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SQUEEZABLE ORTHOGONAL BASES: ACCURACY AND SMOOTHNESS*

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Abstract. We present a method for generating local orthogonal bases on arbitrary partitions of $\mathbf{R}$ from a given local orthogonal shift-invariant basis via what we call a *squeeze map*. We give necessary and sufficient conditions for a squeeze map to generate a nonuniform basis that preserves any smoothness and/or accuracy (polynomial reproduction) of the shift-invariant basis. When the shift-invariant basis has sufficient smoothness or accuracy, there is a unique squeeze map associated with a given partition that preserves this property and, in this case, the squeeze map may be calculated locally in terms of the ratios of adjacent intervals. If both the smoothness and accuracy are large enough, then the resulting nonuniform space contains the nonuniform spline space characterized by that smoothness and accuracy.

Our examples include a multiresolution on nonuniform partitions such that each space has a local orthogonal basis consisting of continuous piecewise quadratic functions. We also construct a family of smooth, local, orthogonal, piecewise polynomial generators with arbitrary approximation order.

Key words. orthogonal bases, nonuniform grids, polynomial reproduction, piecewise polynomial, multiresolution

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1. Introduction.

Finitely generated shift-invariant (FSI) spaces naturally arise in several areas of numerical analysis and approximation theory, including the theory of splines and wavelets. A major advantage of an FSI space is the existence of a convenient basis generated by a (usually) small number of functions. When the basis is local and orthogonal the process of finding the orthogonal projection Pf of $f \in L^2(\mathbf{R})$ onto the space is local so that changing f on a compact interval affects only Pf on a slightly larger interval.

In this paper we introduce and investigate a method for adapting local shift-invariant bases to nonuniform partitions via what we call a *squeeze map*. When the shift-invariant basis is orthogonal, the squeeze map may be chosen so that the nonuniform basis is also orthogonal.

The notion of squeeze maps generalizes ideas introduced in [4], where we gave examples of local orthogonal piecewise polynomial shift-invariant bases that are easily adaptable to arbitrary grids in $\mathbf{R}$. The focus of this paper is on characterizing when a squeeze map generates a nonuniform basis preserving any smoothness and/or accuracy (polynomial reproduction) of the shift-invariant basis. When the shift-invariant basis has sufficient smoothness or accuracy, there is a unique squeeze map associated with a given partition of $\mathbf{R}$ that preserves this property and, in this case, the squeeze map may be calculated locally in terms of the ratios of adjacent intervals. When both the smoothness and accuracy are large enough, we find that the resulting nonuniform

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space contains the nonuniform spline space characterized by that smoothness and accuracy.

Two applications that provide motivation for our work are adaptive least squares and the construction of orthogonal wavelets on semiregular and irregular families of grids:

(1) Since the bases constructed here are local and orthogonal and depend locally on the given grid, it is relatively easy to calculate changes in the orthogonal projection of a given function (onto the span of this basis) resulting from changes in the grid, making them well suited for adaptive least square problems.

(2) While we do not focus on refinable spaces in this paper, it is the refinable case that provides the main motivation for our study. We remark that our methods provide a means to adapt a multiresolution on uniform grids to one on a semiuniform family of grids (that is, an arbitrary coarse grid that is uniformly subdivided). In the example in section 6.3, we start with Daubechies's famous orthogonal scaling function ${}_2\phi$. We find that, given a nonuniform grid, there is a unique squeeze map that preserves the accuracy of the space. In the example in section 6.4, we use ideas from [5] to construct a multiresolution on an arbitrary nonuniform subdivision. (The only requirement is that each interval is subdivided into two subintervals.) Each space has a local orthogonal basis consisting of continuous piecewise quadratic functions.

Finally, in section 7 we construct a family of smooth, local, orthogonal, piecewise polynomial generators with arbitrary approximation order using techniques developed in [6]. These generators have fewer components than the corresponding refinable generators constructed in [6], and so we prefer them when refinability is not required. We mention that a possible application of this family is to code division multiple access (CDMA) technology, where several users share a single channel using orthogonal decompositions.

1.1. Shift-invariant spaces. We call a compactly supported, finite-length (column) vector

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \in L^2(\mathbf{R})^n$$

a *generator*. Note that when it is clear from the context, we also consider a generator Φ to be the set of its components; that is, we also consider $\Phi \subset L^2(\mathbf{R})$. When we refer to the span of Φ we mean the subspace of $L^2(\mathbf{R})$ spanned by the components of Φ.

For a generator Φ, let

$$B(\Phi) := \{\phi_i(\cdot - j) \mid j \in \mathbf{Z}, i = 1, \dots, n\}.$$

If $B(\Phi)$ is an orthogonal set, we say Φ is an *orthogonal generator*. For a generator Φ, let

$$S(\Phi) := \left\{ \sum_{j\in\mathbf{Z}} c(j)^\top \Phi(\cdot - j) \mid c(j) \in \mathbf{R}^n, j \in \mathbf{Z} \right\}.$$

If $V = S(\Phi)$ for some generator Φ, then V is called a *finitely generated shift-invariant* (FSI) space.

1.2. Minimally supported generators. Our procedure for constructing local bases on nonuniform partitions starts with generators supported on $[-1,1]$ satisfying a local linear independence condition on $[0,1]$. In particular, for $k \le n$, we say that a generator

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_k \\ \phi_{k+1} \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \bar{\Phi} \\ \breve{\Phi} \end{pmatrix}$$

(where $\bar{\Phi}$ consists of the first k elements of Φ and $\breve{\Phi}$ consists of the last $n-k$) is a *minimally supported k-generator* (or just *minimally supported*) if

(1) supp $\Phi \subset [-1,1]$;
(2) supp $\breve{\Phi} \subset [0,1]$;
(3) the collection $\breve{\Phi} \cup \bar{\Phi}\chi_{[0,1]} \cup (\bar{\Phi}(\cdot - 1))\chi_{[0,1]}$ is linearly independent.

We denote the collection of all minimally supported k-generators with n components by $\mathcal{G}_k^n$. See section 5 for several illustrative examples of orthogonal minimally supported generators. The notion of generators minimally supported on $[-1,1]$ played a central role in the construction of orthogonal, smooth, piecewise polynomial wavelets given in [5].

For $\Phi \in \mathcal{G}_k^n$, we denote the "left" and "right" pieces of $\bar{\Phi}$ by

$$\Phi_R := \bar{\Phi}\chi_{[0,1]} \text{ and } \Phi_L := \bar{\Phi}\chi_{[-1,0)}.$$

Obviously, condition (3) can be rewritten as $\breve{\Phi} \cup \Phi_R \cup \Phi_L(\cdot - 1)$ is linearly independent. If Φ is minimally supported, then it follows from the local linear independence condition (3) above that $B(\Phi)$ is linearly independent; that is, any $f \in S(\Phi)$ has a unique representation of the form $f = \sum c_j \Phi(\cdot - j)$. In the remainder of this paper, when there is clearly some underlying minimally supported generator with k and n as above, then, for any (row or column) vector v of length n, we let $\bar{v}$ denote the subvector of the first k components of v and $\breve{v}$ the subvector of the last $n-k$ components of v.

Also, for $f, g \in L^2(\mathbf{R})^n$ we define $\langle f, g\rangle := \int_{\mathbf{R}} f(x)g(x)^\top dx \in \mathbf{R}^{n\times n}$, where $v^\top$ denotes the transpose of a (column) vector v.

2. Squeeze maps. Let $a = (a_j)_{j\in\mathbf{Z}}$ be a strictly increasing real-valued sequence with no accumulation point in $\mathbf{R}$, in which case we call a a *knot sequence*. Let $L_j := a_{j+1} - a_j$ denote the length of the jth interval $[a_j, a_{j+1}]$ and let $\tau_j = \tau_j^a$ be given by

$$\tau_j(x) = \begin{cases} (x-a_j)/L_{j-1} & \text{for } x \le a_j, \\ (x-a_j)/L_j & \text{for } x \ge a_j. \end{cases} \tag{2.1}$$

Then τ_j maps the points a_{j-1}, a_j, and a_{j+1} to -1, 0, and 1, respectively.

Suppose Φ is an orthogonal minimally supported generator. Consider

$$B_0 = \bigcup_{j\in\mathbf{Z}} \Phi \circ \tau_j.$$

If Φ is continuous and $k = 1$ (for example, see the example in section 6.1), then (because τ_j is affine on each "overlap" interval $[a_j, a_{j+1}]$ and continuous on $\mathbf{R}$) it follows that B_0 is a continuous orthogonal basis for its span.

On the other hand, if $\Phi \in C^1(\mathbf{R})$ and $\Phi'(0) \neq 0$ (for example, consider the continuously differentiable Φ with $k = 2$ in the example in section 6.2), then the components of B_0 are not in $C^1(\mathbf{R})$ for nonuniform a. In particular, $\bar{\Phi} \circ \tau_j$ is not differentiable at a_j unless $L_{j-1} = L_j$. This leads us to consider a more general construction in which linear combinations of $\bar{\Phi}_L \circ \tau_j$ are pieced together with linear combinations of $\bar{\Phi}_R \circ \tau_j$ via what we call a *squeeze map*.

More specifically, let $A_{\mathrm{L}}^{(j)}$ and $A_{\mathrm{R}}^{(j)}$ be invertible $k \times k$ matrices for $j \in Z$ and let $A_j : \mathbf{R} \to \mathbf{R}^{k\times k}$ denote the matrix-valued function on $\mathbf{R}$ defined by

$$A_j = \chi_{[-1,0)} A_{\mathrm{L}}^{(j)} + \chi_{[0,1]} A_{\mathrm{R}}^{(j)}.$$

Given A and a knot sequence a, we call the sequence of mappings $\sigma = (\sigma_j)_{j\in\mathbf{Z}}$, where $\sigma_j : \mathcal{G}_k^n \to L^2(\mathbf{R})^n$ is given by

$$\sigma_j(\Phi) = \begin{pmatrix} A_j \bar{\Phi} \circ \tau_j \\ \breve{\Phi} \circ \tau_j \end{pmatrix},$$

a *squeeze map (on $\mathcal{G}_k^n$)*.

As before, we let $\bar{\sigma}_j(\Phi)$ denote the vector of the first k components of $\sigma_j(\Phi)$ and $\breve{\sigma}_j(\Phi)$ the remaining $n - k$ components. Observe that

$$\bar{\sigma}_j(\Phi) = \left(\chi_{[-1,0)} A_{\mathrm{L}}^{(j)} \bar{\Phi} + \chi_{[0,1]} A_{\mathrm{R}}^{(j)} \bar{\Phi}\right) \circ \tau_j = \left(A_{\mathrm{L}}^{(j)} \Phi_L + A_{\mathrm{R}}^{(j)} \Phi_R\right) \circ \tau_j$$

and $\mathrm{supp}\bar{\sigma}_j(\Phi) \subset [a_{j-1}, a_{j+1}]$, while $\mathrm{supp}\breve{\sigma}_j(\Phi) \subset [a_j, a_{j+1}]$.

If σ is a squeeze map on $\mathcal{G}_k^n$ and $\Phi \in \mathcal{G}_k^n$, then we define

$$B_\sigma(\Phi) := \bigcup_{j\in\mathbf{Z}} \sigma_j(\Phi)$$

and

$$S_\sigma(\Phi) := \left\{ \sum_{j\in\mathbf{Z}} c(j)^\top \sigma_j(\Phi) \mid c(j) \in \mathbf{R}^n, j \in \mathbf{Z} \right\}.$$

The minimal support of Φ and the invertibility of $A_{\mathrm{L}}^{(j)}$ and $A_{\mathrm{R}}^{(j)}$ imply that $B_\sigma(\Phi)$ is linearly independent.

If σ is a squeeze map with matrix sequences $(A_{\mathrm{L}}^{(j)})$ and $(A_{\mathrm{R}}^{(j)})$, we define

$$R_j = R_j(\sigma) := (A_{\mathrm{L}}^{(j)})^{-1} A_{\mathrm{R}}^{(j)} \qquad (j \in \mathbf{Z}).$$

We say that two squeeze maps σ and ν on $\mathcal{G}_k^n$ are *equivalent* whenever $S_\sigma(\Phi) = S_\nu(\Phi)$ for any $\Phi \in \mathcal{G}_k^n$.

LEMMA 2.1. *Suppose σ and ν are squeeze maps on $\mathcal{G}_k^n$. Then σ and ν are equivalent if and only if*

$$R_j(\sigma) = R_j(\nu) \qquad (j \in \mathbf{Z}). \tag{2.2}$$

Proof. Suppose (2.2) holds. Then

$$\left(A_{\mathrm{L}}^{(j)}\right)^{-1}\bar{\sigma}_j(\Phi)=\left(\tilde{A_{\mathrm{L}}}^{(j)}\right)^{-1}\bar{\nu}_j(\Phi)\qquad(j\in\mathbf{Z}),$$

where σ has matrix sequences $(A_{\mathrm{L}}^{(j)})$ and $(A_{\mathrm{R}}^{(j)})$ and ν has matrix sequences $(\tilde{A_{\mathrm{L}}}^{(j)})$ and $(\tilde{A_{\mathrm{R}}}^{(j)})$. Since $A_{\mathrm{L}}^{(j)}$ and $\tilde{A_{\mathrm{L}}}^{(j)}$ are nonsingular the above shows that $\bar{\sigma}_j(\Phi)$ and $\bar{\nu}_j(\Phi)$ (considered as sets) have the same span. By definition $\breve{\sigma}_j(\Phi)$ and $\breve{\nu}_j(\Phi)$ have the same span showing that $\sigma_j(\Phi)$ and $\nu_j(\Phi)$ have the same span, and hence $S_\sigma(\Phi)=S_\nu(\Phi)$.

On the other hand, if $S_\sigma(\Phi)=S_\nu(\Phi)$, then the local linear independence of $B_\sigma(\Phi)$ and $B_\nu(\Phi)$ shows that $\bar{\sigma}_j(\Phi)$ and $\bar{\nu}_j(\Phi)$ have the same span for each $j\in\mathbf{Z}$. Thus, there must be some nonsingular matrix W_j such that

$$\bar{\nu}_j(\Phi)=W_j\bar{\sigma}_j(\Phi)\qquad(j\in\mathbf{Z}),$$

which implies that (2.2) holds. ▯

Our motivation for considering squeeze maps is that if Φ is a minimally supported orthogonal generator, then we can always find a local orthogonal basis for $S_\sigma(\Phi)\cap L^2(\mathbf{R})$. To see this, note that the elements of $\bar{\sigma}_j(\Phi)$ are orthogonal to the elements of $\bar{\sigma}_{j+1}(\Phi)$:

$$\langle\bar{\sigma}_j(\Phi),\bar{\sigma}_{j+1}(\Phi)\rangle=L_jA_{\mathrm{R}}^{(j)}\langle\Phi_R,\Phi_L(\cdot-1)\rangle\left(A_{\mathrm{L}}^{(j+1)}\right)^{\top}=0\qquad(j\in\mathbf{Z}).$$

It then follows that $\sigma_j(\Phi)$ is orthogonal to $\sigma_{j'}(\Phi)$ for any $j\neq j'\in\mathbf{Z}$. Finally, for each $j\in\mathbf{Z}$, we choose some orthogonal basis for the span of $\bar{\sigma}_j(\Phi)$ (for instance, by applying the Gram–Schmidt process to $\bar{\sigma}_j(\Phi)$). This change of basis corresponds to constructing a squeeze map ν equivalent to σ such that $B_\nu(\Phi)$ is an orthogonal set and is equivalent to performing the following matrix factorization: Let $B_jB_j^{\top}$ be a Cholesky factorization of $\langle\bar{\sigma}_j,\bar{\sigma}_j\rangle$, that is,

(2.3)
$$B_jB_j^{\top}=\langle\bar{\sigma}_j(\Phi),\bar{\sigma}_j(\Phi)\rangle=L_{j-1}A_{\mathrm{L}}^{(j)}\langle\Phi_L,\Phi_L\rangle\left(A_{\mathrm{L}}^{(j)}\right)^{\top}+L_jA_{\mathrm{R}}^{(j)}\langle\Phi_R,\Phi_R\rangle\left(A_{\mathrm{R}}^{(j)}\right)^{\top}.$$

Then ν with matrix sequences $(B_j^{-1}A_{\mathrm{L}}^{(j)})$ and $(B_j^{-1}A_{\mathrm{R}}^{(j)})$ is equivalent to σ, and $B_\nu(\Phi)$ is an orthogonal basis for $S_\sigma(\Phi)\cap L^2(\mathbf{R})$. Thus we have the following lemma.

Lemma 2.2. *Suppose Φ is a minimally supported orthogonal generator and σ is a squeeze map for Φ. Then there is some squeeze map ν equivalent to σ such that $B_\nu(\Phi)$ is an orthogonal basis for $S_\sigma(\Phi)\cap L^2(\mathbf{R})$.*

3. Polynomial reproduction and smoothness. In this section we give necessary and sufficient conditions for a squeeze map σ to preserve the accuracy (polynomial reproduction) and regularity of $S(\Phi)$. Throughout this section Φ is a generator in $\mathcal{G}_k^n$ and σ is a squeeze map on $\mathcal{G}_k^n$ with matrix sequences $(A_{\mathrm{L}}^{(j)})$ and $(A_{\mathrm{R}}^{(j)})$. Recall that $R_j=(A_{\mathrm{L}}^{(j)})^{-1}A_{\mathrm{R}}^{(j)}$ for $j\in\mathbf{Z}$.

First we address the smoothness of $S_\sigma(\Phi)$. Since $S_\sigma(\Phi)$ restricted to bounded intervals has finite dimension it follows that $S_\sigma(\Phi)\subset C^m(\mathbf{R})$ if and only if $\sigma_j(\Phi)\subset C^m(\mathbf{R})$ for all $j\in\mathbf{Z}$.

THEOREM 3.1. *Suppose $\Phi \subset C^m(\mathbf{R})$. Then, for $j \in \mathbf{Z}$, $\sigma_j(\Phi) \subset C^m(\mathbf{R})$ if and only if $\bar{\Phi}^{(q)}(0)$ is either 0 or a right eigenvector of R_j with eigenvalue $(L_j/L_{j-1})^q$ for $0 \le q \le m$, that is, if and only if*

$$R_j \bar{\Phi}^{(q)}(0) = (L_j/L_{j-1})^q \bar{\Phi}^{(q)}(0). \tag{3.1}$$

(Here $\bar{\Phi}^{(q)}$ denotes the qth derivative of $\bar{\Phi}$.)

Hence, $S_\sigma(\Phi) \subset C^m(\mathbf{R})$ if and only if (3.1) holds for all $j \in \mathbf{Z}$.

Proof. The theorem follows from

$$\sigma_j(\Phi)^{(q)}(j^-) = \begin{pmatrix} (L_{j-1})^{-q} A_{\mathrm{L}}^{(j)} \bar{\Phi}^{(q)}(0^-) \\ 0 \end{pmatrix}$$

and

$$\sigma_j(\Phi)^{(q)}(j^+) = \begin{pmatrix} (L_j)^{-q} A_{\mathrm{R}}^{(j)} \bar{\Phi}^{(q)}(0^+) \\ 0 \end{pmatrix}$$

for $0 \le q \le m$ and $j \in \mathbf{Z}$. ☐

Let Π_p, $p \ge 0$, denote the collection of univariate polynomials of degree at most p. A generator Φ is said to have *accuracy* $p+1$ if $\Pi_p \subset S(\Phi)$. If Φ has accuracy $p+1$, then (since $B(\Phi)$ is a linearly independent set), for each $l = 0, \ldots, p$, there is a unique sequence of $1 \times n$ vectors $(\alpha_l(j))_{j \in \mathbf{Z}}$ such that

$$x^l = \sum_{j \in \mathbf{Z}} \alpha_l(j) \Phi(x-j) = \sum_j \bar{\alpha}_l(j) \bar{\Phi}(x-j) + \breve{\alpha}_l(j) \breve{\Phi}(x-j). \tag{3.2}$$

We say $S_\sigma(\Phi)$ has *accuracy* $p+1$ if $\Pi_p \subset S_\sigma(\Phi)$, in which case there exists, for each $l = 0, \ldots, p$, a unique sequence $(\alpha'_l(j))_{j \in \mathbf{Z}}$, such that

$$x^l = \sum_j \alpha'_l(j) \sigma_j(\Phi)(x). \tag{3.3}$$

THEOREM 3.2. *Suppose Φ has accuracy $p+1$ and σ is a squeeze map for Φ. Then $S_\sigma(\Phi)$ has accuracy $p+1$ if and only if $\bar{\alpha}_l(0)$ is either 0 or a left eigenvector of R_j with eigenvalue $(L_j/L_{j-1})^l$ for $l = 0, \ldots, p$ and all $j \in \mathbf{Z}$.*

Proof. Using (3.3) and the definition of $\sigma_j(\Phi)$, observe that $S_\sigma(\Phi)$ having accuracy $p+1$ is equivalent to the existence of sequences $(\alpha'_l(j))_{j \in \mathbf{Z}}$, $l = 0, \ldots, p$, such that

$$x^l = \bar{\alpha}'_l(j) A_{\mathrm{R}}^{(j)} \bar{\Phi} \circ \tau_j(x) + \bar{\alpha}'_l(j+1) A_{\mathrm{L}}^{(j+1)} \bar{\Phi} \circ \tau_{j+1}(x) + \breve{\alpha}'_l(j) \breve{\Phi} \circ \tau_j(x),$$

for $j \in \mathbf{Z}$, and $x \in \tau_j^{-1}([0,1]) = [a_j, a_{j+1}]$. By substituting $\tau_j^{-1}(x)$ for x in the above, we obtain

$$\sum_{i=0}^{l} \binom{l}{i} L_j^i x^i a_j^{l-i} = \bar{\alpha}'_l(j) A_{\mathrm{R}}^{(j)} \bar{\Phi}(x) + \bar{\alpha}'_l(j+1) A_{\mathrm{L}}^{(j+1)} \bar{\Phi}(x-1) + \breve{\alpha}'_l(j) \breve{\Phi}(x),$$

where l and j are as above, but here $x \in [0,1]$. Now, since Φ has accuracy $p+1$, we can use (3.2) to replace x^i in the above. In particular,

$$x^i = \bar{\alpha}_i(0) \bar{\Phi}(x) + \bar{\alpha}_i(1) \bar{\Phi}(x-1) + \breve{\alpha}(0) \breve{\Phi}(x)$$

for $x \in [0,1]$. With this substitution and the minimal support properties of Φ, we find an equivalent system of equations,

$$\begin{aligned} \breve{\alpha}'_l(j) &= \sum_{i=0}^{l} \binom{l}{i} L_j^i a_j^{l-i} \breve{\alpha}_i(0), \\ \bar{\alpha}'_l(j) A_{\mathrm{R}}^{(j)} &= \sum_{i=0}^{l} \binom{l}{i} L_j^i a_j^{l-i} \bar{\alpha}_i(0), \\ \bar{\alpha}'_l(j+1) A_{\mathrm{L}}^{(j+1)} &= \sum_{i=0}^{l} \binom{l}{i} L_j^i a_j^{l-i} \bar{\alpha}_i(1). \end{aligned} \tag{3.4}$$

Now, since $A_{\mathrm{L}}^{(j)}$ and $A_{\mathrm{R}}^{(j)}$ are invertible for all j, the last two of these lead to

$$\sum_{i=0}^{l} \binom{l}{i} L_j^i a_j^{l-i} \bar{\alpha}_i(0) (A_{\mathrm{R}}^{(j)})^{-1} = \sum_{i=0}^{l} \binom{l}{i} L_{j-1}^i a_{j-1}^{l-i} \bar{\alpha}_i(1) (A_{\mathrm{L}}^{(j)})^{-1}. \tag{3.5}$$

Here, we may apply Lemma 3.6 proved at the end of this section, observing that $\alpha_i(0)$ and $\alpha_i(1)$ satisfy (3.11), as, therefore, do $\bar{\alpha}_i(0)$ and $\bar{\alpha}_i(1)$. The "only if" part of the result follows.

All steps in the above argument are reversible except the one from (3.4) to (3.5). The "if" part of the result is achieved by choosing $\breve{\alpha}'$ and $\bar{\alpha}'$ as in (3.4). The choice is consistent with (3.5) and leads to the desired accuracy of $S_\sigma(\Phi)$. ∎

If $\Phi \subset C^m(\mathbf{R})$ and has accuracy $p+1$, then $\breve{\Phi}^{(q)}(0) = 0$ for $0 \le q \le m$ and so

$$\bar{\alpha}_l(0) \bar{\Phi}^{(q)}(0) = \frac{d^q}{dx^q} x^l \bigg|_{x=0} = (q!) \delta_{l,q} \tag{3.6}$$

for $0 \le q \le m$ and $0 \le l \le p$, where $\delta_{l,q}$ denotes the Kronecker delta. For $0 \le q \le m$ and $0 \le l \le p$, we define the following matrices:

$$V_l = \begin{pmatrix} \bar{\alpha}_0(0) \\ \vdots \\ \bar{\alpha}_l(0) \end{pmatrix} \text{ and } W_q = \left(\bar{\Phi}(0) \cdots \bar{\Phi}^{(q)}(0) \right). \tag{3.7}$$

Then (3.6) is equivalent to the matrix equation

$$V_p W_m = D, \tag{3.8}$$

where D is the $(p+1) \times (m+1)$ diagonal matrix whose (l,l)th component is $(l-1)!$. The rank of the right side of (3.8) is $\min(m+1, p+1)$. Also, V_p and W_m have rank at most k which gives the following bound for k.

LEMMA 3.3. *Suppose $\Phi \subset C^m(\mathbf{R})$ and has accuracy $p+1$; then*

$$k \ge \min(m+1, p+1).$$

Next we consider when accuracy or smoothness uniquely determines the squeeze map (up to equivalency) and when accuracy forces smoothness or smoothness forces accuracy.

THEOREM 3.4. *Suppose $\Phi \subset C^m(\mathbf{R})$ and has accuracy $p+1$. Let a be a given knot sequence.*

(i) *If $k \leq p+1$ and the square matrix V_{k-1} is nonsingular, then there exists a unique (up to equivalence) squeeze map σ with knot sequence a such that $S_\sigma(\Phi)$ has accuracy k. In addition, $S_\sigma(\Phi) \subset C^m(\mathbf{R})$.*

(ii) *If $k \leq m+1$ and the square matrix W_{k-1} is nonsingular, then there exists a unique (up to equivalence) squeeze map σ with knot sequence a such that $S_\sigma(\Phi) \subset C^{k-1}(\mathbf{R})$. Furthermore, $S_\sigma(\Phi)$ has accuracy $p+1$.*

Proof. Case (i). Suppose $k \leq p+1$, $V := V_{k-1}$ is nonsingular, and σ is a squeeze map for Φ. Then $\bar{\alpha}_l(0) \neq 0$ for $0 \leq l \leq k-1$, and so Theorem 3.2 asserts that $S_\sigma(\Phi)$ has accuracy k if and only if $\bar{\alpha}_l(0)$ is a left eigenvector of R_j with eigenvalue $(\frac{L_j}{L_{j-1}})^l$ for $0 \leq l \leq k-1$ and $j \in \mathbf{Z}$. The latter condition is equivalent to

$$V\,R_j = \Lambda(L_j/L_{j-1})\,V \qquad (j \in \mathbf{Z}),$$

where $\Lambda(\lambda)$ is a $k \times k$ diagonal matrix whose (l,l)th component is λ^{l-1} for $\lambda \in \mathbf{R}_+$. Thus, $S_\sigma(\Phi)$ has accuracy k if and only if

$$R_j = V^{-1}\,\Lambda(L_j/L_{j-1})\,V \qquad (j \in \mathbf{Z}). \tag{3.9}$$

Equation (3.8) shows that $\Phi^{(q)}(0)$ is the qth column of $V^{-1}D$. Multiplying both sides of (3.9) on the right by $\Phi^{(q)}(0)$ then shows that $\bar{\Phi}^{(q)}(0)$ is a right eigenvector of R_j with eigenvalue $(\frac{L_j}{L_{j-1}})^q$ for $0 \leq q \leq m$. Hence, Theorem 3.1 shows that $S_\sigma(\Phi) \subset C^m(\mathbf{R})$.

Case (ii). Now suppose $k \leq m+1$ and $W := W_{k-1}$ is nonsingular. As in case (i) we find that $S_\sigma(\Phi)$ has accuracy k if and only if

$$R_j = W\,\Lambda(L_j/L_{j-1})\,W^{-1} \tag{3.10}$$

and that $\bar{\alpha}_l(0)$ is a left eigenvector of R_j with eigenvalue $(\frac{L_j}{L_{j-1}})^l$ for $0 \leq l \leq p$. Hence Theorem 3.2 shows that $S_\sigma(\Phi)$ has accuracy $p+1$. ∎

If $k = \min(m+1, p+1)$, then it follows from (3.8) that V_{k-1} and W_{k-1} are both nonsingular, and so both cases in Theorem 3.4 hold. The next theorem shows that $S_\sigma(\Phi)$ contains the spline space

$$S_p^m(a) := \{f \in C^m(\mathbf{R}) \mid f|_{(a_j, a_{j+1})} \in \Pi_p,\ j \in \mathbf{Z}\}$$

when $k = \min(m+1, p+1)$. In this case, it is known from classical spline theory that the accuracy determines the approximation order of $S_\sigma(\Phi)$. Note that $S_p^m(a) = \Pi_p$ if $m \geq p$.

THEOREM 3.5. *Suppose $\Phi \subset C^m(\mathbf{R})$, Φ has accuracy $p+1$ and $k = \min(m+1, p+1)$. Let a be a given knot sequence.*

(i) *There exists a squeeze map σ with knot sequence a such that $S_\sigma(\Phi) \subset C^m(\mathbf{R})$ and has accuracy $p+1$.*

(ii) *If ν is any other squeeze map with knot sequence a such that either $S_\nu(\Phi) \subset C^{k-1}(\mathbf{R})$ or (ν, Φ) has accuracy k, then ν is equivalent to σ.*

(iii) *$S_p^m(a) \subset S_\sigma(\Phi)$. (This is nontrivial only when $m < p$, in which case $k = m+1$.)*

Proof. If $k = \min(m+1, p+1)$, then it follows from (3.8) that V_{k-1} and W_{k-1} are both nonsingular and parts (i) and (ii) follow from Theorem 3.4.

From part (i) we have $\Pi_p \subset S_\sigma(\Phi)$, and so we need only consider the case $m < p$. Since W_{k-1} is nonsingular, it follows from (3.8) that $\bar{\alpha}_l(0) = 0$ for $l = m+1, \ldots, p$.

For simplicity, first suppose that one of the knots, say a_i, is 0. Then (3.4) implies

$$\bar{\alpha}_l'(i) A_{\mathrm{R}}^{(j)} = L_i^l \bar{\alpha}_l(i) = 0 \qquad (l = m+1, \ldots, p).$$

Thus (3.3) becomes

$$x^l = \breve{\alpha}_l'(i)\breve{\sigma}_i(\Phi) + \sum_{j \neq i} \alpha_l'(j)\sigma_j(\Phi).$$

Thus the truncated powers $(x_+)^l$, $l = m+1, \ldots, p$, can be written as

$$(x_+)^l = \breve{\alpha}_l'(i)\breve{\sigma}_i(\Phi) + \sum_{j > i} \alpha_l'(j)\sigma_j(\Phi),$$

and so they are in $S_\sigma(\Phi)$ for $l = m+1, \ldots, p$. Observe that shifting the knots by a constant shift translates the basis $B_\sigma(\Phi)$ by the same amount. Hence $S_\sigma(\Phi)$ contains the truncated powers $((x-a_j)_+)^l$ for $l = m+1, \ldots, p$ and $j \in \mathbf{Z}$. The truncated powers form a basis for $S_p^m(a)$ showing that (iii) holds. $\square$

Finally, we prove the following lemma that was used in the proof of Theorem 3.2.

LEMMA 3.6. *Suppose $a_0, a_1, L_1 \in \mathbf{R}$ and $L_0 = a_1 - a_0$. Further, suppose $\alpha(0)$ and $\alpha(1)$ are sequences of $1 \times k$ vectors such that*

$$\alpha_l(1) = \sum_{i=0}^{l} \binom{l}{i} \alpha_i(0) \tag{3.11}$$

for $l = 0, \ldots, p$. Then the $k \times k$ matrices C and D satisfy the conditions

$$\sum_{i=0}^{l} \binom{l}{i} L_1^i a_1^{l-i} \alpha_i(0) C = \sum_{i=0}^{l} \binom{l}{i} L_0^i a_0^{l-i} \alpha_i(1) D \tag{3.12}$$

for $l = 0, \ldots, p$ if and only if

$$\alpha_l(0)(L_1^l C - L_0^l D) = 0 \tag{3.13}$$

for $l = 0, \ldots, p$.

Proof. For a given l, we may use (3.11) to substitute for $\alpha_i(1)$ in (3.12). Then using routine combinatorial manipulations we find

$$\sum_{i=0}^{l} \binom{l}{i} L_1^i a_1^{l-i} \alpha_i(0) C = \sum_{j=0}^{l} \binom{l}{j} \alpha_j(0) D \sum_{i=j}^{l} \binom{l-j}{i-j} L_0^i a_0^{l-i}. \tag{3.14}$$

By shifting the index on the inner sum by j, the left-hand side becomes

$$\begin{aligned} &\sum_{j=0}^{l} \binom{l}{j} \alpha_j(0) D \sum_{i=0}^{l-j} \binom{l-j}{i} L_0^{i+j} a_0^{l-i-j} \\ &= \sum_{j=0}^{l} \binom{l}{j} \alpha_j(0) L_0^j a_1^{l-j} D, \end{aligned}$$

where the final equality follows from $a_1 = a_0 + L_0$ and the binomial theorem. Thus (3.12) is equivalent to

$$\sum_{i=0}^{l} \binom{l}{i} a_1^{l-i} \alpha_i(0)(L_1^i C - L_0^i D) = 0.$$

From here it is easy to show the equivalence with (3.13) by induction on $l = 0, \ldots, p$. $\square$

4. Constructing the squeeze map. Suppose $\Phi \subset C^m(\mathbf{R})$, Φ has accuracy $p+1$ and $k \leq \max(m+1, p+1)$. Then either case (i) or (ii) of Theorem 3.4 holds and the squeeze map preserving accuracy in case (i) or smoothness in case (ii) is unique up to equivalence. In both cases there is a full set of k eigenvectors for R_j for $j \in \mathbf{Z}$ with specified eigenvalues. These eigenvectors then uniquely determine R_j through either (3.9) or (3.10). In case (i), let $U = V_{k-1}$ and in case (ii) let $U = W_{k-1}^{-1}$, where V_{k-1} and W_{k-1} are given by (3.7). Let

$$R(\lambda) := U^{-1}\,\Lambda(\lambda)\,U \qquad (\lambda > 0). \tag{4.1}$$

Then $R_j = R(\lambda_j)$, where $\lambda_j := L_j/L_{j-1}$ for $j \in \mathbf{Z}$. Thus, the squeeze map is determined (up to equivalence) for an arbitrary knot sequence. Furthermore, each R_j is determined only by the ratio L_j/L_{j-1}.

Now suppose Φ is an orthogonal generator. Let σ be the squeeze map with matrix sequences (I, R_j). Following the proof of Lemma 2.2, an equivalent squeeze map ν so that $B_\nu(\Phi)$ is orthogonal may be found as follows. First, find a Cholesky factorization (see (2.3)):

$$B(\lambda)B(\lambda)^\top = \langle \Phi_L, \Phi_L \rangle + \lambda R(\lambda) \langle \Phi_R, \Phi_R \rangle R(\lambda)^\top. \tag{4.2}$$

Let $B_j = \sqrt{L_{j-1}} B(\lambda_j)$ for $j \in \mathbf{Z}$. Then ν with matrix sequences $A_{\mathrm{L}}^{(j)} = (B_j^{-1})$ and $A_{\mathrm{R}}^{(j)} = (B_j^{-1} R_j)$ gives an orthogonal basis. Again note that for fixed Φ, B_j depends only on L_{j-1} and L_j, and (since a Cholesky factorization is equivalent to an LU factorization using Gaussian elimination) we can find a closed form expression for ν_j in terms of the ratio $\lambda_j = L_j/L_{j-1}$. This makes it simple and quick to construct the squeeze map for an arbitrary knot sequence.

In our examples we consider only $k = 1$ or $k = 2$. When $k = 1$ it is trivial to obtain B_j. Suppose

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is a symmetric positive definite matrix. (That is, $v^\top A v > 0$ for any nonzero 2-vector v.) Then A is positive definite if and only if both a and $\det A$ are positive. One choice for B such that $BB^\top = A$ is given by

$$B = \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ b & \sqrt{\det A} \end{pmatrix}. \tag{4.3}$$

5. Orthogonal minimally supported generators.

5.1. Rescaling orthogonal generators. Any orthogonal compactly supported generator may be used to construct an orthogonal generator supported on $[-1,1]$ as we next describe. If the support of $\Phi = (\phi_1, \dots, \phi_n)^\top$ is contained in $[-1, M]$, then let Φ_M denote the generator consisting of the concatenation of the M generators $\Phi(M\cdot +k)$, $(k = 0, \dots, M-1)$. Then Φ_M is an orthogonal generator supported in $[-1,1]$ and $S(\Phi_M)$ equals $S(\Phi)(M\cdot)$ (that is, the dilation by $1/M$ of the space $S(\Phi)$). The local linear independence conditions for minimal support must then be checked separately. However, when Φ is an orthogonal scalar ($n = 1$) refinable generator it is known that Φ is locally linearly independent (that is, the nonzero restrictions of the shifts of Φ to any open interval are linearly independent), which implies the weaker type of local linear independence we require in the definition of minimal support. The example in section 6.3 is constructed in this way.

5.2. General construction. In [5] the authors developed a method for constructing orthogonal generators. For $W \subset L^2(\mathbf{R})$, let P_W denote the orthogonal projection onto W.

LEMMA 5.1 (see [5]). *Suppose Φ is a minimally supported k-generator. There exists an orthogonal minimally supported k-generator Ψ such that $S(\Psi) = S(\Phi)$ if and only if*

$$(I - P_{S(\breve{\Phi})})\bar{\Phi} \perp (I - P_{S(\breve{\Phi})})\bar{\Phi}(\cdot - 1). \tag{5.1}$$

(That is, $(I - P_{S(\breve{\Phi})})\phi_i \perp (I - P_{S(\breve{\Phi})})\phi_j(\cdot - 1)$ for $1 \le i, j \le k$.)

Proof (sketch of proof). Let $\breve{\Psi}$ be an orthogonal basis for the span of $\breve{\Phi}$ and choose $\bar{\Psi}$ to be an orthogonal basis for the span of $(I - P_{S(\breve{\Phi})})\bar{\Phi}$. Then Ψ is an orthogonal, minimally supported k-generator for $S(\Phi)$ if Ψ and $\Psi(\cdot - 1)$ are orthogonal (or, equivalently, if (5.1) holds). The other direction relies on the observation that if Φ and Ψ are minimally supported k-generators such that $S(\Psi) = S(\Phi)$, then

$$\operatorname{span}\breve{\Phi} = \operatorname{span}\breve{\Psi}$$

and

$$\operatorname{span}\Phi \cup \breve{\Phi}(\cdot + 1) = \operatorname{span}\breve{\Psi} \cup \breve{\Psi}(\cdot + 1) \qquad \square$$

The idea of the construction is to choose $\breve{\Phi}$ so that (5.1) holds. The orthogonal generators in the examples in sections 6.1 and 6.2 and section 7 are constructed in this way.

6. Examples. In this section we present several examples to illustrate our methods. The examples in sections 6.1 and 6.2 first appeared in [4]. In both examples it is the smoothness condition that determines the squeeze map. Also, in these two examples, $k = \min(m+1, p+1)$, and so the resulting $S_\sigma(\Phi)$ contains $S_p^m(a)$ by Theorem 3.5.

In the example in section 6.3, we rescale Daubechies's orthogonal scaling function ${}_2\phi$ as described in section 5.1 to construct a continuous orthogonal refinable generator minimally supported on [-1,1] with $k = n = 2$. The accuracy in this case is $p + 1 = 2$ and, by Theorem 3.4 (i), the squeeze map is uniquely determined by the accuracy condition once a knot sequence is specified. In fact, it is this example that motivated our study of the accuracy of squeezed spaces $S_\sigma(\Phi)$. In the example in section 6.3 we have $m + 1 = 1 < 2 = k$, and so Theorem 3.5 does not apply in this case.

We are also interested in this example because the generator Φ is *refinable*; that is,

$$\Phi(\cdot/2) = \sum_{j\in\mathbf{Z}} c(j)\Phi(\cdot - j) \tag{6.1}$$

for some finitely supported sequence $c : \mathbf{Z} \mapsto \mathbf{R}^{n\times n}$. (In this case the support of c is $\{-2, -1, 0, 1\}$.)

We next remark that such a refinable minimally supported generator Φ generates a *semiregular multiresolution analysis* (that is, a multiresolution consisting of a nonuniform coarse space that is uniformly refined; see [3]) as follows: Let a^0 be an arbitrary knot sequence and let $a^1 \supset a^0$ be given by

$$a^1_{2j} = a^0_j \text{ and } a^1_{2j+1} = (a^0_j + a^0_j)/2 \qquad (j \in \mathbf{Z}).$$

Let σ^0 and σ^1 be the squeeze maps determined (up to equivalence) by the knot sequences a^0 and a^1, respectively. Then one may verify that $S_{\sigma^0}(\Phi) \subset S_{\sigma^1}(\Phi)$. *Thus we provide a way to construct orthogonal semiregular multiresolutions from orthogonal scaling functions in a way that preserves the accuracy and smoothness of the shift-invariant multiresolution.*

In the example in section 4, we construct an *irregular multiresolution analysis* (that is, a fully nonuniform multiresolution; see [3]) such that each space in the multiresolution has a compactly supported orthogonal basis consisting of continuous piecewise quadratic functions. The spaces in this irregular multiresolution are not, strictly speaking, squeezed spaces of the form $S_\sigma(\Phi)$ but instead result from a slight generalization of our notion of the squeeze map.

6.1. $k = 1$, $m = 0$, $p = 1$, $n = 2$. Let h denote the hat function defined by $h(x) = (1 - |x|)^+$ and suppose $w \in L^2(\mathbf{R})$ is nontrivial and supported in the interval $[0, 1]$. Let $\Phi = (h, w)$. Then (5.1) reduces to

$$\langle h, h(\cdot - 1)\rangle = \frac{\langle h, w\rangle\langle w, h(\cdot - 1)\rangle}{\langle w, w\rangle}. \tag{6.2}$$

Thus, any $w \in L^2(\mathbf{R})$ supported in $[0, 1]$ and satisfying (6.2) gives an orthogonal generator Ψ by the process described in Lemma 5.1. For example, let q be the piecewise quadratic function given by $q(x) = x(1-x)\chi_{[0,1]}(x)$. Choose $w \in \operatorname{span}\{q, q^2\}$ so that $w = c_1 q + c_2 q^2$ for some constants c_1, c_2. Substituting into (6.2) yields a quadratic equation in the variable $\alpha := c_2/c_1$:

$$\alpha^2 + 30\alpha + 105 = 0$$

or $\alpha = -15 \pm 2\sqrt{30}$. The graphs of ϕ_1 and ϕ_2 are shown in Figure 6.1 for $\alpha = -15 - 2\sqrt{30}$. (This example was first given in [4].)

For $0 \le x \le 1$, we have

$$\phi_1(x) = \sqrt{3}\ (1-x)\ \left(1 - 2\,x + \left(-3 + \sqrt{30}\right)\ x\ (1 - 5\ (1-x)\ x)\right)$$

and

$$\phi_2(x) = \sqrt{330 - 60\sqrt{30}}\ (1-x)\ x\ \left(-1 + \left(15 + 2\sqrt{30}\right)\ (1-x)\ x\right).$$

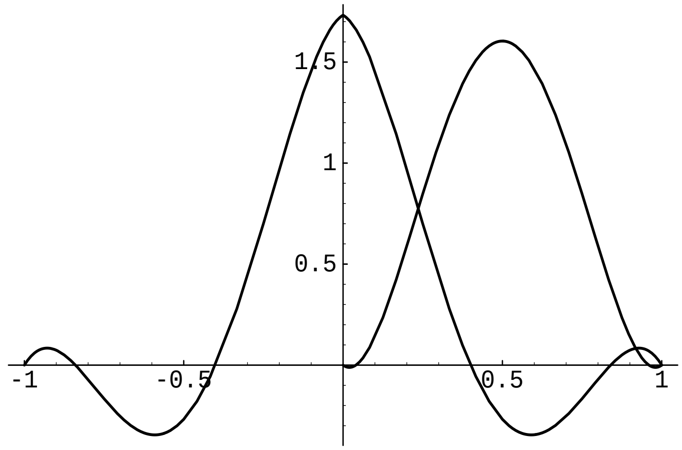


FIG. 6.1. *Continuous orthogonal generator of the example in section* 6.1.

Note that ϕ_1 is even and supported on $[-1, 1]$ and that ϕ_2 has support $[0, 1]$.

In the case $k = 1$ and $m = 0$, the squeeze maps preserving continuity are given by $R_j = 1$ for all $j \in \mathbf{Z}$. By Theorem 3.4, this squeeze map will also preserve the approximation of Φ. By the symmetry of Φ we have

$$\langle \Phi_L, \Phi_L \rangle = \langle \Phi_R, \Phi_R \rangle = 1/2.$$

Using (2.3) we get that σ given by

$$A_L^j = A_R^j = \sqrt{\frac{2}{L_{j-1} + L_j}}$$

generates an orthogonal basis $B_\sigma(\Phi)$.

6.2. $\boldsymbol{k = 2, m = 1, p = 3, n = 4}$. We next construct a continuously differentiable orthogonal generator. We start with the C^1 cubic Hermite spline functions

$$h_1(x) = \begin{cases} (1+x)^2(1-2x), & x \in [-1, 0], \\ (1-x)^2(1+2x), & x \in [0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

$$h_2(x) = \begin{cases} (1+x)^2 x, & x \in [-1, 0], \\ (1-x)^2 x, & x \in [0, 1], \\ 0 & \text{otherwise} \end{cases}$$

and add two continuously differentiable functions w_1 and w_2 supported on $[0, 1]$. (In [5], it is shown that at least two w's are required in this case.) The condition (6.2) is equivalent to the following:

$$\langle h_i, h_j(\cdot - 1) \rangle = \frac{\langle h_i, w_1 \rangle \langle w_1, h_j(\cdot - 1) \rangle}{\langle w_1, w_1 \rangle} + \frac{\langle h_i, w_2 \rangle \langle w_2, h_j(\cdot - 1) \rangle}{\langle w_2, w_2 \rangle}. \tag{6.3}$$

Again let q be the piecewise quadratic function given by $q(x) = x(1-x)\chi_{[0,1]}(x)$. We choose w_1 to be of the form $(c_1 + c_2 q + c_3 q^2)q^2$ and w_2 of the form $(\cdot - 1/2)(c_4 + c_5 q)q^2$

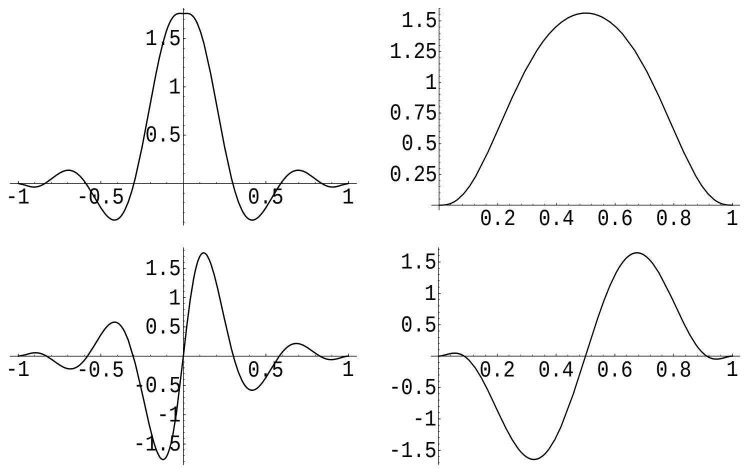


FIG. 6.2. *The C^1 orthonormal generator of the example in section* 6.2. *From upper left, going clockwise:* ϕ_1, ϕ_3, ϕ_4, ϕ_2.

so that w_1 is symmetric about $x = 1/2$ and w_2 is antisymmetric about $x = 1/2$. Substituting into (6.3) yields three quadratic equations in the three variables c_2/c_1, c_3/c_1, and c_5/c_4. Solving these equations numerically and choosing c_1 and c_4 so that $\|w_1\| = \|w_2\| = 1$ yields several solutions. One solution with good properties is given by

c_1	+2.102558692333885
c_2	+214.7707569159831
c_3	-492.4339092336308
c_4	-112.0742772596177
c_5	+1401.893433767276

The graphs of the components of the resulting orthogonal generator $(\phi_1, \phi_2, \phi_3, \phi_4)$ are shown in Figure 6.2.

From the construction of Φ we see that $W = \left(\bar{\Phi}(0)\,\bar{\Phi}'(0)\right)$ is diagonal, and so, using (4.1), we get that $S_\sigma(\Phi) \subset C^1(\mathbf{R})$ if

$$R_j = R(\lambda_j) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_j \end{pmatrix},$$

where $\lambda_j := L_j/L_{j-1}$.

Since Φ is piecewise polynomial, the inner products $\langle \Phi_L, \Phi_L \rangle$ and $\langle \Phi_R, \Phi_R \rangle$ are easily calculated. Using *Mathematica* to perform these calculations, we arrive at the squeeze maps defined by

$$A_L^{(j)} = \frac{1}{\sqrt{L_{j-1}}} \times \begin{pmatrix} \frac{1.414213}{\sqrt{1+\lambda_j}} & 0 \\ \frac{2.829115-2.829115\,\lambda_j^2}{(1+\lambda_j)\sqrt{(0.381634+\lambda_j)\,(1+\lambda_j)\,(2.62031+\lambda_j)}} & \frac{3.162893}{\sqrt{(0.381634+\lambda_j)\,(1+\lambda_j)\,(2.62031+\lambda_j)}} \end{pmatrix}$$

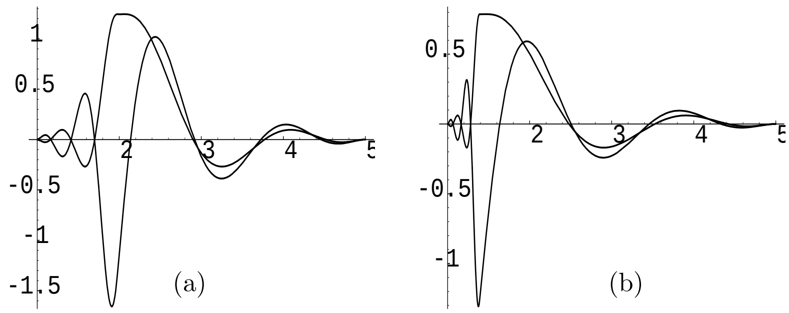


FIG. 6.3. *C^1 basis functions $\bar{\sigma}_j(\Phi)$ from the example in section* 6.2 *for* (a) $\lambda_j = 3$ *(knots at* 1*,* 2*, and* 5*) and* (b) $\lambda_j = 7$ *(knots at* 1*,* 3/2*, and* 5*).*

and

$$A_R^{(j)} = \frac{1}{\sqrt{L_{j-1}}} \times \begin{pmatrix} \frac{1.414213}{\sqrt{1+\lambda_j}} & 0 \\ \frac{2.829115-2.829115\,\lambda_j^2}{(1+\lambda_j)\sqrt{(0.381634+\lambda_j)\,(1+\lambda_j)\,(2.62031+\lambda_j)}} & \frac{3.162893\,\lambda_j}{\sqrt{(0.381634+\lambda_j)\,(1+\lambda_j)\,(2.62031+\lambda_j)}} \end{pmatrix}.$$

We show in Figure 6.3 the resulting $\bar{\sigma}_j(\Phi)$ for several values of λ_j.

6.3. Semiregular multiresolution analysis: $k = 2$, $m = 0$, $p = 1$, $n = 2$. Let ${}_2\phi$ denote the continuous orthogonal scaling function of Daubechies supported on $[0, 3]$ (see [2]) and let

$$\Phi = \sqrt{2}\begin{pmatrix} {}_2\phi(2\cdot +2) \\ {}_2\phi(2\cdot +1) \end{pmatrix}.$$

Then, as discussed in section 5.1, Φ is an orthogonal generator supported on $[-1, 1]$. The local linear independence condition for minimal support may be verified from the support properties of Φ and the fact that the components of Φ_R are orthogonal to the components of Φ_L, thus showing that Φ is a minimally supported generator with $k = 2$. Also, note that Φ is continuous and has accuracy 2. In this example, it is the accuracy that determines the squeeze map.

Recall that ${}_2\phi$ satisfies a refinement equation

$$ {}_2\phi = \sum_{j=0}^{3} c_j\, {}_2\phi(2\cdot -j), \tag{6.4}$$

where

$$c_0 = \frac{1+\sqrt{3}}{4}, \quad c_1 = \frac{3+\sqrt{3}}{4}, \quad c_2 = \frac{3-\sqrt{3}}{4}, \quad c_3 = \frac{1-\sqrt{3}}{4}.$$

Using the refinement equation it is possible to calculate the following coefficients from the zeroth and first moments of ${}_2\phi$ (see [1]):

$$\alpha_0(0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad \alpha_1(0) = \left(\frac{-1-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}}\right)$$

and

$$\langle\Phi_R, \Phi_R\rangle = \begin{pmatrix} \frac{7}{12} + \frac{5}{7\sqrt{3}} & \frac{1}{28\sqrt{3}} \\ \frac{1}{28\sqrt{3}} & \frac{5}{12} + \frac{5}{7\sqrt{3}} \end{pmatrix}, \qquad \langle\Phi_L, \Phi_L\rangle = \begin{pmatrix} \frac{5}{12} - \frac{5}{7\sqrt{3}} & \frac{-1}{28\sqrt{3}} \\ \frac{-1}{28\sqrt{3}} & \frac{7}{12} - \frac{5}{7\sqrt{3}} \end{pmatrix}.$$

Then

$$R(\lambda) = \frac{1}{2}\begin{pmatrix} 1-\sqrt{3}+\left(1+\sqrt{3}\right)\lambda & \left(1+\sqrt{3}\right)(1-\lambda) \\ \left(1-\sqrt{3}\right)(1-\lambda) & 1+\sqrt{3}+\left(1-\sqrt{3}\right)\lambda \end{pmatrix}$$

and

$$\left(\langle\Phi_L, \Phi_L\rangle + \lambda R(\lambda)\langle\Phi_R, \Phi_R\rangle R(\lambda)^\top\right) = \frac{1}{84}\begin{pmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & c(\lambda) \end{pmatrix},$$

where

$$a(\lambda) = 35 - 20\sqrt{3} + 4\left(21 + 8\sqrt{3}\right)\lambda - 4\left(44 + 23\sqrt{3}\right)\lambda^2 + \left(141 + 80\sqrt{3}\right)\lambda^3,$$

$$b(\lambda) = (-1+\lambda)\left(\sqrt{3} - 84\lambda - 31\sqrt{3}\lambda + 42\lambda^2 + 19\sqrt{3}\lambda^2\right),$$

$$c(\lambda) = 49 - 20\sqrt{3} + 4\left(21 + 8\sqrt{3}\right)\lambda - 4\left(19 + 2\sqrt{3}\right)\lambda^2 + \left(27 - 4\sqrt{3}\right)\lambda^3.$$

The factors B_j may then be calculated from (4.3).

6.4. Irregular multiresolution analysis: $k = 1$, $m = 0$, $p = 2$, $n = 3$. Let $\left(a^\ell\right)_{\ell\in\mathbf{Z}}$ be a sequence of nested knot sequences such that $a^{\ell+1}_{2j} = a^\ell_j$ for ℓ, $j \in \mathbf{Z}$ and such that $\{a^\ell_j \mid \ell, j \in \mathbf{Z}\}$ is dense in $\mathbf{R}$. Let $V_\ell = S_{0,2}(a^\ell)$ denote the space of continuous piecewise quadratic splines with break points given by a^ℓ. From the theory of splines it follows that (V_ℓ) is a multiresolution analysis. Here we construct a multiresolution (V'_ℓ) such that

$$V_\ell \subset V'_\ell \subset V_{\ell+1}$$

and each V'_ℓ has a local orthogonal basis. The local orthogonal basis for V'_ℓ is generated with a generalization of the squeeze map idea. Our construction here extends the idea of intertwining multiresolution analyses developed in [5] to the nonuniform case.

Let $\Phi = (h, q)$, where h and q are as in the example in section 6.1. Then $V_\ell = S_{\sigma^\ell}(\Phi)$, where σ^ℓ is the squeeze map with knot sequence a^ℓ given by $R_j = 1$.

Let $I^\ell_j = [a^\ell_j, a^\ell_{j+1}]$. The idea of the construction is to add basis functions $w^\ell_j \in V_\ell$ supported on I^ℓ_j for each $j \in \mathbf{Z}$ to the basis $B_{\sigma^\ell}(\Phi)$ in such a way that the resulting space V'_ℓ has a local orthogonal basis. We first describe the construction when $I = I^\ell_j = [0, 1]$; the general case will follow by rescaling. Then $a := a^{\ell+1}_{2j+1}$ is in $(0, 1)$. Define $q_{1,0}$, $q_{1,1}$, and h_1 by

$$q_{1,0}(x) = q(x/a), \qquad q_{1,1} = q\left(\frac{x-a}{a}\right)$$

and

$$h_1(x) = \begin{cases} x/a & \text{for } x \in [0,a], \\ (1-x)/(1-a) & \text{for } x \in [a,1], \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the space $\mathcal{A}$ of functions in $V^{\ell+1}$ whose support is contained in [0,1] is spanned by $q_{1,0}$, $q_{1,1}$, and h_1. Note that q is in this 3-dimensional space. We choose $w = w_j^\ell$ in the 2-dimensional orthogonal complement of q in $\mathcal{A}$. A basis for this space is given by (with help from *Mathematica*)

$$\begin{aligned} u_0 &= a^2(3a-5)q_{1,0} + (1-a)^2(2+3a)q_{1,1} \\ u_1 &= (-2+3\,(-1+a)\,a^3)q_{1,0} + (-2+3\,(-1+a)^3\,a)q_{1,1} \\ &\quad + \left(\frac{16}{5} - 12\,(-1+a)^2\,a^2\right)h_1. \end{aligned}$$

We choose w in $\mathcal{A}$ and orthogonal to q so that it is of the form

$$w = c_1u_1 + c_2u_2.$$

Define

$$\theta_R = (I - P_{\mathrm{span}(w,q)})h_R$$

and

$$\theta_L = (I - P_{\mathrm{span}(w(\cdot+1),q(\cdot+1))})h_L,$$

where $h_R = h\chi_{[0,1)}$ and $h_L = h\chi_{[-1,0)}$. In order to construct a local orthogonal basis we require

$$\langle \theta_R, \theta_L(\cdot - 1)\rangle = 0,$$

which is equivalent to the following quadratic equation in the variable $c = c_1/c_2$:

$$\begin{aligned} 0 =& 5\left(4 - 5\,(1-a)^2\,a^2\,(15+(1-a)\,a)\right) \\ &- 20\,(2+a\,(9+13\,a\,(-3+2\,a)))\,c + 4\,(1+45\,(1-a)\,a)\,c^2. \end{aligned} \tag{6.5}$$

The discriminant of this equation is

$$80\left(4 - 15\,(1-a)^2\,a^2\right)^2,$$

giving the two solutions

$$c = \frac{20(2+a(9+13a(2a-3))) \pm 4\sqrt{5}(4-15(1-a)^2a^2)}{8(1+45(1-a)a)}.$$

Hence, there are two choices for w for any $a \in (0,1)$. For each $a \in (0,1)$ choose one such w and denote it by W_λ, where $\lambda = (1-a)/a$ is the ratio of the lengths of the two subintervals $[0,a]$ and $[a,1]$. Let $\theta_{R,\lambda} = \theta_R$ and $\theta_{L,\lambda} = \theta_L$ with $w = W_\lambda$. Define

$$\Phi^{\lambda_L,\lambda_R} = \begin{pmatrix} \theta_{L,\lambda_L} + \theta_{R,\lambda_R} \\ q \\ W_{\lambda_R} \end{pmatrix}$$

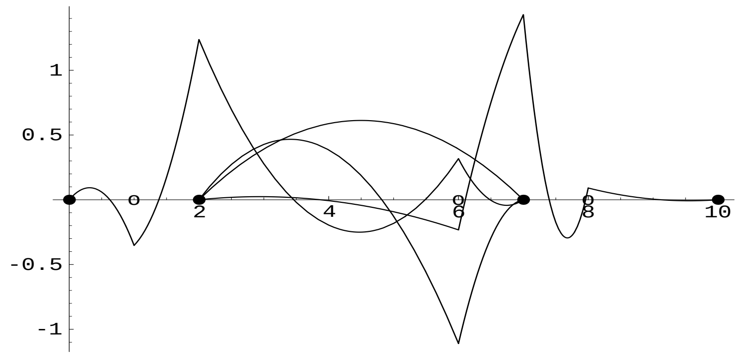


FIG. 6.4. *Continuous, orthogonal piecewise quadratic basis functions from the example in section* 6.4 *with knots* $a_{\ell+1} = \ldots, 0, 1, 3, 6, 7, 8, 10, \ldots$.

and note that $\Phi^{\lambda_L,\lambda_R}$ is continuous and supported on $[-1,1]$. Given $a^{\ell+1}$ we construct basis functions supported on $[a^\ell_{j-1}, a^\ell_{j+1}] = [a^{\ell+1}_{2j-2}, a^{\ell+1}_{2j+2}]$ as follows. Let τ^ℓ_j be as in (2.1) with knot sequence a^ℓ, let $L^\ell_j = a^\ell_{j+1} - a^\ell_j$, and let

$$\lambda^\ell_j = L^{\ell+1}_{2j+1}/L^{\ell+1}_{2j}.$$

Note that the collection of functions

$$B^\ell = \bigcup_{j\in\mathbf{Z}} \Phi^{\lambda^\ell_{j-1},\lambda^\ell_j} \circ \tau^\ell_j$$

is an orthogonal system of functions. Let

$$V'_\ell = \operatorname{span}_{L^2} B^\ell$$

for $\ell \in \mathbf{Z}$. Then

$$V_\ell \subset V'_\ell \subset V_{\ell+1} \subset V'_{\ell+1},$$

from which it follows that $(V'_\ell)_{\ell\in\mathbf{Z}}$ is a multiresolution with local orthogonal basis B^ℓ.

Figure 6.4 shows several of the basis functions (we chose the minus branch of the square root) for $a_\ell = \ldots, 0, 3, 7, 10, \ldots$ and $a_{\ell+1} = \ldots, 0, 1, 3, 6, 7, 8, 10, \ldots$.

7. Higher order accuracy and smoothness. Let S^n_m be the space of polynomial splines of degree n with C^m knots at the integers. If we denote $A^{n,m} = \{g \in S^n_m : \operatorname{supp} g = [0,1]\}$, then it is easy to see [6] that an orthogonal basis for $A^{n,m}$ is provided by $\phi^m_i(t) = t^m(1-t)^m p^{2m+5/2}_{i-2m-2}(2t-1)$, $2m+2 \le i \le n$, where $p^{2m+5/2}_j(t)$ is the monic ultraspherical polynomial of degree j with $\lambda = 2m + 5/2$. If we set $\Phi = (\phi^m_0 \cdots \phi^m_n)^T$, where ϕ^m_i, $i = 0, \ldots, m$, $\operatorname{supp} \phi^m_i = [-1,1]$ are appropriately chosen (i.e., judicious linear combinations of r^i_m and l^i_m, $i = 0, \ldots, m$, with $r^i_m(t) = t^i(1+t)^{m+1}\ -1 \le t \le 0$ and $l^i_m(t) = t^i(1-t)^{m+1} 0 < t \le 1$), then Φ and all its integer translates form a basis for S^n_m. This basis is not orthogonal, so Φ does not generate a local orthogonal basis. We will modify Φ in order to construct an orthogonal set of generators. We do this by adding to Φ, $m+1$ functions w_i chosen so that $W \perp A^{n,m}$ and $\langle (I - P_W)\hat\phi^m_i, (I - P_W)\hat\phi^m_j(\cdot - 1)\rangle = 0\ i,j = 1, \ldots, m+1$. Here $W = \operatorname{span}\{w_i : i = 1, \ldots, m+1\}$, P_W is the orthogonal projection onto

W, and $\hat{\phi}_i^m = (I - P_{\{A^{n,m}, A^{n,m}(+1)\}})\phi_i^m$. In the examples given below we will choose w_i to be linear combinations of $\{\phi_j^m\}_{j>n}$. In this way $w_i \perp A^{n,m}$ since the $\{t^m(1-t)^m p_l^{2m+5/2}(2\cdot -1)\}_{l=0}^{\infty}$ is a set of orthogonal polynomials. Notice that the above w_i will have their knots located at the integers. This is in contrast to the construction carried out in [6] where in order to build a MRA it was necessary to use w_i with half integer knots.

7.1. C^0 example. As a first example we consider the case $m = 0$. Then $r_0(t) = (1+t)$ and $l_0(t) = (1-t)$, and we will choose $w_1^n = \phi_{n+1}^0 + \alpha_n \phi_{n+3}^0$. Since ϕ_i^0 is symmetric or antisymmetric about $1/2$ depending on whether i is even or odd, respectively, we see that w_1^n chosen above will be either symmetric or antisymmetric. With $\hat{r}_0^n(\cdot) = (I - P_{A^{n,0}})r_0(\cdot - 1)$ and $\hat{l}_0^n(t) = (I - P_{A^{n0}})l_0$ we choose α_n so that $\langle (I - P_{w_1^n})\hat{r}_0, (I - P_{w_1^n})\hat{l}_0(t)\rangle = 0$. This gives the following quadratic equation for α_n:

$$\langle \hat{r}_0^n, \hat{l}_0^n\rangle \langle w_1^n, w_1^n\rangle = \langle w_1^n, r_0(\cdot - 1)\rangle \langle w_1^n, l_0\rangle \tag{7.1}$$

or

$$\begin{aligned} &\langle \hat{r}_0^n, \hat{l}_0^n\rangle (\langle \phi_{n+1}^0, \phi_{n+1}^0\rangle + \alpha_n^2 \langle \phi_{n+3}^0, \phi_{n+3}^0\rangle) \\ &= (\langle \phi_{n+1}^0, r_0(\cdot - 1)\rangle + \alpha_n \langle \phi_{n+3}^0, r_0(\cdot - 1)\rangle)(\langle \phi_{n+1}^0, l_0\rangle + \alpha_n \langle \phi_{n+3}^0, l_0\rangle). \end{aligned} \tag{7.2}$$

From [6] we find $\langle \hat{r}_0^n, \hat{l}_0^n\rangle = \frac{(-1)^{n+1}n!}{(n+3)!}$, $\langle \hat{r}_0^n, \hat{r}_0^n\rangle = \frac{1}{n(n+2)}$, and $\langle r_0, \phi_n^0\rangle = 2^{n-2}\frac{n!(n-2)}{2n!}$. Furthermore, since $\langle \phi_n^0, \phi_n^0\rangle = \frac{1}{32}\frac{(n+2)!(n-2)!}{(2n-1)!(2n+1)!!}$, the above equation may be solved for α_n to obtain

$$\alpha_n = -\frac{((2n+7)(2n+3)(n+1) \pm \sqrt{3(2n+7)(2n+3)(n+1)(n+3)}(n+3))(2n+5)}{(n+2)(n+1)(n^2-5n-30)},$$

and $\phi_0^{n,0}$ is given by

$$\phi_0^{n,0}(t) = (I - P_{(w_1^n, w_1^n(\cdot+1))})h(t),$$

where $h(t) = (1-|t|)^+$.

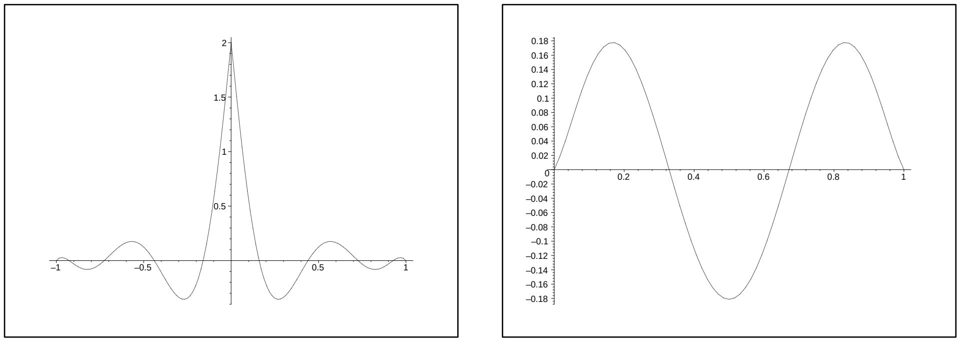
With $\phi_1^{n,0} = w_1^n$ we have the following theorem,

THEOREM 7.1. *For $n \geq 3, \Phi_n = (\phi_0^{n,0}, \phi_1^{n,0}, \phi_2^0 \ldots, \phi_n^0)^T$ constructed as above is a continuous orthogonal generator for $B(\Phi)$. Furthermore, Φ_n has accuracy $n+1$.*

Figure 7.1 shows $\phi_0^{n,0}$ and $\phi_1^{n,0}$ for $n = 3$.

7.2. C^1 example. We now construct a family of C^1 orthogonal compactly supported generators which have varying degrees of accuracy. In this case four ramp functions, $r_1^i = t^i(1+t)^2$, $i = 0, 1$ and $l_1^i = t^i(1-t)^2$, $i = 0, 1$, are needed in the construction of the orthogonal generators with support equal to $[-1, 1]$. We set $\hat{r}_1^{n,i}(\cdot) = (I - P_{A^{n,1}})r_1^i(\cdot - 1)$ and $\hat{l}_1^{n,i}(t) = (I - P_{A^{n,1}})l_1^i$. The necessary integrals to compute the above projections can be found in [6]. In order to make the computations somewhat more tractable we biorthogonalize the above ramp functions. Utilizing the integrals [6]

$$\langle \hat{r}_0^{n,1}, \hat{l}_0^{n,1}\rangle = \frac{4(-1)^{n+1}(n^2+2n-9)(n-2)!}{(n+3)!}, \tag{7.3}$$



$\phi_0^{3,0}$ $\phi_1^{3,0}$

FIG. 7.1. *The functions* ϕ_0 *and* ϕ_3 *from section* 7.1 *for* $n = 3$.

$$\langle \hat{r}_0^{n,1}, \hat{l}_1^{n,1} \rangle = \frac{12(-1)^{n+1}(n-2)!}{(n+3)!}, \tag{7.4}$$

and

$$\langle \hat{r}_1^{n,1}, \hat{l}_1^{n,1} \rangle = \frac{36(-1)^{n+1}(n-3)!}{(n+4)!}, \tag{7.5}$$

we set $r_{n,0} = \hat{r}_0^{n,1}$, $l_{n,0} = \hat{l}_0^{n,1}$, $r_{n,1} = \hat{r}_1^{n,1} - \frac{\langle \hat{r}_1^{n,1}, l_{n,0} \rangle}{\langle r_{n,0}, l_{n,0} \rangle} r_{n,0}$, and $l_{n,1} = \hat{l}_1^{n,1} - \frac{\langle \hat{l}_1^{n,1}, r_{n,0} \rangle}{\langle r_{n,01}, l_{n,1} \rangle} l_{n,1}$. With the help of the inner products given above, we find

$$\langle r_{n,1}, l_{n,1} \rangle = \frac{(-1)^n 36(n-3)!}{(n+4)!(n^2+2n-9)} \tag{7.6}$$

and

$$\langle r_{n,1}, \phi_i^1 \rangle = -\frac{3}{8} \frac{2^{n+i}(n+i)!(n+i-4)!(i^2+i+2ni-n-3)}{(2n+2i)!(n^2+2n-9)}. \tag{7.7}$$

Two functions w_i, $i = 1, 2$ will be needed to construct orthogonal generators from the above ramp functions, and these will be symmetric and antisymmetric with respect to $1/2$ in order to construct symmetric or antisymmetric generators. To this end let $w_1 = v_0(n) + \alpha_1(n) v_2(n)$, where $v_i(n)$ linear combinations of ϕ_{n+1+i}^1 and ϕ_{n+3+i}^1 and chosen so that $\langle v_i(n), r_{n,1} \rangle = 0$. Thus $v_0(n) = -\frac{(5n+9)(n-2)}{2(2n+5)(2n+3)} \phi_{n+1}^1 + \phi_{n+3}^1$ and $v_2(n) = -9 \frac{(n+3)(n+1)n}{2(2n+9)(2n+7)(5n+9)} \phi_{n+3}^1 + \phi_{n+5}^1$. Likewise, $w_2 = v_1(n) + \alpha_2(n) v_3(n)$, where $v_i(n)$ $i = 1, 3$ are orthogonal to $r_{n,0}$. In this case $v_1(n) = -\frac{(n+8)(n+1)n}{2(2n+7)(2n+5)(n+8)} \phi_{n+2}^1 + \phi_{n+4}^1$ and $v_3(n) = v_1(n+2)$. The biorthogonality of the ramps and the construction of v_i, $i = 0, 1, 2, 3$ imply that each $\alpha_i(n)$ must be chosen as a solution to the equation

$$\langle r_{n,i}, l_{n,i} \rangle \langle w_{i+1}, w_{i+1} \rangle = \langle w_{i+1}, r_i^1(\cdot - 1) \rangle \langle w_{i+1}, l_i^1 \rangle.$$

Utilizing (7.6), (7.7), and $\langle \phi_n^1, \phi_n^1 \rangle = \frac{n!(n+8)!}{256(2n+9)!!(2n+7)!!}$ to compute the inner products needed in the above equation we find using *Maple* that

$$\alpha_1(n) = \frac{(5n+9)(2n+7)}{(n+3)} \times \frac{(2n+11)q_1(n) \pm (n+4)(n+5)(5n+9)(2n+7)\left\{\frac{5(2n+11)(n+4)}{n(n+1)(2n+3)}q_2(n)\right\}^{\frac{1}{2}}}{2q_3(n)},$$

where

$$q_1(n) = 41n^5 + 625n^4 + 3733n^3 + 11099n^2 + 17010n + 11340,$$

$$q_2(n) = 17n^5 + 131n^4 - 105n^3 - 2979n^2 - 7884n - 6804,$$

and

$$q_3(n) = 37n^7 + 1376n^6 + 18862n^5 + 139394n^4 + 502291n^3 + 1099160n^2 + 1287090n + 635040.$$

Likewise,

$$\alpha_2(n) = \frac{(2n+9)(n+8)}{(n+3)(n+6)} \times \frac{\left(-(2n+13)q_4(n) \pm (n+5)(n+6)(n+8)(2n+9)\left\{\frac{7(2n+13)(n+4)(n+1)}{(2n+5)(n+2)}q_5(n)\right\}^{\frac{1}{2}}\right)}{2q_6(n)},$$

where

$$q_4(n) = 11n^6 + 115n^5 + 323n^4 + 893n^3 + 8642n^2 + 28968n + 25200,$$

$$q_2(n) = 3n^5 + 27n^4 + 7n^3 - 503n^2 - 1486n - 1400,$$

and

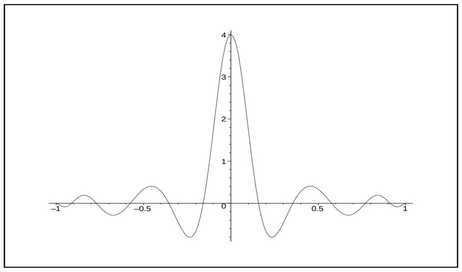
$$q_6(n) = 5n^7 + 39n^6 - 335n^5 - 5129n^4 - 29484n^3 - 112048n^2 - 242304n - 159600.$$

Knowing w_1 and w_2, we are now able to construct the orthogonal C^1 generator. Let $h_0(t) = 2|t|^3 - 3|t|^2 + 1$, if $t \in [-1,1)$, and 0 elsewhere; $h_1(t) = (1-|t|)^2 t$, if $t \in [-1,1)$, and 0 elsewhere; and $\phi_{i+1}^{n,1} = w_i$, $i = 1,2$. Figure 7.2 shows $\phi_0^{n,1}$, $\phi_1^{n,1}$, $\phi_2^{n,1}$, and $\phi_1^{3,1}$ for $n = 6$. Then with

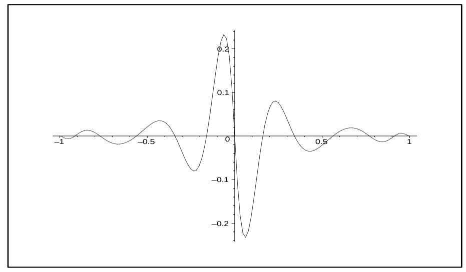
$$\phi_i^{n,1} = \left(I - P_{(\phi_2^1,\dots,\phi_n^1,\phi_2^1(\cdot+1),\dots,\phi_n^1(\cdot+1))}\right) h_i \qquad (i = 0, 1),$$

the above computations give the following theorem.

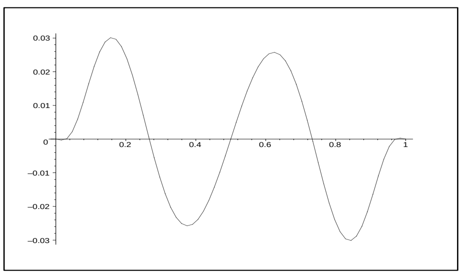
Theorem 7.2. *For $n \geq 5$, and $\alpha_i(n)$ given above, $\Phi^1(n) = (\phi_0^{n,1}, \dots, \phi_n^1)^T$ is a continuously differentiable orthogonal generator for $B(\Phi^1(n))$. Furthermore, the last $n-1$ functions are symmetric or antisymmetric about $1/2$. The first function $\phi_0^{n,1}$ is symmetric about 0, while $\phi_1^{n,1}$ is antisymmetric about 0.*



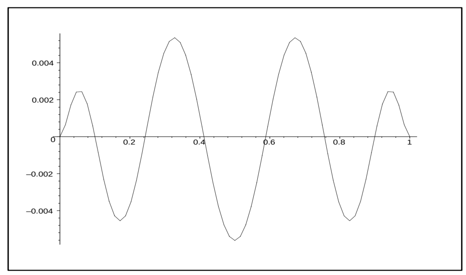
$\phi_0^{6,1}$



$\phi_1^{6,1}$



$\phi_2^{6,1}$



$\phi_3^{6,1}$

FIG. 7.2. *The functions* $\phi_i^{n,1}$ *for* $n = 6$ *and* $i = 0, \dots, 3$.

We now construct the squeeze map associated with $\Phi^1(n)$. Since the last $n-2$ generators are supported on $[0,1]$ we need only concentrate on $\phi_0^{n,1}$ and $\phi_1^{n,1}$. Because of the definition of h_0 and h_1 and the symmetry of $\phi_0^{n,1}$ and $\phi_1^{n,1}$ it is easy to see that $W(n)$ is a diagonal matrix for all n. Therefore $R(n)$ is as in the previous C^1 example, and with $A_L^{(j)}$ a diagonal matrix $R(n)$ is equal to $A_R^{(j)}$. In order to complete the construction of the squeeze map we need to compute the inner products $\langle \Phi_L, \Phi_L \rangle$ and $\langle \Phi_R, \Phi_R \rangle$. From (3.9) in [6] (we would like to point out some errors in that equation; namely, $r_{i+1}^{n,k}$ in the first term on the right-hand side should be $r_i^{n,k}$, the factor multiplying the third term on the right-hand side should be $(n-k-1-i)$, and the factor multiplying the last term should be $(n+k+i+3)$) we find that

$$\langle r_0^{n,1}, r_0^{n,1} \rangle = 4\frac{(n^2+2n-6)(n-2)!}{(n+3)!}, \tag{7.8}$$

$$\langle r_1^{n,1}, r_0^{n,1} \rangle = 6\frac{(n-2)!}{(n+3)!}, \tag{7.9}$$

and

$$\langle r_1^{n,1}, r_1^{n,1} \rangle = 12\frac{(n-3)!}{(n+4)!}. \tag{7.10}$$

To continue on we choose the minus sign in $\alpha_1(n)$ and the plus sign in $\alpha_2(n)$ to compute $\phi_i^{n,1}$ $i = 2,3$. Then (7.7) and the norm squared of ϕ_n^1 can be employed to compute (using *Maple*) the norms of $\phi_i^{n,1}$, $i = 2,3$ and the inner products of these functions with $\hat{r}_i^{n,1}, i = 0,1$. With these in hand, (7.8), (7.9), and (7.10) can be used to compute $\langle \Phi_R, \Phi_R \rangle$, which is

$$\langle \Phi_R, \Phi_R \rangle = \begin{pmatrix} 4\frac{n^5+3n^4-10n^3-21n^2+27n+18}{(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} & -6\frac{n^3-9n+6}{(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} \\ -6\frac{n^3-9n+6}{(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} & 12\frac{n-3}{(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} \end{pmatrix}.$$

Since these functions are either symmetric or antisymmetric $\langle \Phi_L, \Phi_L \rangle$ is the same as the above matrix, except that the off diagonal elements take the opposite sign. Thus (2.3) becomes

$$\begin{aligned} BB^T &= (L_j + L_{j-1}) \\ &\times \begin{pmatrix} 4\frac{(n^5+3n^4-10n^3-21n^2+27n+18)}{(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} & 6\frac{(L_j-L_{j-1})(n^3-9n+6)}{L_{j-1}(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} \\ 6\frac{(L_j-L_{j-1})(n^3-9n+6)}{L_{j-1}(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} & 12\frac{(L_j^2-L_jL_{j-1}+L_{j-1}^2)(n-3)}{L_{j-1}^2(n-2)(n^2+2n+9)(n+1)(n+2)(n+3)} \end{pmatrix}. \end{aligned}$$

The determinant of the above matrix may be written as

$$\begin{aligned} &\det(BB^T) = (L_j + L_{j-1})^2 \\ &\times \frac{12(n^3 - 9n - 18)(L_j^2 + L_{j-1}^2) + 6(5n^5 - 92n^3 + 54n^2 + 423n - 450)L_jL_{j-1}}{(L_{j-1}(n-2)(n^2+2n+9)(n+1)(n+2)(n+3))^2} \end{aligned}$$

so that (4.3) may be used to compute B.

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