Orthogonal matrix polynomials, scalar type Rodrigues' formulas and Pearson equations *

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Abstract

Some families of orthogonal matrix polynomials satisfying second order differential equations with coefficients independent of n have recently been introduced (see [DG1]). An important difference with the scalar classical families of Jacobi, Laguerre and Hermite, is that these matrix families do not satisfy scalar type Rodrigues' formulas of the type $(\Phi^n W)^{(n)} W^{-1}$, where Φ is a matrix polynomial of degree not bigger than 2. An example of a modified Rodrigues' formula, well suited to the matrix case, appears in [DG1].

In this note, we discuss some of the reasons why a second order differential equation with coefficients independent of n does not imply, in the matrix case, a scalar type Rodrigues' formula and show that scalar type Rodrigues' formulas are most likely not going to play in the matrix valued case the important role they played in the scalar valued case. We also mention the roles of a scalar type Pearson equation as well as that of a non-commutative version of it.

1 Introduction

A large class of families of orthonormal matrix polynomials $(P_n)_n$, satisfying second order differential equations of the form

$$P_n''(t)A_2(t) + P_n'(t)A_1(t) + P_n(t)A_0 = \Gamma_n P_n(t) \tag{1.1}$$

has recently been introduced in [DG1].

*The work of the first author is partially supported by D.G.E.S, ref. BFM2000-206-C04-02, FQM-262 (*Junta de Andalucía*), NATO COLLABORATIVE GRANT PST.CLG.979738, that of the second author is partially supported by NSF grants FD9971151 and 1-443964-21160.

Here A_2, A_1 and A_0 are matrix polynomials (which do not depend on n) of degrees less than or equal to 2, 1 and 0, respectively, and Γ_n are Hermitian matrices. As usual, the orthogonality of these families is with respect to a weight matrix W:

Definition 1.1. *We say that an $N \times N$ matrix of measures supported in the real line is a (positive definite) weight matrix if*

1. *$W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$;*
2. *W has finite moments of every order, and*
3. *$\int P(t)dW(t)P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular.*

Condition (3) is necessary and sufficient to guarantee the existence of a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W, P_n of degree n and with nonsingular leading coefficient. Throughout this paper, we always consider weight matrices W having a smooth absolutely continuous derivative W' with respect to Lebesgue measure; assuming that this matrix derivative W' is positive definite at infinitely many real numbers, condition (3) above holds automatically. For other basic definitions and results on matrix orthogonality, see for instance [Be, D2, D1, DP, Ge, K1, K2].

When working with orthogonal matrix polynomials an important concept is that of scalar reducibility: we say that W *reduces to scalar weights* if there exists a nonsingular matrix T for which

$$W(t) = TD(t)T^*, \tag{1.2}$$

with $D(t)$ diagonal.

It is clear that the most interesting matrix examples are those non reducible to scalar weights. In other words, an equivalence relation can be defined for weight matrices: W_1 is similar to W_2 if there exists a nonsingular matrix T (independent of t) such that $W_1 = TW_2T^*$. Weight matrices reducible to scalar weights are, precisely, those corresponding to the class of diagonal weights. Diagonal weights, as a collection of N scalar weights, belong to the study of scalar orthogonality more than to the matrix one. We observe, however, that in [GPT] one finds a notion of similarity for the *pair* consisting of the weight and the differential operator. This notion allows one to distinguish among certain situations that are considered equivalent under the present definition. See example 5.1 in [GPT].

The following example, which it is taken from [DG1], does not reduce to scalar weights, and its sequence of orthonormal matrix polynomials satisfies a second order differential equation as (1.1):

$$W(t) = e^{-t^2} \begin{pmatrix} 1 + a^2t^2 & at \\ at & 1 \end{pmatrix}. \tag{1.3}$$

There is an important difference with respect to the scalar classical families of Jacobi, Laguerre and Hermite: the families introduced in [DG1] do not need to satisfy Rodrigues' formulas of the type

$$P_n(t) = C_n(\Phi^n W)^{(n)} W^{-1}, \quad n \geq 0, \tag{1.4}$$

where Φ is a matrix polynomial of degree not bigger than 2 and C_n, $n \geq 0$, are nonsingular matrices. Instead, the sequence $(P_n)_n$ is going to satisfy some modified Rodrigues' formula; for instance, the expression

$$P_n(t) = \left[e^{-t^2} \left(F(t) + \begin{pmatrix} a^2n/2 & 0 \\ 0 & 0 \end{pmatrix} \right) \right]^{(n)} e^{t^2} F^{-1}(t), \tag{1.5}$$

where F is the matrix polynomial

$$F(t) = \begin{pmatrix} 1 + a^2t^2 & at \\ at & 1 \end{pmatrix},$$

defines a sequence of orthogonal matrix polynomials with respect to the weight matrix (1.3). The example above is already given in [DG1]. For other structural formulas satisfied by this sequence of orthogonal polynomials see [DG2].
One of the purposes of this note is to show that the scalar type Rodrigues' formula (1.4) is (most likely) not going to play in the matrix valued case the important role they played in the scalar valued case (at least when Φ is an scalar polynomial); instead, Rodrigues' formulas like (1.5) are likely going to be more useful.
By setting $n = 1$, the scalar type Rodrigues' formula gives the well-known Pearson equation:

$$(\Phi W)' = \Psi W, \tag{1.6}$$

where Ψ is a matrix polynomial of degree just 1 (the first orthogonal polynomial with respect to W). In the scalar case the converse is also true; moreover the Pearson equation for the weight w is equivalent to the fact that any sequence $(p_n)_n$ of orthogonal polynomials with respect to w satisfies a second order differential equation

$$\phi p_n'' + \psi p_n' = \alpha_n p_n, \quad n \geq 0,$$

where ψ is a polynomial of degree 1 which does not depend on n. Notice that the polynomial ϕ, which appears in the Pearson equation for w, is also the coefficient of the second derivative of p_n.
To prove that a Pearson equation like (1.6) for the weight matrix W implies a scalar type Rodrigues' formula for its sequence of orthogonal matrix polynomials, some commutativity conditions among Φ, Φ', Φ'' and Ψ seems to be needed. We therefore assume that Φ is a scalar polynomial.
When the coefficients of Ψ commute with each other, the Pearson equation (1.6) can be explicitly integrated to get some examples of weight matrices W having orthogonal matrix polynomials satisfying a scalar type Rodrigues' formula. Unfortunately, all these weight matrices reduce to scalar weights (see

(1.2)). We also include an example where we integrate the Pearson equation when the coefficients of Ψ do not commute, although once again this example reduces to escalar weights. However, something more interesting can be done by considering a weaker condition than that of the positive definiteness of the weight matrix ((1) of Definition 1.1): in doing that we get some examples of orthogonal matrix polynomials which are relatives of the classical Bessel scalar polynomials.

Definition 1.2. *We say that a $N \times N$ matrix of measures W supported in the real line is a (Hermitian) weight matrix if*

1. *$W(A)$ is Hermitian for any Borel set $A \subset \mathbb{R}$;*
2. *W has finite moments of every order, and*
3. *$\int P(t)dW(t)P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular.*

Hermitian weight matrices are the analogs of the signed measures of the scalar case.

We prove, in Section 2, the equivalence between the Rodrigues' formula (1.4) for the orthogonal matrix polynomials $(P_n)_n$ and the Pearson equation for the weight matrix W, under the hypothesis that $\Phi(t) = \phi(t)I$, where ϕ is an scalar polynomial of degree not bigger than 2.

The Pearson equation for W trivially implies the following second order differential equation:

$$(\phi W)'' - (\Psi W)' = \theta.$$

Here θ denotes the null matrix.

According to [D2], this second order differential equation implies a second order differential equation for the orthogonal matrix polynomials with respect to W:

$$P_n''(t)\phi(t) + P_n'(t)\Psi(t) = \Gamma_n P_n(t).$$

It is worth noticing that the differential equation (1.1) satisfied by the orthogonal polynomials in the case of our example (1.3) is slightly different to the one above. In fact the coefficient A_0 which appears in (1.1) is essential to guarantee that the weight matrix (1.3) does not reduce to the scalar case.

This allows us to understand why, in the matrix valued case, satisfying a scalar type Rodrigues' formula is no longer equivalent to satisfying a second order differential equation like (1.1). Indeed, in [DG1], it is proved that the orthonormal matrix polynomials $(P_n)_n$ with respect to W satisfy a second order differential equation as (1.1) if and only if $A_2W = WA_2^*$ and

$$(A_2(t)W(t))'' - (A_1(t)W(t))' + A_0W(t) = W(t)A_0^*, \tag{1.7}$$

as well as the extra condition that W satisfies the boundary conditions that

$$A_2(t)W(t) \text{ and } (A_2(t)W(t))' - A_1(t)W(t), \tag{1.8}$$

should have vanishing limits at each of the endpoints of the support of $W(t)$. These conditions on W implies that certain noncommutative Pearson equation has to be satisfied by the weight matrix W:

$$2(A_2(t)W(t))' = A_1(t)W(t) + W(t)A_1^*(t). \tag{1.9}$$

We stress that the equation (1.9) does not imply the stronger one (1.7). In the scalar case both equations (1.7) and (1.9) are equivalent (the second one being the Pearson equation). The noncommutativity of the matrix product implies that, in general, equation (1.9) also differs from the scalar type Pearson equation (1.6). Taking this into account, it is rather understandable that for orthogonal matrix polynomials the second order differential equation (such as (1.1)) does not imply scalar type Rodrigues' formula (such as (1.4)).
In Section 3, we integrate the Pearson equation and show that assuming W to be positive definite, all the examples reduce to the scalar case. In Section 4 and 5, we show however some generic examples of hermitian weight matrices satisfying a Pearson equation as in (1.6) which do not reduce to scalar weights. Structural properties for the families introduced at Section 4 and 5 can be derived as in the case of the classical scalar families (so that we do not include them here).

2 Pearson matrix equation and Rodrigues formula

As we mentioned in the introduction, the scalar type Rodrigues' formula (1.4) for the orthogonal matrix polynomials $(P_n)_n$ with respect to the weight matrix W automatically implies the Pearson equation (1.6) for the weight matrix W: then, $\Psi = P_1$, the first orthogonal matrix polynomials with respect to W; this means that the leading coefficient of Ψ has to be non singular. To prove the converse, some commutativity conditions among Φ, Φ', Φ'' and Ψ seem to be needed. We assume, as mentioned earlier, that $\Phi(t)$ is an scalar polynomial.

Theorem 2.1. *Let W be a weight matrix satisfying the Pearson equation*

$$(\phi(t)W)' = \Psi(t)W(t),$$

where $\phi(t)$ is a scalar polynomial of degree not bigger that 2 and Ψ a matrix polynomial of degree 1 with non singular leading coefficient. We assume that the weight matrix W also satisfies the boundary conditions that $\phi(t)W(t)$ has vanishing limits at each of the endpoints of the support of $W(t)$. If the degree of ϕ is 2 we assume, in addition, that its roots are different (just to avoid the analogs of the Bessel polynomials) and that the spectrum of the leading coefficient of Ψ is disjoint with the set of natural numbers $\mathbb{N}$. Then

$$P_n(t) = (\phi^n(t)W(t))^{(n)}W^{-1}(t)$$

is a sequence of matrix polynomials of degree n with non singular leading coefficients. Moreover, they are orthogonal with respect to W.

Proof. The orthogonality of the sequence follows easily using integration by parts. By a suitable linear change of variable we can assume that ϕ is equal to either $1, t$, or $1-t^2$ when its degree is 0, 1 or 2. We write $\Psi(t) = At + B$ with A non singular.
For $\phi = 1$, we can prove the result by using the formula

$$W^{(n)}W^{-1} = (W')^{(n-1)}W^{-1} = (\Psi W)^{(n-1)}W^{-1} = (\Psi W^{(n-1)} + (n-1)AW^{(n-2)})W^{-1}$$

and complete induction on n.
For $\phi = t$, we use the formula

$$\begin{aligned}
(t^kW)^{(n)} &= ((t^{k-1}tW)')^{(n-1)}W^{-1} \\
&= (k-1)(t^{k-1}W)^{(n-1)}W^{-1} + (t^{k-1}\Psi W)^{n-1}W^{-1} \\
&= [(k-1)(t^{k-1}W)^{(n-1)} + (At^kW)^{(n-1)} + (Bt^{k-1}W)^{(n-1)}]W^{-1} \\
&= [((k-1)I + B)(t^{k-1}W)^{(n-1)} + A(t^kW)^{(n-1)}]W^{-1}
\end{aligned}$$

and induction on n to prove that $(t^kW)^{(n)}W^{-1}$ is a polynomial of degree k with nonsingular leading coefficient. The result then is just the case $k = n$.
For $\phi = 1 - t^2$, we use the formula

$$\begin{aligned}
((1-t^2)^kW)^{(n)} &= (((1-t^2)^{k-1}(1-t^2)W)')^{(n-1)}W^{-1} \\
&= [-2(k-1)(t(1-t^2)^{k-1}W)^{(n-1)} + ((1-t^2)^{k-1}\Psi W)^{n-1}]W^{-1} \\
&= [(-2(k-1)I + A)t(1-t^2)^{k-1}W)^{(n-1)} \\
&\quad +(B(1-t^2)^{k-1}W)^{(n-1)}]W^{-1} \\
&= \Big[(-2(k-1)I + A)\Big(t[(1-t^2)^{k-1}W]^{(n-1)} + (n-1)[(1-t^2)^{k-1}W]^{(n-2)}\Big) \\
&\quad +B[(1-t^2)^{k-1}W]^{(n-1)}]W^{-1}
\end{aligned}$$

and induction on n to prove that $[(1-t^2)^kW]^{(n)}W^{-1}$, $k \geq n$, is a polynomial of degree $2k - n$ with leading coefficient equal to

$$A_{k,n} = (-1)^{k+n}(A - (2k-2)I)(A - (2k-3)I)\cdots(A - (2k-(n+1))I).$$

The result follows now easily. $\square$

3 Integrating the Pearson equation

In this section, we explicitly integrate the Pearson equation for the canonical values $\phi = 1$, $\phi(t) = t$ and $\phi(t) = (1-t^2)$. This can be done easily as soon as we assume that the coefficients of the polynomial Ψ commute. Otherwise the integration of this first order matrix equation is not straightforward. Anyway, even in the case that the coefficients of Ψ do not commute, we conjecture that a weight matrix satisfying (1.6) will reduce to scalar weights; in fact, we include, at the end of this section, an example of this kind.

1. When $\phi = 1$, we can write the Pearson equation (1.6) as

$$W'(t) = (2(B-I)t + A)W(t),$$

which it can be solved explicitly when A and B commute to get:

$$W(t) = e^{-t^2} e^{Bt^2+At} C.$$

To avoid any integrability problem of W at ∞, the real part of the eigenvalues of B have to be less than 1.

2. When $\phi = t$, we can write the Pearson equation (1.6) as

$$W'(t) = \left((A - I) + \frac{B + \alpha I}{t} \right) W(t),$$

which it can be solved explicitly when A and B commute to get:

$$W(t) = t^\alpha e^{-t} e^{At} t^B C.$$

To avoid any integrability problem of W at ∞ and at 0, the (real part of the) eigenvalues of A have to be less than 1 and the (real part of the) eigenvalues of B greater than $-\alpha - 1$, respectively.

3. When $\phi = (1 - t)(1 + t)$, we can write the Pearson equation (1.6) as

$$W'(t) = \left(\frac{A + \alpha I}{1 + t} - \frac{B + \alpha I}{1 - t} \right) W(t),$$

which it can be solved explicitly when A and B commute to get:

$$W(t) = (1 + t)^\alpha (1 - t)^\beta (1 + t)^A (1 - t)^B C.$$

To avoid any integrability problem of W at ± 1, the (real part of the) eigenvalues of A have to be greater than $-\alpha - 1$ and the (real part of the) eigenvalues of B greater than $-\beta - 1$, respectively.

Since the weight matrix W has to be Hermitian, in all the cases we have to impose, in addition to $AB = BA$, the conditions $BC = CB^*$ and $AC = CA^*$. Unfortunately, when C is positive definite (that is W is a positive definite weight matrix), W reduces, in all the cases, to scalar weights (see 1.2). We prove it for $\phi(t) = 1$ (the rest of the cases can be proved analogously). Taking into account the conditions on the matrices A, B and C we can write:

$$\begin{aligned} W(t) &= e^{-t^2} e^{Bt^2+At} C \\ &= e^{-t^2} C^{1/2} e^{C^{-1/2}(Bt^2+At)C^{1/2}} C^{1/2}, \end{aligned}$$

where, $C^{-1/2}BC^{1/2}$ and $C^{-1/2}AC^{1/2}$ are now Hermitian commuting matrices; we can then take an unitary matrix U which simultaneous diagonalizes both matrices. Then, the weight can be written as

$$W(t) = e^{-t^2} C^{1/2} U e^{D_1 t^2 + D_2 t} U^* C^{1/2},$$

with D_1 and D_2 diagonal matrix: that is, W reduces to scalar weights. This is the case of many examples of orthogonal matrix polynomials which can be found in the literature ([CMV], [J1], for instance).
We complete this section integrating a case of Pearson equation where the coefficients of the polynomial Ψ do not commute:
We consider the Pearson equation

$$W'(t) = \left(\frac{A}{t} + \frac{B}{t-1}\right) W(t), \tag{3.1}$$

where the matrices A and B are given by

$$A = \frac{1}{2}\begin{pmatrix} 1-u & u \\ 1-u & u \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} u & -u \\ u-2 & -u+2 \end{pmatrix}, \quad u \in \mathbb{R}.$$

(for convenience, we consider here $\phi = t(1-t)$ instead of $(1-t^2)$, although the example can be easily transformed in one corresponding to $(1-t^2)$).
Although the matrices A and B do not commute, we can integrate the Pearson equation (3.1) to get the solutions:

$$W(t) = \left[\begin{pmatrix} 2u & -2u \\ -2+2u & -2u+2 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 1-2u & 2u \\ 1-2u & 2u \end{pmatrix} t^{1/2} + \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} t\right] C.$$

If we look for a positive definite weight matrix W, a straightforward computation gives that C has to be of the form

$$C = \begin{pmatrix} a & a \\ a & b \end{pmatrix}, \quad b > a,$$

and necessarily $u = 0$.
This gives for W the expression

$$W(t) = \begin{pmatrix} \sqrt{2}at^{1/2} & \sqrt{2}at^{1/2} \\ \sqrt{2}at^{1/2} & \sqrt{2}at^{1/2} + 2(1-t)(b-a) \end{pmatrix},$$

which can be factorized as

$$W(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2(b-a)(1-t) & 0 \\ 0 & \sqrt{2}at^{1/2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix};$$

This shows that W reduces to scalar weights.
This example is very illustrative of what happens in the matrix case: there is a convenient choice of a positive definite matrix D so that the weight matrix

$$F(t) = W(t)DW^*(t)$$

is actually a positive definite matrix polynomial of degree 2. For this precise matrix D, the weight $t^\alpha(1-t)^\beta F$ satisfies not only a noncommutative Pearson equation like that of (1.9) (this happens for any choice of D) but more importantly also satisfies a second order differential equation as that of (1.7). As a consequence the sequence of orthogonal matrix polynomials with respect to $t^\alpha(1-t)^\beta F$ satisfies a second order differential equation of the type (1.1). This weight matrix F does not reduce to scalar weights and its corresponds with the example 5.2 of [GPT].

4 Examples with A or B nilpotent.

The construction above breaks down if C is Hermitian but not positive definite because then C does not have a Hermitian square root. In what follows we implicitly assume that C is non singular (otherwise condition (3) of Definition 1.2 is not fulfilled).

A number of consequences flow from the algebraic conditions imposed on A, B, C by the fact that W is Hermitian. For instance, if A or B are nilpotent, then C can not be positive definite. Indeed, if for instance A is nilpotent of order k and $AC = CA^*$, it follows multiplying by A^{k-1} on the left and by $(A^*)^{k-2}$ on the right that $\theta = A^{k-1}C(A^*)^{k-1}$; since $A^{k-1} \neq \theta$, we deduce that C can not be positive definite.

We show now some examples of this kind.

Take A and B the nilpotent matrices

$$A = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix},$$

The general expression for a Hermitian matrix C such that $AC = CA^*$ and $BC = CB^*$ is

$$C = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}.$$

1. When $\phi = 1$, this gives for W the form

$$W(t) = e^{-t^2} \begin{pmatrix} 1 & ut \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & vt^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} = e^{-t^2} \begin{pmatrix} a + but + bvt^2 & b \\ b & 0 \end{pmatrix}.$$

   A particular case of this example ($a = v = 0, b = u = 1$) can be found in [CMV].

2. When $\phi = t$, this gives

$$W(t) = t^{\alpha} e^{-t} \begin{pmatrix} a + but + bv \log t & b \\ b & 0 \end{pmatrix}.$$

3. Finally the case $\phi = (1-t)(1+t)$ gives the Hermitian weight matrix

$$W(t) = (1+t)^{\alpha}(1-t)^{\beta} \begin{pmatrix} a + bu \log(1+t) + bv \log(1-t) & b \\ b & 0 \end{pmatrix}.$$

5 Examples with A or B square root of a negative semidefinite matrix.

If A or B are a square root of a negative semidefinite matrix, it follows easily that a matrix C such that $AC = CA^*$ (or $BC = CB^*$) can not be positive definite.

Actually we can consider only upper triangular square roots of aI, $a \leq 0$.
Indeed, take an orthonormal basis for which B^2 is diagonal with real entries and B is upper triangular. We prove by induction on N that then $B^2 = aI$, for certain $a \leq 0$.
Indeed, for $N = 2$, it follows straightforwardly that under our hypothesis

$$B = \begin{pmatrix} a_{11} & a_{12} \\ 0 & -a_{11} \end{pmatrix}, \tag{5.1}$$

so that $B^2 = a_{11}^2 I$.
Let us assume now that B has size $N+1$. We split up the matrix B in blocks

$$B = \begin{pmatrix} \tilde{B} & v \\ \theta & b \end{pmatrix},$$

where $\tilde{B}$ is an upper triangular matrix of size $N \times N$, v is a column vector of $\mathbb{C}^N$ and $b \in \mathbb{C}$. Then

$$B^2 = \begin{pmatrix} \tilde{B}^2 & (\tilde{B} + aId.)v \\ \theta & b^2 \end{pmatrix}$$

By the induction hypothesis, $\tilde{B}^2 = aI$, for certain $a \leq 0$. This shows that the eigenvalues of $\tilde{B}$ are $\pm\sqrt{a}$. Since B^2 is diagonal, we deduce that $-b$ is an eigenvalue of $\tilde{B}$, and then also $B^2 = aI$.
All the upper triangular square roots of aI, $a \leq 0$ can be generated recursively. The case $N = 2$ has been already found (see (5.1) above).
For $N = 3$, we look for upper triangular matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & -a_{11} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix};$$

this gives two equations: the product of the second and first row (respectively) by the last column (in the general case of size N, we have $N-1$ equations: the product of the k'th rows, $k = N-1, \cdots, 1$, by the last column):

$$\begin{aligned} a_{23}(a_{33} - a_{11}) &= 0, \\ a_{13}(a_{11} + a_{33}) + a_{12}a_{23} &= 0. \end{aligned}$$

These equations (as well as those of the general case of size N) can be easily solved; in doing so, we find three different solutions (which can not be reduced, in general, to lower size):

- $a_{23} = 0$, $a_{33} = -a_{11}$:
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & -a_{11} & 0 \\ 0 & 0 & -a_{11} \end{pmatrix}.$$

- $a_{33} = a_{11} \neq 0$:
$$\begin{pmatrix} a_{11} & a_{12} & a_{12}a_{23}/(2a_{11}) \\ 0 & -a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}.$$
- $a_{11} = a_{12} = 0$:
$$\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding Hermitian weight matrices can be computed by using that

$$x^A = (\cos(\sqrt{-a}\log x) - 1)I + \frac{A}{\sqrt{-a}}\sin(\sqrt{-a}\log x), \quad a < 0$$

and

$$x^A = I + A\log x, \quad a = 0.$$

The examples $e^{-t}t^AC$ are especially interesting. Indeed, from [DG1] it follows that the (positive definite) weight matrix

$$e^{-t}t^ABt^{A^*},$$

with $A^2 = aI$, $a \leq 0$, and B positive definite, satisfies the second order differential equation:

$$(tW)''(t) + \big[(tI - 2A - I)W(t)\big]' - AW(t) = -W(t)A^*. \tag{5.2}$$

However, W does not satisfy the boundary conditions (1.8). Indeed, a simple calculation gives

$$(tW)' - A_1W = e^{-t}t^A(BA^* - AB)t^{A^*}.$$

The limit of this expression as t tends to 0^+ is θ if and only if $AB = BA^*$. That is not possible when B is positive definite but, as discussed above there exists B hermitian, so that $AB = BA^*$. For such a B the weight matrix $e^{-t}t^ABt^{A^*}$ reduces to $e^{-t}t^{2A}B$, that is, it is of the form considered above.
Since the weight matrix $W(t) = e^{-t}t^ABt^{A^*}$, B positive definite, does not satisfy the boundary conditions, the monic orthogonal matrix polynomials with respect to W do not satisfy the corresponding second order differential equation

$$tP_n''(t) + P_n'(t)(2A + I - tI) - P_n(t)A = nP_n(t).$$

But taking B Hermitian with $AB = BA^*$, we have that the monic orthogonal matrix polynomials with respect to $e^{-t}t^ABt^{A^*}$ now satisfy the second order differential equation

$$tP_n''(t) + P_n'(t)(2A + I - tI) = -nP_n(t).$$

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