Two variable polynomials: intersecting zeros and stability

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Abstract

In order to construct two variable polynomials with a certain zero behavior, the notion of intersecting zeros is studied. We show that generically two-variable polynomials have a finite set of intersecting zeros, and give an algorithm on how to construct a polynomial with the desired intersecting zeros. Relations with the Cayley-Bacharach theorem are addressed. In addition, we will also address the case when stable polynomials are sought.

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1 Introduction

In [?] the notion of intersecting zeroes of a two variable polynomial

$$p(z, w) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i w^j \quad (1.1)$$

was introduced, as being the common zeroes of $p(z, w)$ and $\overleftarrow{p}(z, w)$, where

$$\overleftarrow{p}(z, w) = z^n w^m \overline{p(1/\bar{z}, 1/\bar{w})} = \sum_{i=0}^n \sum_{j=0}^m \bar{p}_{ij} z^{n-i} w^{m-j}.$$

Thus (z, w) is an *intersecting zero* of $p(z, w)$ if

$$p(z, w) = \overleftarrow{p}(z, w) = 0.$$

The particular polynomials studied in [?] were *stable* polynomials (i.e., $p(z, w) \neq 0$, $(z, w) \in \overline{\mathbb{D}}^2$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$). In this case the function $f = \frac{1}{|p|^2}$ is well-defined on the bitorus $\mathbb{T}^2$, where as usual $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The function f is called the *spectral density* function of p , and plays an important role in the design of autoregressive filters. As it turns out the rate of decay of the Fourier coefficients $\hat{f}(k, \ell)$ of f , as $|k| \rightarrow \infty$ and $|\ell| \rightarrow \infty$, depends heavily on the location of the intersecting zeros of p (see [?, Theorem 2.2.1]). Stable polynomials also play an important role in prediction theory and in two variable H^∞ control via certain rational function solutions of the Nevanlinna-Pick problem.

A polynomial of the form (??) will be said to be of *degree* (n, m) . The degree terminology is used somewhat more loosely than usual as we do not require any particular coefficients to be nonzero. Note that the definition of $\overleftarrow{p}$ depends on the degree (the notation $\overleftarrow{p}^{(n, m)}$ would be more accurate, but also more cumbersome). We will specify throughout the paper the degree of our polynomials, which will typically be (n, m) . As in [?] we shall also allow the pairs (∞, z) , (∞, w) , (∞, ∞) to be intersecting zeroes. For this we use the conventions:

$$\begin{aligned} p(\infty, \infty) = 0 &\Leftrightarrow p_{nm} = 0 \\ p(\infty, w) = 0 &\Leftrightarrow \sum_{j=0}^m p_{nj} w^j = 0 \\ p(z, \infty) = 0 &\Leftrightarrow \sum_{i=0}^n p_{im} z^i = 0, \end{aligned}$$

where z and w are complex numbers. Thus, for instance (∞, ∞) is an intersecting zero of p if $p_{00} = 0 = p_{nm}$. With the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$, we now have the simple observation that (z, w) is an intersecting zero for p if and only if $(\frac{1}{\bar{z}}, \frac{1}{\bar{w}})$ is an intersecting zero for p .

One way to determine the intersecting zeros of p is by using matrix valued one variable polynomials. As this works for the common zeros of any pair of polynomials $p(z, w)$ and $q(z, w)$ of degree (n, m) we will explain it for this situation. Denote

$$p(z, w) = \sum_{j=0}^m p_j(z)w^j = \sum_{i=0}^n \tilde{p}_i(w)z^i,$$

$$q(z, w) = \sum_{j=0}^m q_j(z)w^j = \sum_{i=0}^n \tilde{q}_i(w)z^i.$$

Then, if we fix z , we have that $p(z, w) = 0 = q(z, w)$ for some w if and only if the $2m \times 2m$ matrix

$$A(z) := \begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_m(z) & & \\ & \ddots & \ddots & & \ddots & \\ & & p_0(z) & p_1(z) & \cdots & p_m(z) \\ q_0(z) & q_1(z) & \cdots & q_m(z) & & \\ & \ddots & \ddots & & \ddots & \\ & & q_0(z) & q_1(z) & \cdots & q_m(z) \end{pmatrix}$$

is singular. In other words,

$$\begin{aligned} \{z \in \mathbb{C}_\infty : A(z) \text{ is singular}\} &= \\ &= \{z \in \mathbb{C}_\infty; \exists w \in \mathbb{C}_\infty : (z, w) \text{ is common zero for } p \text{ and } q\}. \end{aligned}$$

The statement “ $A(\infty)$ is singular” should be understood as

$$\det \begin{pmatrix} p_{n0} & p_{n1} & \cdots & p_{nm} & & \\ & \ddots & \ddots & & \ddots & \\ & & p_{n0} & p_{n1} & \cdots & p_{nm} \\ q_{n0} & q_{n1} & \cdots & q_{nm} & & \\ & \ddots & \ddots & & \ddots & \\ & & q_{n0} & q_{n1} & \cdots & q_{nm} \end{pmatrix} = 0.$$

Likewise, if we let

$$B(w) = \begin{pmatrix} \tilde{p}_0(w) & \tilde{p}_1(w) & \cdots & \tilde{p}_m(w) & & \\ & \ddots & \ddots & & \ddots & \\ & & \tilde{p}_0(w) & \tilde{p}_1(w) & \cdots & \tilde{p}_m(w) \\ \tilde{q}_0(w) & \tilde{q}_1(w) & \cdots & \tilde{q}_m(w) & & \\ & \ddots & \ddots & & \ddots & \\ & & \tilde{q}_0(w) & \tilde{q}_1(w) & \cdots & \tilde{q}_m(w) \end{pmatrix},$$

we find that

$$\begin{aligned} \{w \in \mathbb{C}_\infty : B(w) \text{ is singular}\} &= \\ &= \{w \in \mathbb{C}_\infty; \exists z \in \mathbb{C}_\infty : (z, w) \text{ is common zero for } p \text{ and } q\}. \end{aligned}$$

It follows from the theory of matrix polynomials (see, e.g., [?]) that when $A(z)$ is regular (i.e., $\det A(z) \neq 0$ for some z) then there are only a finite number of common zeros of p and q . The same is true for $B(w)$, and counting multiplicity there are exactly $2nm$ common zeros of p and q . If, say, $\det A(z)$ has more than $2nm$ roots then it must be identically zero and since $A(z)$ is a resultant it follows by Bezout's theorem that p and q have a common factor.

One of the questions that was left open in [?] is to what extent the intersecting zeros determine the polynomial p . Clearly, the intersecting zeros of p are also intersecting zeros of linear combinations of p and $\overline{p}$, so that the best one can hope for is that the intersecting zeros narrow it down to a 2 dimensional subspace. Note that in one variable the zeros determine a polynomial up to a nonzero multiplicative constant; thus effectively to a one-dimensional subspace. A polynomial p of degree (n, m) has $(n+1)(m+1)$ coefficients. With nm pairs of intersecting zeros (remember, the $2nm$ pairs are symmetric with respect to $\mathbb{T}^2$ so knowing nm of them suffices to reconstruct all of them), which corresponds to $2nm$ complex numbers, it seems that typically only a subset of the zeroes is required to reconstruct the subspace spanned by p and $\overline{p}$. These issues and in general the relations between polynomials and their intersecting roots, will be considered in this paper.

The paper is organized as follows. In Section 2 we give necessary and sufficient conditions for pairs of complex numbers to be the intersecting zeros of a polynomial. As a corollary we prove a Cayley-Bacharach type result. In Section 3 we give an algorithm on how to construct *stable* polynomials with prescribed intersecting zeros.

2 A Kernel Condition

Let $p(z, w) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i w^j$ and $q(z, w) = \sum_{i=0}^n \sum_{j=0}^m q_{ij} z^i w^j$ be polynomials of degree (n, m) , and put

$$A_i = \begin{pmatrix} p_{i0} & p_{i1} & \cdots & p_{im} & & \\ & \ddots & \ddots & & \ddots & \\ & & p_{i0} & p_{i1} & \cdots & p_{im} \\ q_{i0} & q_{i1} & \cdots & q_{im} & & \\ & \ddots & \ddots & & \ddots & \\ & & q_{i0} & q_{i1} & \cdots & q_{im} \end{pmatrix}, \quad i = 0, \dots, n,$$

and

$$B_j = \begin{pmatrix} p_{0j} & p_{1j} & \cdots & p_{nj} & & \\ & \ddots & \ddots & & \ddots & \\ & & p_{0j} & p_{1j} & \cdots & p_{nj} \\ q_{0j} & q_{1j} & \cdots & q_{mj} & & \\ & \ddots & \ddots & & \ddots & \\ & & q_{0j} & q_{1j} & \cdots & q_{mj} \end{pmatrix}, \quad j = 0, \dots, m.$$

Then $A(z) = \sum_{i=0}^n A_i z^i$ and $B(w) = \sum_{j=0}^m B_j w^j$. Assume from now on that $A(z)$ and $B(w)$ are regular polynomials. In that case we shall call p and q a *regular pair* of

polynomials. Thus the equations $\det A(z) = 0$ and $\det B(w) = 0$ have a finite number of roots.

We need to recall some notions on spectral data of matrix polynomials. The main references for this material are [?] and [?]. We shall use the notation

$$\text{col}(F_i)_{i=0}^k = \begin{pmatrix} F_0 \\ \vdots \\ F_k \end{pmatrix}, \quad \text{row}(F_i)_{i=0}^k = (F_0 \cdots F_k).$$

The notation $\sigma(A)$ stands for the spectrum of a square matrix A . Let $L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^q L_q$ be a regular matrix polynomial ($\det L(\lambda) \not\equiv 0$) of size $s \times s$. A pair of matrices (X, T) with X of size $s \times \mu$ and T of size $\mu \times \mu$, is called a *right spectral pair of $L(\lambda)$ at λ_0* if

- (i) $\sigma(T) = \{\lambda_0\}$
- (ii) $\det L(\lambda)$ has a zero λ_0 of multiplicity μ
- (iii) $\text{rank col}(XT^j)_{j=0}^{q-1} = \mu$
- (iv) $L_0 X + L_1 X T + \cdots + L_q X T^q = 0$.

A spectral pair of $L(\lambda)$ at λ_0 is unique up to *similarity*, i.e., if (X, T) and $(\hat{X}, \hat{T})$ are spectral pairs of $L(\lambda)$ at λ_0 then there exist an invertible $\mu \times \mu$ matrix S so that

$$\hat{X} = X S, \quad \hat{T} = S^{-1} T S.$$

If we denote

$$\begin{aligned} \Sigma(L) &:= \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\} \\ &= \{\lambda_1, \dots, \lambda_k\}, \end{aligned}$$

and (X_i, T_i) are right spectral pairs of $L(\lambda)$ at λ_i , $i = 1, \dots, k$, then $X = (X_1, \dots, X_k)$, $T = T_1 \oplus \cdots \oplus T_k$ is called a *finite right spectral pair* of $L(\lambda)$. Moreover, any pair of matrices $(\hat{X}, \hat{T})$ that are similar to (X, T) are called *finite right spectral pairs* of $L(\lambda)$. A pair of matrices (X_∞, T_∞) with X_∞ of size $n \times \mu$ and T_∞ of size $\mu \times \mu$ is called an *infinite right spectral pair* of $L(\lambda)$ if

- (i) $\sigma(T_\infty) = \{0\}$,
- (ii) $\det(\lambda^q L(\lambda^{-1}))$ has a zero at $\lambda_0 = 0$ of multiplicity μ ,
- (iii) $\text{rank col}(X_\infty T_\infty^j)_{j=0}^{q-1} = \mu$,
- (iv) $L_0 X_\infty T_\infty^s + L_1 X_\infty T_\infty^{s-1} + \cdots + L_q X_\infty = 0$.

We will need the right spectral pairs of $A(z)$ and $B(w)$. One way of forming the right spectral pair of $A(z)$, say, at z_0 is to calculate vectors $0 \neq \phi_0^{(i)}, \dots, \phi_{r_i}^{(i)}$ so that

$$\begin{aligned} A(z_0)\phi_0^{(i)} &= 0, \quad A'(z_0)\phi_0^{(i)} + A(z_0)\phi_1^{(i)} = 0, \dots, \\ \frac{1}{r_i!} A^{(r_i)}(z_0)\phi_0^{(i)} + \cdots + A(z_0)\phi_{r_i}^{(i)} &= 0, \end{aligned}$$

and r_i is maximal. The sequence $\phi_0^{(i)}, \dots, \phi_{r_i}^{(i)}$ is called a *right Jordan chain* at z_0 , and r_i is called its *length*. As $A(z_0)$ is a resultant matrix, its kernel is spanned by vectors of the form

$$\begin{pmatrix} 1 \\ w_0 \\ w_0^2 \\ \vdots \\ w_0^{2m-1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 2w_0 \\ \vdots \\ (2m-1)w_0^{2m-2} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 2 \\ 3 \cdot 2w_0 \\ \vdots \\ (2m-1)(2m-2)w_0^{2m-2} \end{pmatrix}, \quad (2.1)$$

etc., where (z_0, w_0) is a common zero of p and q . Thus when the lengths of the Jordan chains are all equal to 1 (the generic case), we may choose the columns of X to be a linear combination of vectors of the form of (??) while T is of the form $T = z_0 I$.

The following is the main result of this section. We will state and prove the result in the case when $A(z)$ and $B(w)$ do not have spectrum at infinity (or, equivalently, when $\det A_n \neq 0$ and $\det B_m \neq 0$). We will explain afterward how the general case may be handled.

Theorem 2.1 *Let $p(z, w) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i w^j$, and $q(z, w) = \sum_{i=0}^n \sum_{j=0}^m q_{ij} z^i w^j$ be two-variable polynomials of degree (n, m) . Introduce the matrix polynomials $A(z)$ and $B(w)$ as in the beginning of this section. Assume that $\det A_n \neq 0$ and $\det B_m \neq 0$. Let $(\hat{X}_A, T_A)$ be a finite right spectral pair for A and $(\hat{X}_B, T_B)$ a finite right spectral pair for B . Let X_A be the $(m+1) \times 2nm$ matrix consisting of the top $m+1$ rows of $\hat{X}_A$ and X_B be the $(n+1) \times 2nm$ matrix consisting of the top $n+1$ rows of $\hat{X}_B$. Furthermore, let*

$$H = \text{col}(X_A T_A^i)_{i=0}^n, \quad \tilde{H} = \text{col}(X_B T_B^i)_{i=0}^m.$$

Then

$$\dim \ker H^T = 2 = \dim \ker \tilde{H}^T.$$

In fact

$$\ker H^T = \left\{ \text{col} \left(\text{col}(p_{ij})_{j=0}^m \right)_{i=0}^n, \text{col} \left(\text{col}(q_{ij})_{j=0}^m \right)_{i=0}^n \right\} \quad (2.2)$$

and

$$\ker \tilde{H}^T = \left\{ \text{col} \left(\text{col}(p_{ij})_{i=0}^n \right)_{j=0}^m, \text{col} \left(\text{col}(q_{ij})_{i=0}^n \right)_{j=0}^m \right\}. \quad (2.3)$$

Moreover, there exists a permutation matrix π and an invertible matrix F so that

$$H = \pi \tilde{H} F.$$

Proof. The inclusion $\supseteq$ in (??) follows directly from examining the first and the $(m+1)$ th rows in the equality

$$A_0 \hat{X}_A + A_1 \hat{X}_A T_A + \dots + A_n \hat{X}_A T_A^n = 0.$$

The inclusion $\supseteq$ in (??) follows similarly.

For the converse we introduce the resultants

$$\text{Res}_N(A) = \begin{pmatrix} A_0 & \cdots & A_n & & \\ & \ddots & & \ddots & \\ & & A_0 & \cdots & A_n \end{pmatrix}, \quad \text{Res}_M(B) = \begin{pmatrix} B_0 & \cdots & B_m & & \\ & \ddots & & \ddots & \\ & & B_0 & \cdots & B_m \end{pmatrix}$$

which have N and M block rows, respectively. From [?] we know that

$$\ker \text{Res}_N(A) = \text{Ran } \text{col}(\hat{X}_A T_A^i)_{i=0}^{N+n-1}, \quad N \geq n, \quad (2.4a)$$

$$\ker \text{Res}_M(B) = \text{Ran } \text{col}(\hat{X}_B T_B^i)_{i=0}^{M+m-1}, \quad M \geq m. \quad (2.4b)$$

Here Ran stands for the range. In addition, note that there exist permutations $\tilde{\pi}$ and $\hat{\pi}$ so that

$$\text{Res}_n(A) = \tilde{\pi} \text{Res}_m(B) \hat{\pi}^{-1}.$$

But then it follows that

$$\hat{\pi} \text{col}(\hat{X}_B T_B^i)_{i=0}^{2m-1} F = \text{col}(\hat{X}_A T_A^i)_{i=0}^{2n-1}, \quad (2.5)$$

where F is some invertible $2nm \times 2nm$ matrix. In fact the action by the permutation $\hat{\pi}$ in (??) is taking the j th row in the i th block, and making it the i th row in the j th block. But then it easily follows that

$$H = \pi \tilde{H} F$$

for a related permutation π .

Let now $h = \text{col}(\text{col}(h_{ij})_{j=0}^m)_{i=0}^n$ be some vector in $\ker H^T$. Put $h_{ij} = 0$, for $(i, j) \notin \{0, \dots, n\} \times \{0, \dots, m\}$. Then it follows immediately that

$$\text{col} \left(\text{col}(h_{ij})_{j=0}^{2m-1} \right)_{i=-k}^{N+n-k-1} \in \ker \left[\text{col}(\hat{X}_A T_A^i)_{i=0}^{N+n-1} \right]^T, \quad k = 0, \dots, N-1.$$

By (??), and taking into account the nature of the permutation $\hat{\pi}$, we get

$$\text{col} \left(\text{col}(h_{ij})_{i=-k}^{2n-k-1} \right)_{j=0}^{2m-1} \in \ker \left[\text{col}(\hat{X}_B T_B^i)_{i=0}^{2m-1} \right]^T.$$

But then it follows that

$$\text{col} \left(\text{col}(h_{ij})_{i=-k}^{2n-k-1} \right)_{j=-\ell}^{2m-\ell-1} \in \ker \left[\text{col}(\hat{X}_B T_B^i)_{i=0}^{2m-1} \right]^T, \quad \ell = 0, \dots, m-1.$$

Using (??) again, we find

$$\text{col} \left(\text{col}(h_{ij})_{j=-\ell}^{2m-\ell-1} \right)_{i=-k}^{N+n-k-1} \in \ker \left[\text{col}(\hat{X}_A T_A^i)_{i=0}^{N+n-1} \right]^T, \quad \ell = 0, \dots, m-1, k = 0, \dots, N-1, \quad (2.6)$$

holds for $N = n$. But then it also holds for $N \geq n$. From (??) we obtain now

$$\text{col} \left(\text{col}(h_{ij})_{j=-\ell}^{2m-\ell-1} \right)_{i=-k}^{N+n-k-1} \in \text{Ran} [\text{Res}_N(A)]^T, \quad \ell = 0, \dots, m-1, k = 0, \dots, N-1.$$

Note that $\text{Res}_N(A)$ is of size $(2mN) \times 2m(N+n)$ and has a kernel of dimension $2nm$. Thus its rows are linearly independent (another way of seeing this is that $\det A_n \neq 0$, however the above reasoning may be easily adjusted for the general case).

In a similar way we obtain

$$\text{col} \left(\text{col}(h_{ij})_{i=k}^{2n-1-k} \right)_{j=-\ell}^{M+m-\ell-1} \in \text{Ran} [\text{Res}_M(B)]^T, \quad k = 0, \dots, n-1, \ell = 0, \dots, M-1,$$

and $\text{Res}_M(B)$ has linearly independent rows. Express now

$$\tilde{h} = \text{row} \left(\text{row}(h_{ij})_{j=0}^{2m-1} \right)_{i=0}^{2n-1}$$

as a linear combination of the rows of $\text{Res}_n(A)$, i.e.,

$$\tilde{h} = (a \ b) \text{Res}_n(A), \quad (2.7)$$

where

$$a = \text{row} \left(\text{row}(a_{ij})_{j=0}^{n-1} \right)_{i=0}^{m-1},$$

$$b = \text{row} \left(\text{row}(b_{ij})_{j=0}^{n-1} \right)_{i=0}^{m-1},$$

for some a_{ij} and b_{ij} . If we can show that $a_{ij} = 0 = b_{ij}$ for $(i, j) \neq (0, 0)$, we are done. Assume the contrary. Without loss of generality we can assume that $a_{i_0, j_0} \neq 0$ for some $1 \leq i_0 \leq n-1$ and $0 \leq j_0 \leq m-1$. Consider the vector

$$\hat{h} = \text{row} \left(\text{row}(h_{ij})_{j=0}^{2m-1} \right)_{i=-n+i_0}^{2n-1}$$

which is linear combination of the rows of $\text{Res}_{2n-i_0}(A)$. If we let

$$\check{h} = \text{row} \left(\text{row}(h_{ij})_{j=0}^{2m-1} \right)_{i=-n+i_0}^{2n-i_0-1}$$

we have that $\hat{h} = (\check{h} \ 0)$, and that $\check{h}$ is a linear combination of the rows of $\text{Res}_n(A)$, say,

$$\check{h} = (c \ d) \text{Res}_n(A).$$

But then

$$\hat{h} = (c \ d \ 0) \text{Res}_{2n-i_0}(A). \quad (2.8)$$

On the other hand, as $\hat{h}$ is a shifted version of $\tilde{h}$ padded with zeros, we obtain from (??) an equality

$$\hat{h} = (\hat{a} \ \hat{b} \ \hat{c}) \text{Res}_{2n-i_0}(A). \quad (2.9)$$

As $a_{i_0, j_0} \neq 0$, we obtain that $\hat{c} \neq 0$. But then (??) and (??) yield that

$$(\hat{a} - c, d - \hat{b}, \hat{c}) \text{Res}_{2n-i_0}(A) = 0.$$

As $\hat{c} \neq 0$, this means that the rows of $\text{Res}_{2n-i_0}(A)$ are not linearly independent. Contradiction. $\square$

Remark 2.2 In Theorem ?? we do not allow for spectrum at infinity. One way of dealing with this case is to consider the reverse polynomials instead. That is, if p and q have common spectrum at ∞ , consider then $\overleftarrow{p}$ and $\overleftarrow{q}$ instead. In case p and q have both spectrum at 0 and ∞ , one may first perform a translation $(z, w) \mapsto (z + \alpha, w + \beta)$, where $(-\alpha, -\beta)$ is a point outside of the common spectrum of p and q . If the resulting polynomials, $\hat{p}$ and $\hat{q}$, say, still have common roots at ∞ , one may consider $\overleftarrow{\hat{p}}$ and $\overleftarrow{\hat{q}}$.

The above theorem allows the following Cayley-Bacharach type corollary.

Corollary 2.3 *Let $p(z, w)$ and $q(z, w)$ be polynomials of degree (n, m) with $2nm$ common roots (counting multiplicity and allowing infinity). Then there exist $nm + n + m - 1$ root conditions among these $2nm$ so that when a third polynomial $r(z, w)$ satisfies these $nm + n + m - 1$ root conditions, it will automatically satisfy all $2nm$ of them. In fact, in that case $r(z, w)$ is a linear combination of $p(z, w)$ and $q(z, w)$.*

Proof. With the polynomials build the matrix H as in Theorem ?. This matrix has size $(nm + n + m + 1) \times 2nm$ and has rank $nm + n + m - 1$ (as its left kernel has dimension 2 by Theorem ?). Consequently, one may choose $nm + n + m - 1$ of its columns to span the column space. If now a row vector $\text{row}(\text{row}(r_{ij})_{j=0}^m)_{i=0}^n$ annihilates these $nm + n + m - 1$ columns, it will automatically be in the left kernel of H . But then, by Theorem ?, it follows that $r(z, w)$ is a linear combination of $p(z, w)$ and $q(z, w)$. $\square$

When we specify this corollary to the case when $n = m = 3$ then we see that when $r(z, w)$ satisfy 14 conditions out of the 18, it will automatically satisfy all 18. The classical Cayley-Bacharach Theorem concerns the case of $p(z, w)$, $q(z, w)$, and $r(z, w)$ of total degree 3. So $p_{13} = p_{22} = p_{23} = p_{31} = p_{32} = p_{33} = 0$ and the same conditions for q and r . So automatically $p(z, w)$, $q(z, w)$ and $r(z, w)$ have 6 root conditions in common. Thus $r(z, w)$ only needs to satisfy an additional $14 - 6 = 8$ conditions, to obtain that $r(z, w)$ is a linear combination of $p(z, w)$ and $q(z, w)$. The Cayley-Bacharach Theorem indeed requires $r(z, w)$ to have 8 zero conditions in common with $p(z, w)$ and $q(z, w)$, to conclude that $r(z, w)$ then necessarily satisfies all the finite common root conditions of $p(z, w)$ and $q(z, w)$. We refer to [?] for an extensive exposition on the different versions of the Cayley-Bacharach theorem and their history. It is interesting to note that Hilbert [?] used the Cayley-Bacharach theorem to prove that not all nonnegative polynomials of more than one real variable are a sum of squares.

3 Stable polynomials

In this section we consider the question of building a *stable* polynomial with prescribed intersecting zeros. We will first develop some auxiliary results on stable polynomials. We refer to [?], [?] and [?] for a fuller account of multivariable stability results.

A single variable polynomial $p(w) = \sum_{i=0}^m p_i w^i$ is called *stable* when all its roots lie outside the closed unit disk. The well-known Schur-Cohn test states that a polynomial

p is stable if and only if

$$\begin{pmatrix} p_0 & & \circ \\ \vdots & \ddots & \\ p_{m-1} & \cdots & p_0 \end{pmatrix} \begin{pmatrix} \bar{p}_0 & \cdots & \bar{p}_{m-1} \\ & \ddots & \vdots \\ \circ & & \bar{p}_0 \end{pmatrix} - \begin{pmatrix} \bar{p}_m & & \circ \\ \vdots & \ddots & \\ \bar{p}_1 & \cdots & \bar{p}_m \end{pmatrix} \begin{pmatrix} p_m & \cdots & p_1 \\ & \ddots & \vdots \\ \circ & & p_m \end{pmatrix} \quad (3.1)$$

is positive definite. For a two-variable polynomial $p(z, w) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i w^j$ we write as before

$$p(z, w) = \sum_{j=0}^m p_j(z) w^j = \sum_{i=0}^n \tilde{p}_i(w) z^i. \quad (3.2)$$

Introduce the Schur-Cohn type expressions

$$\begin{aligned} E_p(z) := & \begin{pmatrix} p_0(z) & & \circ \\ \vdots & \ddots & \\ p_{m-1}(z) & \cdots & p_0(z) \end{pmatrix} \begin{pmatrix} \bar{p}_0(1/z) & \cdots & \bar{p}_{m-1}(1/z) \\ & \ddots & \vdots \\ \circ & & \bar{p}_0(1/z) \end{pmatrix} \\ & - \begin{pmatrix} \bar{p}_m(1/z) & & \circ \\ \vdots & \ddots & \\ \bar{p}_1(1/z) & \cdots & \bar{p}_m(1/z) \end{pmatrix} \begin{pmatrix} p_m(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ \circ & & p_m(z) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \tilde{E}_p(w) := & \begin{pmatrix} \tilde{p}_0(w) & & \circ \\ \vdots & \ddots & \\ \tilde{p}_{n-1}(w) & \cdots & \tilde{p}_0(w) \end{pmatrix} \begin{pmatrix} \bar{\tilde{p}}_0(1/w) & \cdots & \bar{\tilde{p}}_{n-1}(1/w) \\ & \ddots & \vdots \\ \circ & & \bar{\tilde{p}}_0(1/w) \end{pmatrix} \\ & - \begin{pmatrix} \bar{\tilde{p}}_n(1/w) & & \circ \\ \vdots & \ddots & \\ \bar{\tilde{p}}_1(1/w) & \cdots & \bar{\tilde{p}}_n(1/w) \end{pmatrix} \begin{pmatrix} \tilde{p}_n(w) & \cdots & \tilde{p}_1(w) \\ & \ddots & \vdots \\ \circ & & \tilde{p}_n(w) \end{pmatrix}. \end{aligned}$$

Notice that when p is stable, we must have that $E_p(z) > 0$, $|z| = 1$, where the notation $M > 0$ means that M is positive definite. Indeed, in that case we have that for every $|z| = 1$ the polynomial $p(z, w)$ is stable as a polynomial in w . The positive definiteness of $E_p(z)$ follows now from the Schur-Cohn test. Similarly, when p is stable we get that $\tilde{E}_p(w) > 0$, $|w| = 1$. In fact, we have the following equivalences.

Proposition 3.1 *Let p be a polynomial of degree (n, m) . The following are equivalent:*

- (i) p is stable;
- (ii) p_0 is stable and $E_p(z) > 0$, $z \in \mathbb{T}$;
- (iii) $\tilde{p}_0$ is stable and $\tilde{E}_p(w) > 0$, $w \in \mathbb{T}$;
- (iv) $E_p(z) > 0$, $z \in \mathbb{T}$, and $\tilde{E}_p(w) > 0$, $w \in \mathbb{T}$.

Proof. (i) $\rightarrow$ (ii): Clearly, if p is stable, then p_0 must be stable. The positive definiteness of $E_p(z)$ follows from the observations above.

(ii) $\rightarrow$ (i): Suppose $E_p(z) > 0$, $z \in \mathbb{T}$. We claim that $p(z, w) = 0$ implies that $|z| \neq 1$ or $|w| \neq 1$. Suppose otherwise, namely that $p(z, w) = 0$ for some $|z| = 1 = |w|$. Then $\overleftarrow{p}(z, w) = 0$ as well. Recall from [?, Section 4] or [?, see (2.3.26)]:

$$\frac{p(z, w) \overline{p(1/\bar{z}, w_1)} - \overleftarrow{p}(z, w) \overline{\overleftarrow{p}(1/\bar{z}, w_1)}}{1 - w\bar{w}_1} = (1 \ \dots \ w^{m-1})E_p(z) \begin{pmatrix} 1 \\ \vdots \\ \bar{w}_1^{m-1} \end{pmatrix}. \quad (3.3)$$

Thus $p(z, w) = 0 = \overleftarrow{p}(z, w)$ for some $|z| = 1 = |w|$ implies that the right hand side of (3.3) equals 0 for all w_1 . But then $E_p(z)$ has a nontrivial kernel, contradicting its positive definiteness.

In addition, we claim that $p(z, w) \neq 0$ for all $(z, w) \in \mathbb{T} \times \mathbb{D}$. If not, suppose $p(z_0, w_0) = 0$ for some $|z_0| = 1$ and $|w_0| < 1$. Set $z = z_0$ and $w = w_1 = w_0$. Then by (3.3) we get that

$$(1 \ \dots \ w_0^{m-1})E_p(z_0) \begin{pmatrix} 1 \\ \vdots \\ \bar{w}_0^{m-1} \end{pmatrix} = -\frac{|\overleftarrow{p}(z_0, w_0)|^2}{1 - |w_0|^2} \leq 0,$$

again contradicting the positive definiteness of $E_p(z_0)$.

Combining these two observations with the stability of p_0 , we get by Theorem 2.1.5(iii) in [?] (with the roles of z and w and a and b interchanged) that p is stable.

The proofs of (i) $\rightarrow$ (iii) and (iii) $\rightarrow$ (i) are the same as above with the roles of z and w interchanged.

For (i) $\rightarrow$ (iv) observe that (i) implies both (ii) and (iii), which together yield (iv).

Finally, for (iv) $\rightarrow$ (i) observe that the arguments above show that the positive definiteness of $E_p(z)$ and $\tilde{E}_p(w)$ imply that $p(z, w) \neq 0$ for all $(z, w) \in (\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T})$. But then Theorem 2.1.5(iii) in [?] implies that p is stable. $\square$

From the equivalence of (i) and (ii) in Proposition ?? we get the following two-variable Schur-Cohn test. We say that the self-adjoint block matrices $(A_{ij})_{i,j=0}^n$ and $(B_{ij})_{i,j=0}^n$ have the same block-diagonal sums if for $j = 0, \dots, n$ we have $\sum_{i=j}^n A_{i,i-j} = \sum_{i=j}^n B_{i,i-j}$.

Theorem 3.2 *Let p be a polynomial of degree (n, m) . Then p is stable if and only if*

$$\alpha\alpha^* - \beta^*\beta > 0, \quad (3.4)$$

and there exists a positive semidefinite $F = (F_{ij})_{i,j=0}^n$ with $F_{00} > 0$ so that F and

$$\begin{pmatrix} C_0 \\ \vdots \\ C_n \end{pmatrix} (C_0^* \ \dots \ C_n^*) - \begin{pmatrix} D_n^* \\ \vdots \\ D_0^* \end{pmatrix} (D_n \ \dots \ D_0) \quad (3.5)$$

have the same block-diagonal sums. Here

$$\alpha = \begin{pmatrix} p_{00} & \circ & \\ \vdots & \ddots & \\ p_{n-1,0} & \dots & p_{00} \end{pmatrix}, \quad \beta = \begin{pmatrix} p_{n0} & \dots & p_{10} \\ & \ddots & \vdots \\ \circ & & p_{n0} \end{pmatrix},$$

and for $i = 0, \dots, n$,

$$C_i = \begin{pmatrix} p_{i0} & & \bigcirc \\ \vdots & \ddots & \\ p_{i,m-1} & \cdots & p_{i0} \end{pmatrix}, \quad D_i = \begin{pmatrix} p_{im} & \cdots & p_{i1} \\ & \ddots & \vdots \\ \bigcirc & & p_{im} \end{pmatrix}.$$

Proof. By the one-variable Schur-Cohn test, (??) is equivalent to the stability of $p_0(z)$. Next, note that if we write $E_p(z) = \sum_{i=-n}^n E_i z^i$ we have that $E_i = \sum_{j=i}^n C_j C_{j-i}^* - \sum_{j=i}^n D_{j-i}^* D_j$, $i = 0, \dots, n$. But then by Section 1.2 of [?], we get that $E_p(z) > 0$ is equivalent to (??). Now the result follows from Proposition 2.1. $\square$

Clearly, by using the other equivalences in Proposition 2.1 one may state other variations of this two-variable Schur-Cohn test. We leave this for the reader. It should be observed that the existence of an F as in Theorem 2.2 may easily be determined numerically by using semidefinite programming (see, e.g., [?]).

Returning to the question of constructing a stable polynomial with prescribed intersecting zeros, we need to look at linear combinations of a polynomial p and its reverse. As it turns out a linear combination of p and $\overleftarrow{p}$ is stable if and only if p or $\overleftarrow{p}$ is. This is the content of the following proposition.

Proposition 3.3 *Let $p(z, w)$ be a polynomial of degree (n, m) . Then $c_1 p(z, w) + c_2 \overleftarrow{p}(z, w)$ is stable for some complex numbers c_1 and c_2 if and only if p or $\overleftarrow{p}$ is stable.*

Proof. The only if part is trivial. For the only if part we observe that the matrix polynomials $E_{c_1 p + c_2 \overleftarrow{p}}$ and E_p satisfy the following simple relationship:

$$E_{c_1 p + c_2 \overleftarrow{p}}(z) = (|c_1|^2 - |c_2|^2) E_p(z). \quad (3.6)$$

Similarly,

$$\tilde{E}_{c_1 p + c_2 \overleftarrow{p}}(w) = (|c_1|^2 - |c_2|^2) \tilde{E}_p(w). \quad (3.7)$$

Assume now that $c_1 p + c_2 \overleftarrow{p}$ is stable, and thus $E_{c_1 p + c_2 \overleftarrow{p}}(z) > 0$, $|z| = 1$, and $\tilde{E}_{c_1 p + c_2 \overleftarrow{p}}(w) > 0$, $|w| = 1$. By (??) it follows that $|c_1| \neq |c_2|$. In case $|c_1| > |c_2|$ we get from (??) that $E_p(z) > 0$, $|z| = 1$, and from (??) that $\tilde{E}_p(w) > 0$, $|w| = 1$. But then p is stable by Proposition ???. Similarly, if $|c_1| < |c_2|$ we get (??) that $E_p(z) < 0$, $|z| = 1$, and from (??) that $\tilde{E}_p(w) < 0$, $|w| = 1$. Since $E_{\overleftarrow{p}}(z) = -E_p(z)$, $|z| = 1$, and $\tilde{E}_{\overleftarrow{p}}(w) = -\tilde{E}_p(w)$, $|w| = 1$, it follows that $\overleftarrow{p}$ is stable. $\square$

The stability of p may be determined directly in terms of the right spectral pairs of $A(z)$ and $B(w)$ (as defined in Section 2 with $q(z, w) = \overleftarrow{p}(z, w)$) by examining the relation between these factors and the spectral factorization of $E_p(z)$ and $\tilde{E}_p(w)$. The following proposition gives the relation.

Proposition 3.4 *Let $a(z), b(z), c(z)$ and $d(z)$ be $m \times m$ matrix polynomials so that $c(z), d(z)$ and their derivatives commute. Let*

$$X(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \quad H(z) = a(z)d(z) - b(z)c(z).$$

Let $z_0 \in \mathbb{C}$ so that z_0 is not an eigenvalue of $d(z)$. Then z_0 is an eigenvalue of $X(z)$ of multiplicity μ if and only if z_0 is an eigenvalue of $H(z)$ of multiplicity μ . In addition, if (J, Y) is a left Jordan pair for $X(z)$ corresponding to z_0 , then $(J, Y \begin{pmatrix} I_m \\ 0 \end{pmatrix})$ is a left Jordan pair for $H(z)$ corresponding to z_0 . Conversely, if (J, Z) is a left Jordan pair for $H(z)$ corresponding to z_0 , then there exists a left Jordan pair (J, Y) for $X(z)$ corresponding to z_0 with $Z = Y \begin{pmatrix} I_m \\ 0 \end{pmatrix}$. If in addition $\deg H(z) = 2\deg X(z) = 2\deg d(z)$, then the same statement holds at ∞ .

Proof. Since $\det d(z_0) \neq 0$, we have that

$$\det X(z_0) = \det(a(z_0) - b(z_0)d(z_0)^{-1}c(z_0)) \det d(z_0) = \det H(z_0),$$

where we first use a standard Schur complement argument and next the commutativity of $c(z_0)$ and $d(z_0)$.

Recall (see [?]) that $\phi_0, \dots, \phi_t \in \mathbb{C}^{1 \times r}$ is called a *left Jordan chain* of the $r \times r$ matrix polynomial $L(z)$ at z_0 if $\phi_0 \neq 0$ and

$$(\phi_t \quad \dots \quad \phi_0) \begin{pmatrix} L(z_0) & 0 & \dots & 0 \\ L'(z_0) & L(z_0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t!}L^{(t)}(z_0) & \frac{1}{(t-1)!}L^{(t-1)}(z_0) & \dots & L(z_0) \end{pmatrix} = 0.$$

Moreover, the left Jordan pairs are built up from left Jordan chains (see [?] for more details). It therefore suffices to show the correspondence between the left Jordan chains of $X(z)$ and $H(z)$.

First let $\phi_i = (x_i \ y_i)$, $i = 0, \dots, r$, be a Jordan chain for $X(z)$ at z_0 . We claim that x_i , $i = 0, \dots, r$, is a Jordan chain for $H(z)$ at z_0 . Indeed, since $x_0 X(z_0) = 0$, we get that

$$x_0 a(z_0) + y_0 c(z_0) = 0, x_0 b(z_0) + y_0 d(z_0) = 0.$$

First note that if $x_0 = 0$, then also $y_0 = -x_0 b(z_0) d(z_0)^{-1} = 0$. But this would contradict $\phi_0 \neq 0$ and thus $x_0 \neq 0$. Next,

$$x_0 H(z) = x_0 (a(z_0) - b(z_0)d(z_0)^{-1}c(z_0))d(z_0) = (x_0 a(z_0) + y_0 c(z_0))d(z_0) = 0.$$

By the same calculation, we get that if $x_0 H(z) = 0$, then

$$(x_0 \ -x_0 b(z_0)d(z_0)^{-1}) X(z) = 0.$$

Furthermore, if $\phi_0 X'(z_0) + \phi_1 X(z_0) = 0$, then it is straightforward to check that

$$y_1 = x_1 b(z_0) d(z_0)^{-1} - x_0 b'(z_0) d(z_0)^{-1} + x_0 b(z_0) d(z_0)^{-1} d'(z_0) d(z_0)^{-1}.$$

Using the commutativity of $c(z_0)$, $d(z_0)$, $c'(z_0)$ and $d'(z_0)$, it is easy to check that $x_0 H'(z_0) + x_1 H(z_0) = 0$. Moreover, one can go back as well (define y_0 and y_1 as above).

Proceeding in this way one obtains the desired correspondence between the left Jordan chains of $X(z)$ and $H(z)$ at z_0 .

For the statement at ∞ use that if $n = \deg X(z)$ then

$$z^n X(1/z) = \begin{pmatrix} z^n a(1/z) & z^n b(1/z) \\ z^n c(1/z) & z^n d(1/z) \end{pmatrix},$$

and

$$z^{2n} H(1/z) = (z^n a(1/z))(z^n d(1/z)) - (z^n b(1/z))(z^n c(1/z)),$$

satisfy the conditions of the first part at $z_0 = 0$. $\square$

Recall that if

$$E(z) = F(1/\bar{z})^* F(z), \quad (3.8)$$

with $\det F(z) \neq 0$, $z \in \overline{\mathbb{D}}$, then (??) is called a *right spectral factorization* of $E(z)$, and $F(z)$ is called a *right spectral factor*. One obtains a *left spectral factorization* when one considers the equation $E(z) = F(z)F(1/\bar{z})^*$. It is well known (see, e.g., [?]) that $E(z)$ has a right (left) spectral factorization if and only if $E(z) > 0$, $z \in \mathbb{T}$. In addition, the right (left) spectral factor is unique up to multiplication with a constant unitary matrix on the left (right).

Using Proposition ?? we may now construct a spectral factorization as follows. Let $(\hat{X}_A, T_A)$ be the right spectral pair (including the infinite part) associated with $A(z)$. Let $z_1, \dots, z_t$ be the eigenvalues of $A(z)$ that lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$, $(\tilde{X}_{z_i}, T_{z_i})$ be right Jordan pairs of $A(z)$ corresponding to z_i , and $(\tilde{X}_\infty, T_\infty)$ be the infinite right Jordan pair of $A(z)$ (which may be void). Put

$$T_G = T_{z_1}^{-1} \oplus \dots \oplus T_{z_t}^{-1} \oplus T_\infty, X_G = (X_{z_1} \quad \dots \quad X_{z_t} \quad X_\infty).$$

where X_{z_i} are the top m rows of $\tilde{X}_{z_i}$. Then by Proposition ?? (X_G, T_G) is the comonic right Jordan pair of the matrix polynomial $G(z) := F(0)^{-1}F(z)$ and $G^*(\bar{z})$ the comonic part of the left spectral factor $F(z)$ of $E(z)$. Next, set

$$C = \lim_{z \rightarrow 0} (E(z)G(1/z)). \quad (3.9)$$

and $M^*M = C$, then $(MG(\bar{z}))^*$ is a left spectral factor of $E(z)$. We now factor E as,

$$E(z) = G(\bar{z})^* C G(1/z),$$

The matrix polynomial G can be computed from the formula (see [?]),

$$G(z) = I - X_G T_G^n (V_1 z^n + \dots + V_n z), \quad (3.10)$$

where

$$\text{row}(V_i)_{i=1}^n = [\text{col}(X_G T_G^i)_{i=0}^{n-1}]^{-1}.$$

In an analogous manner we can compute $\tilde{C}$ so that

$$\tilde{E}(w) = \tilde{G}(\bar{w})^* \tilde{C} \tilde{G}(1/w),$$

where $\tilde{G}^*(\bar{w})$ is the comonic left stable factor of $\tilde{E}(w)$. Note that

$$\hat{G} = z^n G(1/z)|_{z=0} = X_G T_G^n X_G^{-1}$$

and if $\hat{G}$ is invertible then the above formula for C can be written as

$$C = z^n E(z)|_{z=0} \hat{G}^{-1}.$$

Proposition ?? and Theorem ?? allow us to show how the intersecting zeros determine a stable polynomial.

Theorem 3.5 *Let $(z_i, w_i), i = 1, \dots, nm$ be distinct pairs in $\mathbb{C}_\infty^2$ with $|z_i| \leq 1, i = 1, \dots, nm$. Let $(z_{i+nm}, w_{i+nm}) = (1/\bar{z}_i, 1/\bar{w}_i), i = 1, \dots, nm$, where $1/0 = \infty$. There exists a stable polynomial of degree (n, m) with intersecting zeros $(z_i, w_i), i = 1, \dots, 2nm$, if and only if*

(i) $|z_i| < 1$ and $|w_i| > 1, i = 1, \dots, nm$;

(ii) $\dim \ker H^T = 2$, where H is the $(n+1)(m+1) \times 2nm$ matrix whose i th column equals

$$(1 \ w_i \ \dots \ w_i^m \ \dots \ z_i^n \ z_i^n w_i \ \dots \ z_i^n w_i^m)^T, \text{ when } z_i, w_i \in \mathbb{C},$$

$$(0 \ \dots \ 0 \ 1 \ \dots \ 0 \ \dots \ 0 \ z_i^n)^T, \text{ when } z_i \in \mathbb{C}, w_i = \infty,$$

$$(0 \ \dots \ 0 \ \dots \ 1 \ w_i \ \dots \ w_i^m)^T, \text{ when } z_i = \infty, w_i \in \mathbb{C},$$

$$(0 \ \dots \ 0 \ \dots \ 0 \ \dots \ 0 \ 1)^T, \text{ when } z_i = w_i = \infty;$$

(iii) the matrices C and $\tilde{C}$ constructed above are definite with the same sign.

Proof. If $p \in \ker H^T$ is stable then by [?, Theorem 2.1.5] there are nm points (z_i, w_i) counting multiplicity with $|z_i| < 1$ and $|w_i| > 1$ such that $p(z_i, w_i) = 0 = \overleftarrow{p}(z_i, w_i)$. (Note by symmetry that $(1/\bar{z}_i, 1/\bar{w}_i)$ is also a root). Now suppose that p is in $\ker H^T$ and the intersecting zeros are located as given in the hypothesis. Compute C and $\tilde{C}$ as above. Theorem 9.4.2 in [?] implies that $G^*(0) = (X_G T_G^n X_G^{-1})^*$, and $\tilde{G}^*(0) = (X_{\tilde{G}} T_{\tilde{G}}^m X_{\tilde{G}}^{-1})^*$ are invertible so that the hypotheses that C and $\tilde{C}$ are definite with the same sign implies that from Proposition ?? and Proposition ?? that either p or $\overleftarrow{p}$ is stable. $\square$

The stable polynomial in the above theorem, if it exists, will have intersecting zeros that are all simple. As we will see in some of the examples, with an appropriate definition of the matrix H cases of intersecting zeros with higher multiplicity may be handled as well. Besides obtaining a stable polynomial it is necessary to check that its reverse is also in the kernel of H^T . We do not yet have a general statement for this case.

Using the results developed in this section we obtain the following algorithm for finding stable polynomials with given intersecting zeros.

Algorithm. Given are pairs $(z_i, w_i) \in ((\mathbb{C} \setminus \overline{\mathbb{D}}) \times \mathbb{D}) \cup (\mathbb{D} \times (\mathbb{C} \setminus \overline{\mathbb{D}})), i = 1, \dots, 2nm$, that lie symmetrically with respect to $\mathbb{T}^2$.

Step 1: Construct intersecting right spectral pairs and form H . In the case of distinct roots H is the $(nm + n + m + 1) \times 2nm$ two-variable Vandermonde matrix whose j th column equals

$$(1 \ w_j \ \dots \ w_j^m \ z_j \ z_j w_j \ \dots \ z_j w_j^m \ \dots \ z_j^n \ z_j^n w_j \ \dots \ z_j^n w_j^m)^T.$$

Step 2: Check that $\dim \ker H^T = 2$. If so, continue. If not, stop (no solution exists by Theorem ??).

Step 3: Find a basis for $\ker H^T$ of the form $\text{span}\{v, Z\bar{v}\}$, where Z is the matrix with an antidiagonal of 1's and zeros elsewhere.

Step 4: Writing $v = \text{col} \left(\text{col}(p_{ij})_{j=0}^m \right)_{i=0}^n$, introduce $p(z, w) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i w^j$. Then check whether p or $\bar{p}$ is stable. If not there is no stable polynomial associated with the given pairs.

Remark 3.6 The above algorithm may also be used in case only some of the intersecting zeros are given. To make the above algorithm work for a degree (n, m) polynomial, one needs to start with at least $(n + 1)(m + 1) - 2$ given pairs of intersecting zeros.

Example 1. Let $n = 1$ and $m = 1$ with intersecting zeros $(1/z_1, w_1)$ with $|z_1| < 1$ and $|w_1| < 1$. In this case

$$H^T = \begin{pmatrix} 1 & w_1 & 1/z_1 & w_1/z_1 \\ 1 & 1/\bar{w}_1 & \bar{z}_1 & \bar{z}_1/\bar{w}_1 \end{pmatrix},$$

and it is easy to check that the vector $[1 - |z_1 w_1|^2, -\bar{w}_1(1 - |z_1|^2), -(1 - |w_1|^2)z_1, 0]^T$ is in $\ker H^T$. This vector gives rise to the stable polynomial $p(z, w) = 1 - |z_1 w_1|^2 - \bar{w}_1(1 - |z_1|^2)w - z_1(1 - |w_1|^2)z$. Also in this case C and $\tilde{C}$ are scalars given by

$$C = (1 - |w_1|^2)(1 - |z_1|^2|w_1|^2),$$

and

$$\tilde{C} = (1 - |z_1|^2)(1 - |z_1|^2|w_1|^2),$$

Thus in the $n = 1, m = 1$ case every set of intersecting roots yields a stable polynomial.

Example 2. Let $n = 1, m = 2$ and pairs $(1/5, 2), (1/3, 7/3), (5, 1/2), (3, 3/7)$ be given. In this case

$$\ker H^T = \text{span}\left\{\left(-\frac{45}{17}, 0, 1, \frac{2679}{935}, -\frac{4448}{935}, -\frac{27}{935}\right)^T, \left(-\frac{27}{17}, 1, 0, \frac{1417}{935}, -\frac{2679}{935}, \frac{504}{935}\right)^T\right\}.$$

Thus the polynomial

$$p(z, w) = -\frac{45}{17} + w^2 + \frac{2679}{935}z - \frac{4448}{935}zw - \frac{27}{935}zw^2,$$

has the intersecting roots given above. Since $p(1, w) = \frac{12}{555} - \frac{4448}{935}w + \frac{908}{935}w^2$ has one root inside the unit circle and the other outside, neither p nor $\bar{p}$ are stable and from

Proposition ?? there are no stable polynomials of degree $(1, 2)$ with these intersecting roots.

Example 3. Let $n = 1$, $m = 2$ and pairs $(1/5, 2), (1/7, 3), (5, 1/2), (7, 1/3)$ be given. In this case we find

$$\ker H^T = \text{span}\left\{\left(\frac{11}{6} \ 0 \ 1 \ -\frac{11}{6} \ \frac{25}{3} \ -11\right)^T, \left(-11 \ \frac{25}{3} \ -\frac{11}{6} \ 1 \ 0 \ \frac{11}{6}\right)^T\right\},$$

and thus

$$p(z, w) = \frac{11}{6} + w^2 - \frac{11}{6}z + \frac{25}{3}zw - 11zw^2.$$

Since $\overleftarrow{p}(z, 1) = (17z - 105)/6$ and

$$\tilde{E}_{\overleftarrow{p}}(w) = \frac{55}{18}(2 - w)(3 - w)(2 - 1/w)(3 - 1/w),$$

we find that $\overleftarrow{p}$ is stable.

Example 4. Let $n = 1$, $m = 2$ and the pair $(2, 1/4)$ with multiplicity 2 in w . In this case H^T is given by,

$$H^T = \begin{pmatrix} 1 & 1/4 & 1/16 & 2 & 1/2 & 1/8 \\ 0 & 1 & 1/2 & 0 & 2 & 1 \\ 1 & 4 & 16 & 1/2 & 2 & 8 \\ 0 & 1 & 8 & 0 & 1/2 & 4 \end{pmatrix}.$$

The vector $[-5/4, 0, 5/4, 1/2, 1, -21/8]^T$ in the $\ker H^T$ gives the polynomial

$$p(z, w) = -5/4 + 5/4w^2 + 1/zw + zw - 21/8zw^2.$$

In this case, $T = \text{diagonal}(2, 2)$, and

$$X = \begin{pmatrix} 1 & 0 \\ 1/4 & 1 \end{pmatrix},$$

so that

$$C = \begin{pmatrix} -85/16 & 5/2 \\ 5/2 & -85/16 \end{pmatrix}$$

Since $\tilde{E}(w) = -1/64(w - 4)^2(4 - 1/w)^2$ we find that $\overleftarrow{p}$ is stable.

Example 5. Let $n = 1$ and $m = 3$ with intersecting roots $(5, 5/13), (4, 3/7), (3, 1/2)$. Substitution of these pairs and their reciprocals into the 6×8 matrix H^T gives a matrix with rank 5 so there is not a stable polynomial associated with these roots. A basis for the kernel is,

$$\{(-25, 0, 1, 0, -5, 24, 5, 0)^T, (-125, 0, 0, 1, -25, 120, 24, 5)^T, (-5, 1, 0, 0, -1, 5, 0, 0)^T\},$$

and the polynomials associated with this basis all have a common factor of $5zw + w - z - 5$.

Example 6. Let $n = 2 = m$ with intersecting roots $(4, 1/3), (4, 1/6), (8/5, 3/5), (-131/16, -3/17)$. Substitution of these pairs and their complex conjugate reciprocals into the 8×9 matrix

H^T with rank 7. The vector $[-5135/28, 23/8, 1, 27/7, 39177/112, 0, 5/4, -205/112, -175]^T$ is in $\ker H^T$ and gives rise to the polynomial $p(z, w) = -5135/28 + 23/8z + z^2 + 27/7w + 39177/112zw + 5/4w^2 - 205/112zw^2 - 175z^2w^2$. In this case, $T = \text{diagonal}(4, 4, 8/5, -131/16)$ and

$$X_G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1/3 & 1/6 & 3/5 & -3/17 \end{pmatrix},$$

Thus,

$$C := \frac{1}{1232} \begin{pmatrix} 3400605 & 119205 \\ 119205 & 130787 \end{pmatrix},$$

which is positive definite. A similar calculation shows that $\tilde{C}$ is given by

$$\tilde{C} = \begin{pmatrix} 196043625/68992 & -53325/308 \\ -53325/308 & 65745/616 \end{pmatrix}.$$

Thus p is a stable polynomial.

Example 7. This next $n = 2, m = 2$ example is a bit more involved than the previous ones and is to illustrate the intersection structure indicated earlier. The polynomial we are interested in finding has a intersection of multiplicity 3 in z and w with its reverse polynomial at $(z_1, w_1) = (1/2, 3)$ and another at $(z_2, w_1) = (3550/4703, 3)$. The higher order intersection in z at $z = 1/2$ implies that for this intersection the 3×3 matrix T_{w_1} has Jordan block form with w_1 on the diagonal and 1's on the first upper diagonal. The matrix X_{z_1} given by,

$$X_{z_1} = [x_1(z_1), 1/5x_2(z_1), 1/50x_3(z_1)]^T,$$

where $x_1(z) = [1, z, z^2]$, $x_2(z) = [0, 1, 2z, 3z^2]$, and $x_3(z) = [0, 0, 2, 6z]$ accommodates this intersection structure. The structure at $(2, 1/3)$ is given by T_{1/w_1} having the same form at T_{w_1} above and,

$$X_{1/z_1} = [x_1(1/z_1), 35/5x_2(1/z_1), 108/5x_2(1/z_1) + 648/25x_3(1/z_1)]^T.$$

The structure at the remaining roots is given by the structure of a simple root. Thus

$$H^T = \begin{pmatrix} X_{z_1}^T & (X_{z_1}T_{w_1})^T & (X_{z_1}T_{w_1}^2)^T \\ x_1(z_2) & w_1x_1(z_2) & w_1^2x_1(z_2) \\ x_1(1/z_2) & x_1(1/z_2)/w_1 & x_1(1/z_2)/w_1^2 \\ X_{1/z_1}^T & (X_{1/z_1}T_{1/w_1})^T & (X_{1/z_1}T_{1/w_1}^2)^T \end{pmatrix}.$$

With the values of z and w indicated the $\dim \ker H^T = 2$. The vector

$$L = [178608, -263262, 108552, -172090, 346405, -202090, 37518, -86217, 55302]$$

is in $\ker H^T$ and gives rise to the polynomial $[1, z, z^2, w, wz, wz^2, w^2, zw^2, z^2w^2]L$ which is

$$p(z, w) = (-3 + w)(-59536 + 37518w - 86217wz + 55302wz^2 + 87754z - 36184z^2).$$

Since $\tilde{p}_0 = (3 - w)(59536 - 37518w)$ and

$$\tilde{C} = \begin{pmatrix} 466366545/3544534296 & -188559711/1772267648 \\ -188559711/1772267648 & 104009805/886133824 \end{pmatrix},$$

we see that p is stable. It is not difficult to check that $\overleftarrow{p}$ is also in the $\ker H^T$

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