) \end{cases}$$

((1.28) is just a restatement of (1.26), and (1.27) is obtained by multiplying (1.25) by $(1-(\bar{u}-\bar{v})^2)$ and using (1.26).) Therefore at a stationary point, we find that

$\phi_b(\bar{u}, \bar{v}) = 2(\alpha-1)\bar{u}^2 - 2(\alpha-1)\bar{u}\bar{v} + 1 = (\bar{v}-\bar{u})((\alpha-1)\bar{v} - (2\alpha-1)\bar{u})$, and hence if $\phi_b(\bar{u}, \bar{v}) = 0$, we find (1.22). The conclusion is that, if $\alpha \geqq 2 + \sqrt{2}$, then $\bar{u}^2 + \bar{v}^2 \leqq 1$ is immediately obtained from (1.22), and we thus conclude that $(\bar{u}, \bar{v}) \in \bar{B} \cap C_+$.

Fact 4: We now wish to show that $(\bar{u}, \bar{v})$ is the absolute minimum of ϕ_{b_2} in C_+. To see this, observe first that a direct computation gives $b_2 < 2$. Therefore if $(u, v) \in C_+$, if $b \leqq b_2 < 2$ and if we let $u = v + t$, we find that $t \in (-1, 1)$ and that $\phi_b(u, v) = v^2[2(\alpha-1) - b(\alpha-1)(1-t^2)^{\alpha-2}] + 4(\alpha-1)vt + 2(\alpha-1)t^2 + 1$. The above identity leads immediately to the fact that if $(u, v) \in C_+$, then

$$\lim_{v \to \infty} \phi_b(u, v) = +\infty. \tag{1.29}$$

With the help of (1.29) we are now ready to conclude. Let

$$\mu(b) = \inf\{\phi_b(u, v): (u, v) \in C_+\}. \tag{1.30}$$

As above, we observe that $\mu(0) = 1$, $\mu(+\infty) = -\infty$ and that μ is continuous and strictly decreasing. Therefore there exists a unique $\bar{b}$ such that $\mu(\bar{b}) = 0$. In view of (1.22), $\bar{b} \leqq b_2$. Therefore using (1.29), we find that the minimum is attained in (1.30) by a certain $(\bar{u}, \bar{v}) \in C_+$. Hence $\phi_{\bar{b}}(\bar{u}, \bar{v}) = 0$ and grad $\phi_{\bar{b}}(\bar{u}, \bar{v}) = 0$, which by (1.22) leads to $\bar{b} = b_2$, therefore establishing Fact 4.

Fact 5: This is trivial once it has been observed that if $u^2 + v^2 = 1$, then $\phi_b(u, v) = 1 + 2(\alpha-1)u^2 - b(\alpha-1)\,|2uv|^{\alpha-2}v^2$. This concludes Step 3 and thus part (2) of Theorem 1.6.

Proof of part 1. As mentioned above, we consider only the case $\alpha \geqq 2$. Also let $a_0 = 2^{\alpha-1}b_0$.

(ii)$\Rightarrow$(i). We let $\bar{f}(\xi) = |\xi|^{2\alpha} - a_0\,|\det \xi|^\alpha$ and $\bar{h}(z) = h(z) + a_0\,|z|^\alpha$ for $z \in \mathbb{R}$. We may then write $f(\xi) = \bar{f}(\xi) + \bar{h}(\det \xi)$. In view of part (2), $\bar{f}$ is rank one convex and since h satisfies (1.12) we deduce that $\bar{h}$ is convex and thus f is rank one convex.

(i)$\Rightarrow$(ii). We proceed by contradiction. Assume that there exists $\bar{z} \in \mathbb{R}$ such that

$$h''(\bar{z}) + a_0\alpha(\alpha-1)\,|\bar{z}|^{\alpha-2} < 0. \tag{1.31}$$

By continuity of h'' we may assume that $\bar{z} \neq 0$. As seen from (1.17) and (1.21) and by Proposition 1.1, we have that there exists $\xi \in \mathbb{R}^4$ such that

$$L(\xi) \equiv 2\alpha(2\alpha-2)\,|\xi|^{2\alpha-4}\xi_1^2 - a_0\alpha(\alpha-1)\,|\det \xi|^{\alpha-2}\xi_4^2 + 2\alpha\,|\xi|^{2\alpha-2} = 0. \tag{1.32}$$

In view of (1.32) we may even ensure that $\bar{z}\,.\det \xi > 0$. We then let $\bar{\xi} = (\bar{z}/\det \xi)^{1/2}\xi$. Thus by homogeneity of L in (1.32) we obtain

$$L(\bar{\xi}) = 0, \qquad \det \bar{\xi} = \bar{z}. \tag{1.33}$$

If f were rank one convex, we should have by Proposition 1.1

$$M(\xi) \equiv 2\alpha(2\alpha-2)\,|\xi|^{2\alpha-4}\,\xi_1^2 + h''(\det \xi)\xi_4^2 + 2\alpha\,|\xi|^{2\alpha-2} \geqq 0, \tag{1.34}$$

for every $\xi \in \mathbb{R}^4$. However, choosing $\xi = \bar{\xi}$, using (1.31) and (1.33), we find that $M(\bar{\xi}) < 0$, which contradicts (1.34) and thus establishes Theorem 1.6. □

Proof of Theorem 1.8

We proceed as in Theorem 1.6 and we decompose the proof into three steps.

Step 1. Using the homogeneity of f and Proposition 1.1, we find that the rank one convexity of f is equivalent to $(1+(1/\alpha))+2(\alpha+1)u^2-2\gamma uv-\gamma y(1+2(\alpha-1)u^2)\geqq 0$, for every $u^2+v^2\leqq 1$, $(u+v)^2-1\leqq 2y\leqq 1-(u-v)^2$. Since y attains its maximum at $\frac{1}{2}(1-(u-v)^2)$, we find that the above inequality is equivalent to

$$\phi_\gamma(u,v)\equiv\left(1+\frac{1}{\alpha}\right)-\frac{\gamma}{2}+2(\alpha+1)u^2-2\gamma uv$$
$$+\frac{\gamma}{2}(u-v)^2-\gamma(\alpha-1)(1-(u-v)^2)u^2\geqq 0 \quad (1.35)$$

in $\bar{B}$ where $B=\{(u,v)\in\mathbb{R}^2\colon u^2+v^2<1\}$.

Denoting by

$$\sigma(\gamma)=\min\{\phi_\gamma(u,v)\colon (u,v)\in\bar{B}\}, \quad (1.36)$$

we find that σ is a continuous strictly decreasing function such that $\sigma(0)=1+\frac{1}{\alpha}>0$, $\sigma(+\infty)=-\infty$. Therefore there exists a unique $\gamma_0>0$ such that

$$\sigma(\gamma_0)=0. \quad (1.37)$$

Observe also that

$$0<\frac{\gamma_0}{2}<1+\frac{1}{\alpha}. \quad (1.38)$$

Consequently f is rank one convex if and only if $0\leqq\gamma\leqq\gamma_0$. It is the aim of the next steps to show that γ_0 is as stated in Theorem 1.8.

Step 2. We first claim the following facts which will be proved in Step 3.

Fact 1: We have that

$$\min\{\phi_\gamma(u,v)\colon (u,v)\in\partial B\}\geqq 0\Leftrightarrow 0\leqq\gamma\leqq\gamma_1. \quad (1.39)$$

Fact 2: If $\alpha=1$ and grad $\phi_\gamma(\bar{u},\bar{v})=0$, then $\phi_\gamma(\bar{u},\bar{v})>0$.

Fact 3: Let $\alpha>1$, then

$$\phi_\gamma(\bar{u},\bar{v})=0 \quad\text{and}\quad \operatorname{grad}\phi_\gamma(\bar{u},\bar{v})=0$$
$$\Leftrightarrow\gamma=\gamma_2,\ \bar{u}^2=\frac{\alpha-1+\sqrt{4\alpha^2-2\alpha-2}}{2(\alpha^2-1)},\ \bar{v}=\frac{3\bar{u}+2(\alpha-1)\bar{u}^3}{1+2(\alpha-1)\bar{u}^2}. \quad (1.40)$$

Furthermore,

$$\bar{u}^2+\bar{v}^2\leqq 1\Leftrightarrow\alpha\geqq\frac{9+5\sqrt{5}}{4}. \quad (1.41)$$

Fact 4: Let $(\bar{u},\bar{v})$ be as in (1.40); then $(\bar{u},\bar{v})$ is the absolute minimum of ϕ_{γ_2} over the whole of $\mathbb{R}^2$.

With the help of the above facts we may now conclude.

Case 1 ($\alpha \geqq (9+5\sqrt{5})/4$). We have that if $(\bar{u}, \bar{v})$ is as in (1.40), then $0=\phi_{\gamma_2}(\bar{u}, \bar{v})=\min\{\phi_{\gamma_2}(u, v)\colon (u, v)\in \bar{B}\}$, which when combined with (1.37) immediately gives $\gamma_0=\gamma_2$.

Case 2 ($1 \leqq \alpha < (9+5\sqrt{5})/4$). In this case we have $\gamma=\gamma_1$. Indeed since $\bar{B}$ is compact, we have that there exists $(\bar{u}, \bar{v})\in \bar{B}$ such that $\phi_{\gamma_0}(\bar{u}, \bar{v})=\sigma(b_0)=0$. Observe that $(\bar{u}, \bar{v})\in \partial B$, otherwise we would have $(\bar{u}, \bar{v})\in B$, $\phi_{\gamma_0}(\bar{u}, \bar{v})=0$ and $\operatorname{grad}\phi_{\gamma_0}(\bar{u}, \bar{v})=0$, which would imply by Fact 3 that $\gamma_0=\gamma_2$, $(\bar{u}, \bar{v})=(\bar{u}, \bar{v})\notin \bar{B}$, which is absurd.

We therefore have $0=\sigma(\gamma_0)=\min\{\phi_{\gamma_0}(u, v)\colon (u, v)\in \bar{B}\}=\min\{\phi_{\gamma_0}(u, v)\colon (u, v)\in \partial B\}$. By Fact 1 and (1.37) we have immediately $\gamma_0=\gamma_1$, and this achieves the proof of Step 2.

Step 3. We now have to prove the above facts.

Fact 1: In (1.35), we let $u^2+v^2=1$ and we set $u=\cos\theta$, $v=\sin\theta$, $t=\tan\theta$; we then obtain

$$\begin{aligned}\phi_\gamma(u, v)&=\left(1+\frac{1}{\alpha}\right)+2(\alpha+1)\cos^2\theta-3\gamma\cos\theta\sin\theta-2\gamma(\alpha-1)\cos^3\theta\sin\theta\\&=\left(1+\frac{1}{\alpha}\right)+\frac{2(\alpha+1)}{1+t^2}-3\gamma\frac{t}{1+t^2}-2\gamma(\alpha-1)\frac{t}{(1+t^2)^2}.\end{aligned}$$

Therefore, $\phi_\gamma(u, v)\geqq 0$ over $u^2+v^2=1$ if and only if

$$h(t)=\left(1+\frac{1}{\alpha}\right)t^4-3\gamma t^3+2(\alpha+1)\left(1+\frac{1}{\alpha}\right)t^2-\gamma(2\alpha+1)t+\left(1+\frac{1}{\alpha}\right)(2\alpha+1)\geqq 0.$$

Since trivially $h(t)\geqq 0$ for $t\leqq 0$, we have that $h(t)\geqq 0$ if and only if

$$\gamma_1\equiv\left(1+\frac{1}{\alpha}\right)m(\alpha)=\left(1+\frac{1}{\alpha}\right)\min_{t>0}\left\{\frac{t^4+2(\alpha+1)t^2+2\alpha+1}{3t^3+(2\alpha+1)t}\right\}\geqq\gamma,$$

which is exactly the claimed result. Note also that

$$\frac{\gamma_1}{2}\leqq 1+\frac{1}{\alpha}. \tag{1.42}$$

Fact 2: We now let $\alpha=1$; then $\phi_\gamma(u, v)=2-\frac{\gamma}{2}+(4+(\gamma/2))u^2-3\gamma uv+\frac{\gamma}{2}v^2$. One sees immediately that if $\gamma\neq 0, 1$, then the unique stationary point is $u=v=0$ (and therefore $\phi_\gamma(0, 0)=2-\frac{\gamma}{2}>0$ by (1.38)). If $\gamma=0$, then $\phi_\gamma(u, v)\geqq 2$ and if $\gamma=1$, then $\phi_\gamma(u, v)\geqq\frac{3}{2}$. Thus Fact 2 is established.

Fact 3: We first compute $\operatorname{grad}\phi_\gamma(\bar{u}, \bar{v})=0$; we have

$$\frac{\partial\phi_\gamma}{\partial u}=2\left(2(\alpha+1)-\gamma\alpha+\frac{3\gamma}{2}\right)\bar{u}-3\gamma\bar{v}+2\gamma(\alpha-1)(\bar{u}-\bar{v})^2\bar{u}+2\gamma(\alpha-1)\bar{u}^2(\bar{u}-\bar{v})=0 \tag{1.43}$$

$$\frac{\partial\phi_\gamma}{\partial v}=\gamma(\bar{v}-3\bar{u}+2(\alpha-1)\bar{u}^2(\bar{v}-\bar{u}))=0. \tag{1.44}$$

We first observe that if either $\gamma=0$ or $\bar v=\bar u=0$, then $\phi_\gamma(\bar u,\bar v)>0$. We have from (1.44) and from adding (1.43) and (1.44) that

$$\bar v=\frac{3\bar u+2(\alpha-1)\bar u^3}{1+2(\alpha-1)\bar u^2}, \tag{1.45}$$

$$\bar u(2(\alpha+1)-\gamma\alpha)-\gamma\bar v+\gamma(\alpha-1)(\bar u-\bar v)^2\bar u=0. \tag{1.46}$$

Using (1.45) in (1.46) and dividing by $\bar u$, we obtain

$$(2(\alpha+1)-\gamma\alpha)-\gamma\left(1+\frac{2}{1+2(\alpha-1)\bar u^2}\right)+\gamma(\alpha-1)\frac{4\bar u^2}{(1+2(\alpha-1)\bar u^2)^2}=0.$$

We therefore obtain $\gamma[(\alpha+1)(1+2(\alpha-1)\bar u^2)^2+2]=2(\alpha+1)(1+2(\alpha-1)\bar u^2)^2$, which leads to

$$\gamma=2-\frac{4}{2+(\alpha+1)(1+2(\alpha-1)\bar u^2)^2}. \tag{1.47}$$

We now return to the expression (1.35) of ϕ_γ and use (1.46) to obtain

$$\phi_\gamma(\bar u,\bar v)=\left(1+\frac{1}{\alpha}-\frac{\gamma}{2}\right)+\frac{\gamma}{2}(\bar v-3\bar u)(\bar v-\bar u).$$

On inserting (1.45) in the above identity, we have

$$\phi_\gamma(\bar u,\bar v)=\left(1+\frac{1}{\alpha}-\frac{\gamma}{2}\right)-\frac{4\gamma(\alpha-1)\bar u^4}{(1+2(\alpha-1)\bar u^2)^2}. \tag{1.48}$$

Finally, using (1.47) in (1.48), we obtain

$$\phi_\gamma(\bar u,\bar v)=0\Leftrightarrow(\alpha^2-1)\bar u^4-(\alpha-1)\bar u^2-\tfrac{3}{4}=0. \tag{1.49}$$

Combining (1.45), (1.47) and (1.49), we have indeed obtained (1.40). To conclude the proof of Fact 3, it remains to show (1.41), but this is easily seen by combining (1.40) and (1.45). Indeed, $u^2+v^2\leqq 1$ is equivalent to

$$1+\left(1+\frac{2(\alpha+1)}{2\alpha+\sqrt{4\alpha^2-2\alpha-2}}\right)^2\leqq\frac{2(\alpha^2-1)}{(\alpha-1)+\sqrt{4\alpha^2-2\alpha-2}}.$$

Suppressing the denominator, we obtain

$$1+(2\alpha+1-\sqrt{4\alpha^2-2\alpha-2})^2\leqq\tfrac{2}{3}(\sqrt{4\alpha^2-2\alpha-2}-(\alpha-1)),$$

which leads to

$$\begin{aligned}12\alpha^2+4\alpha-1=(6\alpha-1)(2\alpha+1)&\leqq(6\alpha+4)\sqrt{4\alpha^2-2\alpha-2}\\&=(6\alpha+4)\sqrt{(2\alpha+1)(2\alpha-2)}.\end{aligned}$$

The last inequality turns out to be equivalent to (bearing in mind that $\alpha\geqq 1$) $4\alpha^2-18\alpha-11\geqq 0\Leftrightarrow\alpha\geqq(9+5\sqrt5)/4$. Hence $u^2+v^2\leqq 1$ if and only if $\alpha\geqq(9+5\sqrt5)/4$.

Fact 4: In view of Fact 3, it is sufficient to show that there exists a, $b>0$ such that

$$\phi_{\gamma_2}(u,v)>\frac{1}{\alpha}>0, \tag{1.50}$$

in $\Omega_{a,b}=\{(u,v)\in\mathbb{R}^2: |u|>a \text{ or } |v|>b\}$ (since the only stationary points are $(0,0)$ with $\phi_{\gamma_2}(0,0)>0$ and $(\bar{u},\bar{v})$ with $\phi_{\gamma_2}(\bar{u},\bar{v})=0$).

We fix $\alpha\geqq 1$ and we let ε be such that $\gamma_2=2-2\varepsilon$. Observe that $0<\varepsilon\leqq\frac{1}{2}$. To prove the existence of a and b, we first find a such that (1.50) holds for every $|u|>a$ and every $v\in\mathbb{R}$, then find b such that (1.50) holds for every $|u|\leqq a$ and every $|v|>b$.

(1) Observe first that if $u=0$, we immediately obtain (1.50), since $\phi_{\gamma_2}(0,v)=1/\alpha+\varepsilon+(1-\varepsilon)v^2>1/\alpha$. We may then assume that $u\neq 0$ and we let $v=tu$ with $t\in\mathbb{R}$. We obtain

$$\phi_{\gamma_2}(u,tu)=\frac{1}{\alpha}+\varepsilon+h(t)u^2+2(1-\varepsilon)(\alpha-1)(1-t)^2u^4, \qquad (1.51)$$

where $h(t)=(1-\varepsilon)t^2-6(1-\varepsilon)t+(5-3\varepsilon+2\varepsilon\alpha)$. Observe that $h(t)\geqq -4$ and that since $h(1)=2\varepsilon(\alpha+1)$, there exists $\delta=\delta(\alpha)>0$ such that $h(t)\geqq\varepsilon(\alpha+1)$ for every $t\in[1-\delta,1+\delta]$.

Case 1. If $t\in[1-\delta,1+\delta]$, we then obtain (1.50), since $\phi_{\gamma_2}(u,tu)\geqq 1/\alpha+\varepsilon+\varepsilon(\alpha+1)u^2>1/\alpha$.

Case 2. If $t\notin[1-\delta,1+\delta]$, then $\phi_{\gamma_2}(u,tu)\geqq 2(1-\varepsilon)(\alpha-1)\delta^2u^4-4u^2+1/\alpha+\varepsilon$.

Then choosing a sufficiently large so that $|u|>a$ implies $2(1-\varepsilon)(\alpha-1)\delta^2u^4-4u^2\geqq 0$, we immediately obtain (1.50).

(2) We may now assume that $|u|\leqq a$; we then obtain

$$\begin{aligned}\phi_{\gamma_2}(u,v)&=\left(\frac{1}{\alpha}+\varepsilon\right)+(5-3\varepsilon+2\varepsilon\alpha)u^2+(1-\varepsilon)v^2-6(1-\varepsilon)uv\\&\quad+2(1-\varepsilon)(\alpha-1)(u-v)^2u^2\\&\geqq\frac{1}{\alpha}+\varepsilon+(1-\varepsilon)(v^2-6uv)\geqq\frac{1}{\alpha}+\varepsilon+(1-\varepsilon)(v^2-6a\,|v|).\end{aligned}$$

Therefore we choose b sufficiently large so that $v^2-6a\,|v|\geqq 0$. We deduce (1.50) immediately and thus the theorem. □

2. Numerical results

We now turn to the example of Dacorogna and Marcellini [8], i.e. that of Theorem 1.8 with $\alpha=1$; more precisely, let

$$f_\gamma(\xi)=|\xi|^4-\gamma\,|\xi|^2\det\xi. \qquad (2.1)$$

We have seen in Theorem 1.8 as well as in Remark 1.9 that:

(1) If $0\leqq\gamma\leqq\frac{4}{\sqrt{3}}$, then f_γ is rank one convex;

(2) if $2<\gamma\leqq\frac{4}{\sqrt{3}}$, then f_γ is not polyconvex.

In view of the general diagram presented in the introduction, f_γ polyconvex $\Rightarrow f_\gamma$ quasiconvex $\Rightarrow f_\gamma$ rank one convex, we deduce that the function f_γ is a candidate for answering Morrey's conjecture: f rank one convex $\not\Rightarrow f$ quasiconvex.

Unfortunately, it seems very hard to decide whether the function f_γ is quasiconvex or not when $\gamma = 4/\sqrt{3}$. We could not even decide if the function f_γ is quasiconvex at 0, which means that we could not decide whether $\inf\{\int_\Omega f_\gamma(\nabla\phi(x))\,dx\colon\ \phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)\}$ is 0 or $-\infty$. We have therefore undertaken some numerical computations which we present below. Our results tend to indicate that f_γ is quasiconvex whenever $\gamma \leqq 4/\sqrt{3}$.

We now set the problem. For $\xi \in \mathbb{R}^{2\times 2}$ and $\phi \in W_0^{1,4}(\Omega; \mathbb{R}^2)$, we let

$$J_\gamma(\xi, \phi) = \int_\Omega [f_\gamma(\xi + \nabla\phi(x)) - f_\gamma(\xi)]\,dx. \tag{2.2}$$

We choose $\Omega = (0,1) \times (0,1)$. The quasiconvexity of f_γ is then equivalent to

$$\inf_{\xi \in \mathbb{R}^{2\times 2}} \ \inf_{\phi \in W_0^{1,4}(\Omega; \mathbb{R}^2)} \{J_\gamma(\xi, \phi)\} = 0. \tag{2.3}$$

Remarks 2.1. (i) Note that because of the homogeneity of f_γ, the infimum in (2.3) is either 0 or $-\infty$.

(ii) It follows from Theorem 1.8 that, if $\gamma > 4/\sqrt{3}$, then f_γ is not rank one convex and therefore in (2.3) the infimum is $-\infty$. This can also be seen by choosing $\xi = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$ and a highly oscillating ϕ. It is for this reason that in our numerical computations we study with special care the case $\xi = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$.

We now describe our numerical approximation. We let N be a positive integer and $h = 1/N$. We partition Ω into $\Omega_{ij} = (ih, (i+1)h) \times (jh, (j+1)h)$, $0 \leqq i, j \leqq N-1$. each of these Ω_{ij} is then subdivided into two triangles as in Figure 1. We let τ_h denote this triangulation of Ω and the triangles by K. We let

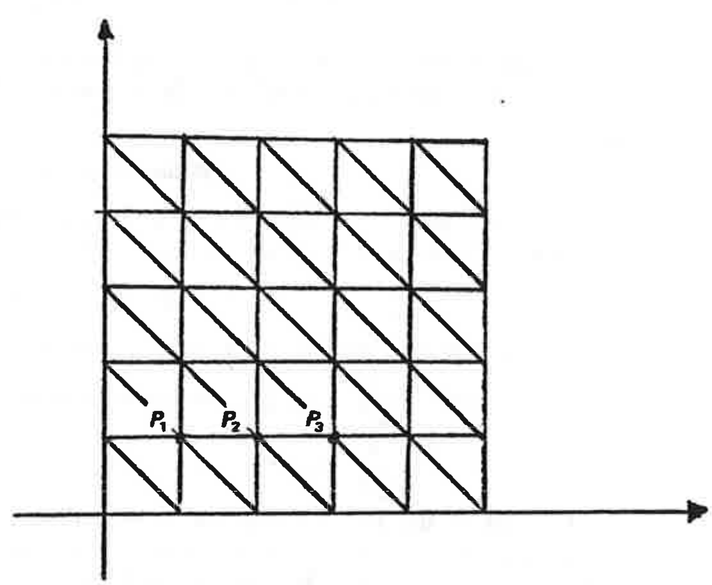

Figure 1

$P_1, P_2, \ldots, P_M$, $M=(N-1)^2$, be the internal nodes. We next set

$$V_h=\{u\in C^0(\bar{\Omega}): u \text{ is affine on each } K\in\tau_h \text{ and } u=0 \text{ on } \partial\Omega\},$$
$$W_h=V_h\times V_h\subset W_0^{1,4}(\Omega;\mathbb{R}^2).$$

Remark 2.2. It is well known that for every $u\in W_0^{1,4}(\Omega;\mathbb{R}^2)$ we have $\lim_{h\to 0}\inf_{u_h\in W_h}\|u-u_h\|_{W^{1,4}}=0$.

We first fix $\xi\in\mathbb{R}^{2\times 2}$ and we minimise J_γ over W_h. To do this, we use the gradient method which we now describe. Let w^l, $l=1,2,\ldots,L$ be given and $d^l=\nabla J_\gamma(w^l)$, $g^l(\alpha)=J_\gamma(w^l+\alpha d^l)$, $w^{l+1}=w^l+\bar{\alpha}d^l$, where $\bar{\alpha}$ is obtained by solving $\dfrac{dg^l}{d\alpha}=0$, using only one step in Newton's method with starting point $\alpha=0$.

We now give our results. We fix $N=10$. We have done five types of experiment. Each time when we choose w^1, we mean that we have chosen $w^1(P_m)$ for $1\leqq m\leqq(N-1)^2$ and where P_m are the internal nodes. We recall also that $4/\sqrt{3}\approx 2{\cdot}3094$.

(I) *ξ random and w^1 random.* For each of the values of γ, we have chosen four different random values of ξ and four different random values of w^1. The conclusions are that $J_\gamma(w^L)=0$ up to the order 10^{-12} and usually for L around 500 and this for the values of $\gamma=2{\cdot}31/2{\cdot}32/2{\cdot}3225/2{\cdot}3275$.

(II) $\xi=(1,0,0,\sqrt{3})$ *and w^1 random.* In fact we have chosen a multiple of ξ $(100.\xi)$. We found that for $\gamma=2{\cdot}31/2{\cdot}32/2{\cdot}3225/2{\cdot}3275$, $J_\gamma(w^L)=0$. However, if $\gamma=2.33$ we found for four different w^1 chosen in a random way, $J_\gamma(w^{782})=-44.05; -354.01; -440.55; -73.552$.

(III) $\xi=(1,0,0,\sqrt{3})$ *and oscillating w^1.* We have chosen $w^1(P_m)=(-1)^{m+1}(1,1)$ and we found that for $\gamma=2{\cdot}31/2{\cdot}32/2{\cdot}3225/2{\cdot}3275$, $J_\gamma(w^L)=0$. However, if $\gamma=2.33$ we found that $J_\gamma(w^{300})=-261.72$.

(IV) $\xi=(0,0,0,0)$ *and w^1 random.* We chose as before four different random values of w^1 and we found that up to $\gamma=3{\cdot}25$, $J_\gamma(w^L)=0$ up to the order 10^{-12} usually for $L=300$. However, for $\gamma=3{\cdot}3$ we found $J_\gamma(w^{115})=-60$.

(V) $\xi=(0,0,0,0)$ *and oscillating w^1.* We have chosen $w^1(P_m)=(-1)^{m+1}(1,1)$ and we found that up to $\gamma=3.5$, $J_\gamma(w^L)=0$, while at $\gamma=3{\cdot}60$, $J_\gamma(w^{280})=-5.3$.

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