MATH 6021 - Exercises

Problem 1: Let $p, q > 0$ be such that $1/p + 1/q = 1$. Prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(Use hint given in class).

Problem 2: Let (X, ρ) be a metric space. Let $A \subseteq X$. Show that $\mathrm{dist}(x, A) = 0$ if and only if $x \in \bar{A}$.

Problem 3 (Due date 1/28, Posted on 1/21): Find two closed sets $A, B \subseteq \mathbb{R}^2$ (with the usual metric) such that $A \cap B = \emptyset$ and $\mathrm{dist}(A, B) = 0$.

Problem 4: Let (X, ρ) be a metric space. Let $\sigma(x, y) = \rho(x, y)/(1 + \rho(x, y))$. Prove that (X, σ) is a metric space.

Problem 5 (Due date 1/28, Posted on 1/21): Let (X, ρ) be a metric space. Let $A \subseteq X$ prove that the function $f(x) = \mathrm{dist}(x, A)$ is continuous.

Problem 6 (Due date 1/28, Posted on 1/21): Let $\mathcal{C}[0, 1]$ denote the set of real valued continuous functions on $[0, 1]$. For $f, g \in \mathcal{C}[0, 1]$ we define $\rho(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Show that $(\mathcal{C}[0, 1], \rho)$ is a metric space

Problem 7: Let (X, ρ) be a metric space. Let $x \in X$ and $r > 0$. Show that $U_r(x)$ is open.

Problem 8: Let (X, ρ) be a metric space. Let $x \in X$ and $r > 0$. Show that the closure of $U_r(x)$ is $\bar{U}_r(x) = \{y \in X : \rho(x, y) \leq r\}$ and $bd(U_r(x)) = \{y \in X : \rho(x, y) = r\}$.

Problem 9: Let (X, ρ) be a metric space, and let $A \subseteq X$. Prove that the following conditions are equivalent:

1) A is closed
2) $U_\varepsilon(x) \cap A \neq \emptyset$ for all $\varepsilon > 0$ implies $x \in A$
3) $\mathrm{dist}(x, A) = 0$ implies $x \in A$

Problem 10: Let (X, ρ) be a metric space. Prove that:

1) $\emptyset$ and X are open
2) the intersection of two open sets is an open set
3) the union of any family (finite or infinite) of open sets is an open set

Problem 11: Let λ, ρ, σ be the following metrics on $\mathbb{R}^n$: $\rho(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$, $\lambda(x, y) = \sum_{i=1}^n |x_i - y_i|$ and $\sigma(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$. Prove that these three metrics are equivalent.

Problem 12: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x$. Prove that f is continuous.

Problem 13: 1) Let $A = [0, 1]$. Let $x_n = 0.5 + 0.5(-1)^n$. Show that (x_n) is eventually in A.

2) Let $A = [0, 1]$. Let $x_n = 1 + 0.5(-1)^n$. Show that (x_n) is frequently in A.

3) Let $x_n = 1/n$. Show that $x_n \to 0$.

4) Let $x_n = 1/n + (-1)^n$. What are the cluster points of (x_n).

Problem 14 (Due date 1/28, Posted on 1/21): 1) Show that the open intervals in $\mathbb{R}$ form a base.

2) Show that the open intervals with rational endpoints in $\mathbb{R}$ form a base.

Problem 15 (Due date 1/28, Posted on 1/21): What topology must X have if every real-valued function defined on X is continuous?

Problem 16 (Due date 2/4, Posted on 1/28): Show that in a Hausdorff topological space, every sequence converges at most to one point.

Problem 17 (Due date 2/4, Posted on 1/28): If the space is not Hausdorff, can a convergent sequence have more than one limit point? (Justify).

Problem 18: Show that every finite set in a Hausdorff space is closed.

Problem 19 (Due date 2/4, Posted on 1/28): Show that every metric space is Hausdorff.

Problem 20: Let X, Y, Z be topological spaces. Show that if X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z.

Problem 21 (Due date 2/4, Posted on 1/28): Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1$ and $(x, y) \neq (1, 0)\}$. Show that A is homeomorphic $\mathbb{R}$ and $\mathbb{R}$ is homeomorphic to $(0, 1)$.

Problem 22: Let X be a topological space. Probe that X is second countable implies that x is first countable for every $x \in X$.

Problem 23 (Due date 2/4, Posted on 1/28): Does x is first countable for every $x \in X$ imply that X is second countable?

Problem 24: Show that every metric space is first countable at every point.

Problem 25 (Due date 2/11, Posted on 1/31): Let X, Y be topological spaces. Let $f : X \to Y$ be continuous and bijective. Show that $f(G)$ is open for every open G implies that f an homeomorphism.

Problem 26: Let X, Y be topological spaces. Let $f : X \to Y$ be continuous and bijective. Does $f(F)$ is closed for every closed F imply that f an homeomorphism? (Justify).

Problem 27 (Due date 2/11, Posted on 1/31): For $x, y \in \mathbb{R}$, let $\rho(x, y) = |x - y|/(1 + |x - y|)$.

a) Show that $(\mathbb{R}, \rho)$ is a metric space

b) Is $\mathbb{R}$ bounded with this distance?

c) Is $\mathbb{R}$ totally bounded with this distance?

d) Is $\mathbb{R}$ compact with this distance?

Problem 28 (Due date 2/11, Posted on 2/1): Let X, Y be topological spaces. Let $f : X \to Y$ be continuous. Show that X compact implies $f(X)$ compact.

Problem 29 (Due date 2/11, Posted on 2/1): Let X be a metric space. Let $F \subseteq X$ closed and $K \subseteq X$ compact. Show that $F \cap K = \emptyset$ implies $\text{dist}(F, K) > 0$.

Problem 30: Let $Y = [0, 1]$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \left\{ x \in \mathbb{R} : \frac{1}{2} < x \leq 1 \right\},$$

$$B = \left\{ x \in \mathbb{R} : \frac{1}{2} < x < 1 \right\},$$

$$C = \left\{ x \in \mathbb{R} : \frac{1}{2} \leq x < 1 \right\},$$

$$D = \left\{ x \in \mathbb{R} : \frac{1}{2} \leq x \leq 1 \right\}.$$

Problem 31 (Due date 2/11, Posted on 2/1): Let X be topological space. Which of the following statements are true (prove that they are true or give a counterexample)?

a) For all $(A_\alpha)_{\alpha \in \Lambda}$, collections of subsets of X, we have $\overline{\cup_{\alpha \in \Lambda} A_\alpha} \subseteq \cup_{\alpha \in \Lambda} \overline{A_\alpha}$.

b) For all $(A_\alpha)_{\alpha \in \Lambda}$, collections of subsets of X, we have $\overline{\cup_{\alpha \in \Lambda} A_\alpha} = \cup_{\alpha \in \Lambda} \overline{A_\alpha}$.

c) For all $(A_\alpha)_{\alpha \in \Lambda}$, collections of subsets of X, we have $\cup_{\alpha \in \Lambda} \overline{A_\alpha} \subseteq \overline{\cup_{\alpha \in \Lambda} A_\alpha}$.

Problem 32: Let $(x_j) \subseteq \mathbb{R}^n$ bounded.

a) For each positive integer t, construct $F_t \subseteq \mathbb{R}^n$ such that :

1. $F_t = [a_{1t}, b_{1t}] \times \ldots \times [a_{nt}, b_{nt}]$,
2. $F_{t+1} \subseteq F_t$
3. $\text{diam}(F_t) \to 0$ (recall $\text{diam} A = \sup_{x,y \in A} \rho(x, y)$, where in this case $\rho(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.
4. $I_t = \{j : x_j \in F_t\}$ is an infinite set.

b) Let $a = (a_1, \ldots, a_n)$, where $a_i =$ least upper bound of $\{a_{i,t} : t \text{ positive integer}\}$. Let j_t be such that $j_t \in I_t$ and $j_{t+1} > j_t$. Show that $x_{j_t} \to a$.

c) Show that, if $A \subseteq \mathbb{R}^n$ is closed and $x_j \in A$ for all j, then $a \in A$.

Problem 33: Show that every convergent sequence is Cauchy.

Problem 34: Show that every Cauchy sequence is bounded.

Problem 35: Show that every Cauchy sequence with a convergent subsequence converges.

Problem 36: Show that totally bounded implies bounded.

Problem 37 (Due date 2/18, Posted on 2/7): Give an example of a bounded set that is not totally bounded.

Problem 38: Let $(x_j) \subseteq X$, where X is totally bounded. Show that we can select $(y_t) \subseteq X$ such that $I_1 = \{j : x_j \in U_1(y_1)\}$ is an infinite set and $I_t = \{j \in I_{t-1} : x_j \in U_{1/n}(y_n)\}$ is an infinite set for all $t > 1$.

Problem 39: Exercises in the notes about the completion theorem.

Problem 40: Let (X, ρ) be a metric space. Let Y and Y' be two completions of X, i.e. there exists $f : X \to Y$ and $f' : X \to Y'$ isometric embeddings such that $f(X)$ is dense in Y and $f'(X)$ is dense in Y'. Show that there exists an isometric homeomorphism $h : Y \to Y'$.

Problem 41 (Due date 2/18, Posted on 2/8): What are the connected sets in $\mathbb{R}$? (Prove your anwser).

Problem 42 (Due date 2/18, Posted on 2/8): Let X, Y be homeomorphic topological spaces. Show that X is connected if and only if Y is connected.

Problem 43: Let X, Y be topological spaces and $f : X \to Y$ an homeomorphism. Show that, for any $A \subseteq X$, $X - A$ is homeomorphic to $Y - f(A)$.

Problem 44 (Due date 2/18, Posted on 2/8): Let X, Y be topological spaces and $f : X \to Y$ continuous. Show that X is connected implies $f(X)$ is connected.

Problem 45: Show that the product topology and the one induce by the distance $\rho(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ in $\mathbb{R}^n$ are the same.

Problem 46: Show that the product of two Hausdorff spaces is Hausdorff.

Problem 47 (Due date 2/18, Posted on 2/8): Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Problem 48: $A \subseteq X$, X topological space. Recall, A^0 is the interior of A and $bd(A)$ is the boundary of A and $\overline{A}$ is the closure of A. Show that:
a) $A^0 \cap bd(A) = \emptyset$,
b) $\overline{A} = A^0 \cup bd(A)$,
c) $bd(A) = \emptyset$ if and only if A is both open and closed,
d) A is open if and only if $bd(A) = \overline{A} - A$.

Problem 49: Show that:
a) $[0, 1]$ is arcwise connected.
b) $\mathbb{R}^2 - \{(0, 0)\}$ is arcwise connected.
c) $\mathbb{R}^2 - \mathbb{Q}^2$ is arcwise connected.
d) $\{(x, \sin(1/x) : x > 0\} \cup \{(0, 0)\}$ is connected but not arcwise connected.

Problem 50 (Due date 2/25, Posted on 2/11): Show that $\mathbb{R}^n$ is locally path connected.

Problem 51 (Due date 2/25, Posted on 2/11): Let X be a topological space. Let $f : [0, 1] \to X$ be a path and $h : [0, 1] \to [0, 1]$ an homeomorphism with $h(0) = 0$. Show that f and $f \circ h$ are homotopic.

Problem 52: Show that the homotopic relation is an equivalence relation.

Problem 53 (Due date 2/25, Posted on 2/11): Let $X = A \cup B$, where A and B are closed. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Let

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Show that h is well defined and h is continuous.

Problem 54: Let $f, g : [0, 1] \to X$ paths such that $f(1) = g(0)$. Show that $f \cdot g$ is a path, i.e. $f \cdot g : [0, 1] \to X$ is a continuous function.

Problem 55 (Due date 2/25, Posted on 2/11): Let $F, G : [0, 1]^2 \to X$ be continuous functions such that $F(1, s) = G(0, s)$ for all $s \in [0, 1]$. Let $H : [0, 1]^2 \to X$ be defined by

$$H(t, s) = \begin{cases} F(2t, s) & \text{if } t \in [0, 1/2] \\ G(2t - 1, s) & \text{if } t \in [1/2, 1]. \end{cases}$$

Show that H is a continuous function.

Problem 56: Let $f, g, h : [0, 1] \to X$ be paths such that $f(1) = g(0)$ and $g(1) = h(0)$. Show that $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$.

Problem 57: Show that $[e_x] \cdot [f] = [f] \cdot [e_y] = [f]$ for all paths $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Problem 58 (Due date 2/25, Posted on 2/16): Let $X \subseteq \mathbb{R}^n$ convex and $x \in X$. Show that $\pi(X, x) = 0$.

Problem 59: Let X, Y topological spaces and $h : X \to Y$ continuous. Show that $h_\star$ defined in class is well defined and is a group homomorphism.

Problem 60: Let X, Y topological spaces and $h : X \to Y$ continuous. Show an example where h is 1-1 but $h_\star$ is not.

Problem 61 (Due date 3/4, Posted on 2/22): Let X, Y topological spaces and $h : X \to Y$ continuous. Show an example where h is onto but $h_\star$ is not.

Problem 62: Find a retraction that is not a deformation retraction.

Problem 63: Show that if r is a deformation retraction, the $r_\star$ is an isomorphism.

Problem 64: Let $\| \cdot \|$ be a norm. Show that $\rho(x, y) = \|x - y\|$ is a distance.

Problem 65: Show that the following are norms in $\mathcal{C}[0, 1]$: 1) $\|f\| = \max_{x \in [0,1]} |f(x)|$ and 2) $\|f\| = \sqrt{\in_0^1 |f(x)|^2 \mathrm{d}x}$.

Problem 66: Show that in a normed space $U_r(x) = x + rU_1(0)$.

Problem 67 (Due date 3/4, Posted on 2/22): Show that $(B(X,Y), \|\cdot\|)$ is a normed space, where, as defined in class, $\|T\| = \inf_{\{k\geq 0 : \|T(x)\| \leq K\|x\| \text{ for all } x\}} K$.

Problem 68 (Due date 3/4, Posted on 2/22): Let $X = \mathcal{C}[0,1]$ (real valued) with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Let $T : X \to X$ be defined as $T(f)(x) = \int_0^x f(t)\mathrm{d}t$. Show that $T \in B(X,X)$ and compute $\|T\|$.

Problem 69 (Due date 3/4, Posted on 2/25): Let X be a real normed space. Let $f : X \to \mathbb{R}$ be a linear functional. Show that $\mathrm{Ker}(f) = f^{-1}(\{0\})$ is a subspace of codimension 1.

Problem 70 (Due date 3/4, Posted on 2/25): Let X be a real normed space. Let $f : X \to \mathbb{R}$ be a linear functional. Show the following:

1. f is bounded if and only if $\mathrm{Ker}(f)$ is closed
2. f is not bounded if and only if $\mathrm{Ker}(f)$ is dense.

Problem 71 (Due date 3/18, Posted on 3/11): Is $X = \mathcal{C}[0,1]$ (real valued) with the inner product $\int_0^1 f(x)g(x)\mathrm{d}x$ a Hilbert space? Justify.

Problem 72 (Due date 3/18, Posted on 3/11): Show that A and B convex implies $A+B$ is convex.

Problem 73 (Due date 3/18, Posted on 3/11): We say that $w_0, \ldots, w_k$ is affinely independent if $\sum_{i=0}^k \lambda_i w_i = 0$ with $\sum_{i=0}^k \lambda_i = 0$ implies $\lambda_i = 0$ for all i.

Let W be an affine subspace. Prove that the $\dim W = k$ if and only if there exists $w_0, \ldots, w_k$ affinely independent such that $W = \mathrm{aff}\{w_0, \ldots, w_k\}$.

Problem 74 (Due date 3/18, Posted on 3/11): Prove that the convex hull of $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Problem 75 (Due date 3/18, Posted on 3/11): Show that in $\mathbb{R}^n$, the convex hull of a compact set is compact and the convex hull of an open set is open.

Problem 76 (Due date 4/8, Posted on 3/31): Let $A \subseteq X$ convex. Let $x, y \in X - A$. Prove that $x \in \mathrm{conv}(A \cup \{y\})$ and $y \in \mathrm{conv}(A \cup \{x\})$ then $x = y$.

Problem 77 (Due date 4/8, Posted on 3/31): Let $w_0, \ldots, w_n \in \mathbb{R}^n$ be affinely independent. Show that $\mathrm{conv}(\{w_0, \ldots, w_n\})$ has non-empty interior.

Problem 78 (Due date 4/8, Posted on 3/31): Let $S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1/4,1/4,1/4)\}$ What are the faces of $\mathrm{conv}S$?

Problem 79 (Due date 4/8, Posted on 3/31): Which of the following sets is convex:
a) $\{(x,y) : y \geq 2x^2 \text{ and } y \leq x^2 + 1\}$
b) $\{(x,y) : y \geq 2x^2 \text{ and } y \leq -x^2 + 1\}$

Problem 80: Let

$$Q = [-1,1]^2 = \{(x,y) : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}.$$

Let f be the projection operator of Q. What if $f(x, y)$ (i.e. give the formula for f)?

Problem 81: Let $A \subseteq \mathbb{R}^n$ be closed and convex. Let f be the projection operator of A. Let $x, y \in \mathbb{R}^n$ and $a \in A$. Assume that $f(x) = f(y) = a$ prove that $f(\lambda x + (1-\lambda)y) = a$ for all $0 \leq \lambda \leq 1$.

Problem 82: Let

$$A = \{(x, y) : y \geq 2x^2 \text{ and } y \leq x^2 + 1\} \text{ and } B = \{(0, 3/2))\}.$$

Prove that A and B can not be separated by a hyperplane.

Problem 83 (Due date 4/8, Posted on 3/31): What are the faces and the primitive faces of

$$\{(x, y) : -1 < x \leq 1,\ x^2 \leq y \text{ and } x \geq y\}.$$

Problem 84 (Due date 11/8, Posted on 3/8): Let $S \subseteq \mathbb{R}^n$. Show that

$$\left\{ \sum_{i=1}^{m} \lambda_i a_i : a_i \in S \text{ and } \lambda_i \geq 0 \text{ for all } i \right\}$$

is a cone.

Problem 85 (Due date 11/8, Posted on 3/8): Let $A, B \subseteq \mathbb{R}^n$. A compact and B closed implies $A + B$ closed.