Asymptotics of the information entropy of the Airy function

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Abstract. The Boltzmann-Shannon information entropy of the linear potential wavefunctions is known to be controlled by the information entropy of the Airy function $\text{Ai}(x)$. Here, the entropy asymptotics is analyzed so that the leading term (previously calculated in the WKB approximation) as well as the following term (already conjectured) are derived by using only the specific properties of the Airy function.

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1. Introduction.

The Boltzmann-Shannon information entropy of an individual microstate of a one-dimensional physical system is defined by

$$S[\Psi] := - \int |\Psi(x)|^2 \log |\Psi(x)|^2 dx, \quad (1)$$

where $\Psi(x)$ is the quantum-mechanical wavefunction of the state [1, 2]. The exact computation of this quantity is a formidable, practically impossible task. This is because either the Schrödinger equation of the system cannot be exactly solved or, when it is, the wavefunction is a known special function whose information entropy cannot be analytically evaluated and its numerical computation is highly unstable. Most efforts have been focussed on classical hypergeometric functions of the type $\Psi(x) = \sqrt{\omega(x)} y_n(x)$, where $\{y_n(x); n = 0, 1, \dots\}$ denotes a set of polynomials orthogonal with respect to the weight function $\omega(x)$. In this case, the computation of the entropy (1) boils down to evaluating the so-called information entropy of the orthogonal polynomials $y_n(x)$, given by

$$S[y_n] := - \int y_n^2(x) [\log y_n^2(x)] \omega(x) dx.$$

Briefly, up to now, this mathematical quantity has been exactly calculated only for Chebyshev polynomials, although numerous results for other classical orthogonal polynomials (Hermite, Laguerre, Jacobi) have been found, mainly from an asymptotical point of view. See Ref. [3] for a recent summary of the main achievements. The numerical computation of this entropic integral on finite intervals is most conveniently done by the effective method of Buyarov *et al.* [4].

Recently, the evaluation of the physical entropy (1) has been attacked for wavefunctions controlled by special functions other than orthogonal polynomials. This occurs for systems such as the circular membrane [5], the Toda-like potential [6] and confining power-type potential [7]. For these cases the entropy has been evaluated in the asymptotic region, that is for highly energetic quantum states, either by use of only the properties of the involved special functions or by means of the semiclassical or WKB approximation. In doing so, we have calculated the asymptotics of the information entropies of the Bessel functions [5] and the McDonald functions [6].

In this paper we shall consider the computation of the Boltzmann-Shannon entropy for a linear potential, i.e. for the quantum-mechanical problem of a particle under the influence of a constant (nonvanishing) force. This potential for vanishing orbital quantum number and the harmonic oscillator potential are the only two power-type confinement potentials [8, 9], so much used with quark models and more particularly the charmonium model [10] of the J/ψ particle [11, 12], which are exactly solvable in terms of known special functions. The linear potential has the same importance in particle physics as the Coulomb potential in atomic physics and the harmonic potential in solid state physics. In Section 2 it is pointed out that the linear potential wavefunctions for vanishing orbital quantum number are controlled by the convergent

solutions of Airy's equation, i.e. by the Airy function $\text{Ai}(z)$. So, the calculation of the physical entropy of the highly energetic states of a particle in the linear potential boils down to the asymptotics of the information entropy of the Airy function. The relation between these physical and mathematical quantities was explicitly given by J. Sánchez-Ruiz [13]. This author not only determined the leading terms of the entropy by use of the WKB approximation, but also conjectured the following term in the form $K'n'^{-1/3}$, with $K' \simeq 0.709$ if we consider $n' = 1, 2, 3 \dots$ running only over odd states, or $K \simeq 2^{1/3}K' = 0.8933$ if we consider $n = 1, 2, 3 \dots$ running over odd and even states. Notice that $K'n'^{-1/3} = Kn^{-1/3} = K(2n' + 1)^{-1/3} \simeq K2^{-1/3}n'^{-1/3}$.

The main purpose of this paper is the analytical derivation of these two terms by means of the specific properties of the Airy functions, although the value of K is numerically calculated. The structure of this paper follows so that in Section 3, the asymptotics of the entropy of the Airy function S_n^{Ai} is investigated in detail by use of its specific properties. Then, in Section 4, the constant K is numerically evaluated. Finally, a brief summary and some open problems are pointed out.

2. Linear potential wavefunctions and the Airy function: information entropies.

The three-dimensional motion of a particle under a linear potential for the case of vanishing orbital quantum number reduces to the one-dimensional problem of the potential $V(x) = F|x|$, where F denotes the field strength [14, 15]. The wavefunctions of the quantum-mechanical states of the particle in such a potential are the solutions of the associated Schrödinger equation,

$$-\frac{1}{2} \frac{d^2 \Psi(x)}{dx^2} + F|x|\Psi(x) = E\Psi(x),$$

where units $\hbar = m = 1$ have been used. Various authors (see e.g. [14, 15]) have shown that the eigenfunctions can be expressed in terms of the Airy function as

$$\Psi_n(x) = N_n \text{Ai}(\alpha|x| + \beta_n), \quad (2)$$

where the normalization constant N_n and the parameters β_n and α are given by

$$N_n = \begin{cases} \sqrt{\frac{\alpha}{2}} \frac{1}{\sqrt{-a'_{n/2+1}} \text{Ai}(a'_{n/2+1})} & \text{even } n \\ \text{sign}(x) \sqrt{\frac{\alpha}{2}} \frac{1}{\text{Ai}'(a_{(n+1)/2})} & \text{odd } n \end{cases},$$

$$\beta_n = \begin{cases} a'_{n/2+1} & \text{even } n \\ a_{(n+1)/2} & \text{odd } n \end{cases},$$

$$\alpha = (2F)^{1/3},$$

where a_s and a'_s are the zeros of the Airy function $\text{Ai}(x)$ and its derivative $\text{Ai}'(x)$ respectively.

The associated energy eigenvalues are

$$E_n = -\frac{F}{\alpha}\beta_n. \quad (3)$$

The spreading of the linear potential wavefunctions, $\Psi_n(x, t) = \Psi_n(x) \exp(-\frac{i}{\hbar}E_n t)$, (3)-(2), is best described [1] by the Boltzmann-Shannon information entropy (1), which gives the spatial distribution of the associated quantum-mechanical Born density $|\Psi_n(x, t)|^2 = |\Psi_n(x)|^2$. Moreover, it gives the uncertainty of the position of the particle. Taking into account Eq. (2), this entropy is expressed as

$$S[\Psi_n] = -\int_{-\infty}^{\infty} |\Psi_n(x)|^2 \log |\Psi_n(x)|^2 dx = -\log N_n^2 + 2\frac{N_n^2}{\alpha} S_n^{\text{Ai}}, \quad (4)$$

where S_n^{Ai} is the information entropy of the Airy function on the interval $[\beta_n, \infty)$:

$$S_n^{\text{Ai}} = -\int_{\beta_n}^{\infty} \text{Ai}^2(x) \log \text{Ai}^2(x) dx.$$

3. Entropy asymptotics.

Taking into account the asymptotic expansion for the Airy function [16],

$$\text{Ai}(-z) \sim \frac{1}{\sqrt{\pi z^{1/4}}} \left(\cos\left(\zeta - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s}}{\zeta^{2s}} + \sin\left(\zeta - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-)^s \frac{u_{2s+1}}{\zeta^{2s+1}} \right),$$

with $\zeta = \frac{2}{3}z^{3/2}$ and $|\text{ph}z| < \frac{2}{3}\pi$, we have that

$$\exists M, x_0 < 0 \quad \text{such that} \quad |\text{Ai}(x) - \overline{\text{Ai}}(x)| < \frac{M}{x_0}, \quad x < x_0,$$

where

$$\overline{\text{Ai}}(x) = \frac{(-x)^{-1/4}}{\sqrt{\pi}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right).$$

So we can write

$$\begin{aligned} S_n^{\text{Ai}} &= -\int_{x_0}^{\infty} \text{Ai}^2(x) \ln \text{Ai}^2(x) dx - \int_{\beta_n}^{x_0} \text{Ai}^2(x) \ln \text{Ai}^2(x) dx \\ &\simeq -\int_{x_0}^{\infty} \text{Ai}^2(x) \ln \text{Ai}^2(x) dx - \int_{\beta_n}^{x_0} \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx \\ &= -\int_{x_0}^{\infty} \text{Ai}^2(x) \ln \text{Ai}^2(x) dx + \int_{x_0}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx \\ &\quad - \int_{\beta_n}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx. \end{aligned}$$

Then,

$$S_n^{\text{Ai}} \simeq -\int_{\beta_n}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx + \kappa, \quad (5)$$

where

$$\kappa = \lim_{x_0 \rightarrow -\infty} \left(-\int_{x_0}^{\infty} \text{Ai}^2(x) \ln \text{Ai}^2(x) dx + \int_{x_0}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx \right). \quad (6)$$

Using the L^q -norm method (see e.g. [17, 18]) we can write

$$-\int_{\beta_n}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx = - \left[\frac{d}{dq} \int_{\beta_n}^0 \left(\overline{\text{Ai}}^2(x) \right)^q dx \right]_{q=1}. \quad (7)$$

Considering the asymptotics of the zeros β_n [16],

$$a_s = -T \left(\frac{3}{8} \pi (4s - 1) \right), \quad T(t) \sim t^{2/3} \left(1 + \frac{5}{48} t^{-2} - \dots \right),$$

$$a'_s = -U \left(\frac{3}{8} \pi (4s - 3) \right), \quad U(t) \sim t^{2/3} \left(1 - \frac{7}{48} t^{-2} + \dots \right),$$

as well as the change of variable, $x = \beta_n(y/\pi)^{2/3}$, and applying the Lemma 2.1 of Aptekarev *et al.* [17], which states that

$$\int_0^\pi g(n\theta + \gamma(\theta)) f(\theta) d\theta \rightarrow_{(n \rightarrow \infty)} \frac{1}{\pi} \int_0^\pi g(\theta) d\theta \int_0^\pi f(\theta) d\theta,$$

where $g \in C(\mathbf{R})$, $g(\theta + \pi) = g(\theta)$, $f \in L^1([0, \pi])$, and $\gamma(\theta)$ measurable almost everywhere and finite in $[0, \pi]$; we have from Eq. (7),

$$\begin{aligned} \int_{\beta_n}^0 \left(\overline{\text{Ai}}^2(x) \right)^q dx &= \\ \frac{1}{\pi^q} \frac{2}{3} \left[\left(\frac{3}{4} n \right)^{\frac{2-q}{3}} + O\left(n^{-\frac{1+q}{3}}\right) \right] &\left[\int_0^\pi \left[\sin^2 \left(\frac{1}{2} ny + \frac{1}{4}(y + \pi) \right) \right]^q y^{-\frac{q+1}{3}} dy + O(n^{-1}) \right] \\ \rightarrow_{(n \rightarrow \infty)} \frac{1}{\pi^q} \frac{2}{3} \left(\frac{3}{4} n \right)^{\frac{2-q}{3}} &\times \left[\frac{1}{\pi} \int_0^\pi \left[\sin^2(y) \right]^q dy \int_0^\pi y^{-\frac{q+1}{3}} dy + \tau(q) n^{\sigma(q)} \right], \end{aligned} \quad (8)$$

It can be seen that the limit set up by this lemma has an error term that goes as $\tau(q)n^{\sigma(q)}$, where we make the numerically-based hypothesis $\sigma(q) = \frac{q-2}{3}$ (see Table 3 in Section 4), so that the multiplicative factor $n^{\frac{2-q}{3}}$ cancels the n -dependence in the error term but not the q -dependence.

Thus, according to Eqs. (7) and (8), differentiating with respect to q and taking $q = 1$, we obtain the additional constant κ' in the limit,

$$-\int_{\beta_n}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx = C \left[\frac{1}{3} n^{\frac{1}{3}} \ln n + D n^{\frac{1}{3}} \right] + \kappa' + O(n^{-\frac{2}{3}} \ln n), \quad (9)$$

where $C = \frac{1}{\pi} \left(\frac{3}{4} \pi \right)^{\frac{1}{3}}$, $D = \frac{1}{3} \ln(48\pi^4) - 2$, and

$$\kappa' = -\frac{1}{3\pi} \left(\frac{2}{9} \right)^{\frac{1}{3}} \left[\left(\log \frac{4}{3} - 3 \log \pi \right) \tau(1) + 3\tau'(1) \right].$$

Eqs. (5) and (9) provide the following asymptotical behavior of the entropy of the Airy function,

$$S_n^{\text{Ai}} = C \left[\frac{1}{3} n^{\frac{1}{3}} \ln n + D n^{\frac{1}{3}} \right] + \kappa + \kappa' + O(n^{-\frac{2}{3}} \ln n). \quad (10)$$

For completeness, let us write down that the Boltzmann-Shannon information entropy (4) of the linear potential wavefunctions for the highly excited particle state is consequently given by

$$S[\Psi_n] = \frac{2}{3} \ln n + \left(\ln \left(\frac{2(6\pi)^{2/3}}{\alpha} \right) - 2 \right) + \frac{\kappa + \kappa'}{C} n^{-\frac{1}{3}} + O(n^{-1} \ln n). \quad (11)$$

As already pointed out, the leading term was previously computed within the framework of WKB approximation by Sánchez-Ruiz [13]. Moreover, the same author on the basis of some numerical experiments, also conjectured the following term in the form $Kn^{-1/3}$ with $K = 0.709$ by taking into account only the odd-parity eigenfunctions; one can easily find (see e.g. Section 1) with the same method that $K = 0.8933$ when both odd-parity and even-parity levels are considered.

The value of our work is to have determined the first two asymptotical terms of the information entropy by means of only the specific properties of the Airy function. However, we have not been able to explicitly compute the coefficient of the $n^{-1/3}$ -term, i.e. the constant parameters κ and κ' . This will be done numerically in the next section.

4. Numerical analysis.

Now we turn to the numerical problem of computing the constants κ and κ' . We achieve this by doing various computations in order to obtain their values from different approaches.

First, we try to compute κ from Eq. (5),

$$\kappa = \lim_{n \rightarrow \infty} \left(S_n^{\text{Ai}} + \int_0^{\beta_n} \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx \right). \quad (12)$$

The numerical study of this expression is shown in Table 1, where one can realize that $\kappa \simeq 0.2265$.

An alternative way to compute this constant is directly from its definition (6)

$$\kappa = \lim_{x_0 \rightarrow -\infty} \left(- \int_{x_0}^{+\infty} \text{Ai}^2(x) \log \text{Ai}^2(x) dx + \int_{x_0}^0 \overline{\text{Ai}}^2(x) \log \overline{\text{Ai}}^2(x) dx \right), \quad (13)$$

and the resulting values are given in Table 2, where it is apparent that $\kappa \simeq 0.2265$.

At this it is pertinent to perform a numerical test of the hypothesis $\sigma(q) = \frac{q-2}{3}$. This is shown in Table 3, where we realize the high quality of the hypothesis. To obtain the $\sigma(q)$ values, a linear dependence on q of the quantity σ was assumed, and the numerical analysis of Eq. 8 was consequently performed.

The constant κ' can be obtained from the expression (9),

$$\kappa' = \lim_{n \rightarrow \infty} \left(- \int_{\beta_n}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx - C \left[\frac{1}{3} n^{\frac{1}{3}} \ln n + D n^{\frac{1}{3}} \right] \right), \quad (14)$$

whose numerical analysis is carried out in Table 4, which shows that $\kappa' \simeq 0.152$.

With these computations we find that $\kappa + \kappa' \simeq 0.378$. Finally, two more tests for the $\kappa + \kappa'$ value are possible. Firstly, by use of Eq. (10), one has

$$\kappa + \kappa' = \lim_{n \rightarrow \infty} \left(S_n^{\text{Ai}} - C \left[\frac{1}{3} n^{\frac{1}{3}} \ln n + D n^{\frac{1}{3}} \right] \right). \quad (15)$$

The numerical study of this equation is done in Table 5, which also illustrates that $\kappa + \kappa' \simeq 0.3785$.

The second and final test uses Eq. (11), which allows us to write down

$$\kappa + \kappa' = \lim_{n \rightarrow \infty} \left(\left[S[\Psi_n] - \frac{2}{3} \ln n + \left(\ln \left(\frac{2(6\pi)^{2/3}}{\alpha} \right) - 2 \right) \right] C n^{\frac{1}{3}} \right). \quad (16)$$

The numerical study of this equation (see Table 6) leads again to the value $\kappa + \kappa' \simeq 0.378$.

So, the coefficient for $n^{-1/3}$ term is $K = 0.8936$, quite close to the $K = 0.8933$ value given in [13].

5. Summary and open problems.

We have determined the asymptotics of the information entropy of the Airy function $\text{Ai}(x)$, whose usefulness for the computation of the Boltzmann-Shannon entropy of the highly-energetic (Rydberg) quantum-mechanical states of the linear potential with vanishing angular momenta is also shown. In doing so the only existing related results [13], which are based in the WKB approximation, are rigorously corroborated.

Let us point out, for completeness, that the physical entropies of the non-vanishing angular momentum Rydberg states of the linear potential [19–21], so much useful for the theory of quark confinement, require the previous knowledge of the information entropies of the first order derivatives of the Airy function $\text{Ai}'(x)$ as well as the combination of $\text{Ai}(x)$ and $\text{Ai}'(x)$. These two problems have not yet been solved.

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Table 1. Numerical study of Eq. (12).

n	S_n^{Ai}	$-\int_0^{\beta_n} \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx$	κ
1	0.8491255877133055	0.6180634559349288	0.2310621317783767
2	1.016910182186539	0.7907991368271019	0.2261110453594368
5	1.415460253243194	1.188199380524827	0.2272608727183667
10	1.862037026152716	1.635594254035353	0.2264427721173636
20	2.491976409618374	2.265497102260039	0.2264793073583347
50	3.703731382087362	3.477237747557407	0.2264936345299549
100	5.012945106156195	4.786446149442159	0.2264989567140363
200	6.783473208118115	6.556977015888961	0.2264961922291544
500	10.09256633814676	9.866065338606688	0.2265009995400753
1000	13.59354960534267	13.36705097438357	0.2264986309591066
10000	35.84815107795398	35.62158882912692	0.2265622488270580
100000	91.88389679603820	91.65736391453106	0.2265328815071399

Table 2. Numerical study of Eq. (13).

x_0	$-\int_{x_0}^{+\infty} \text{Ai}^2(x) \log \text{Ai}^2(x) dx$	$\int_{x_0}^0 \overline{\text{Ai}}^2(x) \log \overline{\text{Ai}}^2(x) dx$	κ
-1	0.5222161751978991	0.2961890404907877	0.2260271347071114
-10	2.077338083778954	1.850652129535547	0.2266859542434068
-100	9.397399253933362	9.170902650261821	0.2264966036715403
-200	14.69392229097821	14.46742682065904	0.2264954703191755
-500	26.27530667657610	26.04880146831822	0.2265052082578833
-1000	40.49014289623582	40.26363079325318	0.2265121029826460

Table 3. Numerical analysis of the hypothesis $\sigma(q) = \frac{q-2}{3}$

q	$\sigma(q)$	$(q-2)/3$
0.5	-0.4990839447495011	-0.5
0.9	-0.3659978669311457	-0.366666
0.95	-0.3493617633586690	-0.35
0.99	-0.3360528305771765	-0.336666
0.995	-0.3343892108870942	-0.335
0.999	-0.3330583146412395	-0.333666
1.0	-0.3327255905122667	-0.333333
1.001	-0.3323928663558781	-0.333
1.005	-0.3310619694535155	-0.331666
1.01	-0.3293983477085469	-0.33
1.05	-0.3160893490880808	-0.316666
1.1	-0.2994530387476477	-0.3
1.5	-0.1663586884534043	-0.166666

Table 4. Numerical study of Eq. (14).

n	$-\int_{\beta_n}^0 \overline{\text{Ai}}^2(x) \ln \overline{\text{Ai}}^2(x) dx$	κ'
1	0.6180634559349288	0.2721346681876156
2	0.7907991368271019	0.2316547650211527
5	1.188199380524827	0.2081046520126155
10	1.635594254035353	0.1899098287874792
20	2.265497102260039	0.1784040275788774
50	3.477237747557407	0.1680146216219147
100	4.786446149442159	0.1628399946896453
200	6.556977015888961	0.1592838456040866
500	9.866065338606688	0.1562518872093719
1000	13.36705097438357	0.1548086738545216
10000	35.62158882912692	0.1526412373166366
100000	91.65736391453106	0.1520980222152417
1000000	229.8043910824926	0.1524238466430177

Table 5. Numerical study of Eq. (15).

n	S_n^{Ai}	$\kappa + \kappa'$
1	0.8491255877133055	0.5031967999659922
2	1.016910182186540	0.4577658103805904
5	1.415460253243193	0.4353655247309812
10	1.862037026141317	0.4163526008934437
20	2.491976409144777	0.4048833344636149
50	3.703731017196829	0.3945078912613367
100	5.012945102393711	0.3893389476411979
200	6.783473041810359	0.3857798715254850
500	10.09256605054762	0.3827525991503027
1000	13.59354798908833	0.3813056885592871
10000	35.84808970400448	0.3791421121941951
100000	91.88385127364121	0.3785853813253937
1000000	230.0304243971540	0.3784571613044250
10000000	565.1890275970421	0.3785182856238407

Table 6. Numerical study of Eq. (16).

n	$S(\Psi_n)$	$\kappa + \kappa'$
1	1.710183690538731	0.4487154478408955
2	1.914670354838653	0.4278693661686791
5	2.294432719271654	0.4133267764578933
10	2.625943420152888	0.4015923330453082
20	2.990723472091418	0.3940847941662819
50	3.507127798799131	0.3874632791011496
100	3.916389626903620	0.3842961155825298
200	4.337317744851943	0.3822036132202163
500	4.907064452326643	0.3805121161752789
1000	5.345631824193879	0.3797471090953328
10000	6.832532675670682	0.3786967516781765
100000	8.345340678998884	0.3784637933050010
1000000	9.870081112950714	0.3784129459145020