Obs: all the solutions of eq (1) are the linear combinations of the n solutions we found last lecture.

Example 1) $y'' - 3y' + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0 \qquad \lambda = \frac{3 \pm \sqrt{9-4(2)}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$

Solutions: e^x, e^{2x}

General solution: $y(x) = c_1 e^x + c_2 e^{2x} \qquad c_1, c_2 \in \mathbb{R}$

2) $y'' - 4y' + 4y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$

$\lambda^2 - 4\lambda + 4 = 0 \qquad (\lambda - 2)^2 = 0$

$\lambda_1 = 2 \qquad n_1 = 2 \qquad$ Solutions: e^{2x}, xe^{2x}

$y(x) = c_1 e^{2x} + c_2 x e^{2x}$

$y(0) = c_1 = 0 \qquad y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$

$y'(0) = 2c_1 + c_2 = c_2 = 1$

$\boxed{y = xe^{2x}}$

3) $y'' + 2y' + 2y = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda_1 = -1 + i \qquad \lambda_2 = -1 - i$

$e^{(-1+i)x} = e^{-x}\cos x + i e^{-x} \sin x$

$y_1(x) = e^{-x} \cos x$

$y_2(x) = e^{-x} \sin x$

$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

$\lambda_1 = -1+i \qquad \lambda_2 = -1-i$

Keep $\lambda_1 = -1+i$, $n_1 = 1$

$e^{\lambda_1 x} = e^{(-1+i)x}$

$y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$

4) $y^{(4)} + 2y'' + y = 0$

$\lambda^4 + 2\lambda^2 + 1 = 0$

$(\lambda^2+1)^2 = 0$

$(\lambda+i)^2(\lambda-i)^2 = 0$

$\lambda_1 = i \qquad \lambda_2 = -i$

$n_1 = 2 \qquad n_2 = 2$

Solutions: e^{ix}, $x e^{ix}$ (complex)

Real solutions: $\cos x$, $\sin x$, $x\cos x$, $x\sin x$

Inhomogeneous linear constant coefficient odes

(2) $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f$

$a_0, \ldots, a_{n-1} \in \mathbb{R} \qquad f = x^{\ell} e^{\beta x}$

ℓ is a non-negative integer, $\beta \in \mathbb{C}$

First goal: Find one solution of eq. (2)

Step 1: Check if β is a root of $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$

Let $k=0$ if β is not a root of $P(\lambda)$

Otherwise, let $k =$ multiplicity of β as a root of $P(\lambda)$

Set $\boxed{y = x^k (c_0 + c_1 x + \cdots + c_\ell x^\ell)\, e^{\beta x}}$

Plug y into eq. (2) and $c_0, \dots, c_\ell$ that give you a solution.

Examples: 1) $y' + y = e^{3x}$

$\beta = 3$ | $P(\lambda) = \lambda + 1$ | $3ce^{3x} + ce^{3x} = e^{3x}$

$k = 0$ | $y = ce^{3x}$ | $4c = 1 \quad c = \frac{1}{4}$

$\ell = 0$ | | $\boxed{y = \frac{1}{4} e^{3x}}$

2) $y'' + y = xe^{ix}$

$\beta = i$ | $P(\lambda) = \lambda^2 + 1$ | $y = (c_0 x + c_1 x^2) e^{ix}$

$\ell = 1$ | $P(\lambda) = (\lambda - i)(\lambda + i)$ | $y' = (c_0 + 2c_1 x) e^{ix} + i(c_0 x + c_1 x^2) e^{ix}$

$k = 1$ | $y = x(c_0 + c_1 x) e^{ix}$ |

$$y' = (c_0 + (2c_1 + ic_0)x + ic_1 x^2)\, e^{ix}$$

$$y'' = \left[(2c_1 + ic_0) + 2ic_1 x\right] e^{ix} + (c_0 + (2c_1 + ic_0)x + ic_1 x^2)\, ie^{ix}$$

$$y'' = \left[(2c_1 + 2ic_0) + (-c_0 + 4ic_1)x - c_1 x^2\right] e^{ix}$$

$$y'' + y = \left[(2c_1 + 2ic_0) + 4ic_1 x\right] e^{ix} = xe^{ix}$$

$$(2c_1 + 2ic_0) + 4ic_1 x = x$$

$$2c_1 + 2ic_0 = 0 \Rightarrow c_0 = ic_1 = \frac{1}{4}$$

$$4ic_1 = 1 \Rightarrow c_1 = \frac{-i}{4}$$

$$4ic_1 = 1 \Rightarrow c_1 = \frac{-1}{4}$$

$$y = x\left(\frac{1}{4} - \frac{i}{4}x\right)e^{ix}$$

<u>Obs</u>: $f(x) = f_1(x) + i f_2(x)$ f_1 & f_2 real valued.

Let $y = y_1 + i y_2$. Then, y is a solution of

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f$$

If and only if y_1 is a solution

$$y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_1 y_1' + a_0 y_1 = f_1$$

and y_2 is a solution of

$$y_2^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_1 y_2' + a_0 y_2 = f_2$$

<u>Example</u>: $y'' + y = x\cos x$

Note $x\cos x = \mathrm{Re}(xe^{ix})$

First solve $z'' + z = xe^{ix}$, then $y = \mathrm{Re}(z)$

One solution is $z = \frac{x}{4}(1 - ix)e^{ix}$. Then, one solution

of $y'' + y = x\cos x$ is $y = \mathrm{Re}(z) = \frac{x}{4}(\cos x + x\sin x)$

<u>Obs</u>: To find a solution of

(a) $y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = x^l e^{ax}\cos(bx)$ $a, b \in \mathbb{R}$

(3) $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = x^{\ell} e^{ax} \cos(bx)$

$a, b \in \mathbb{R}$

Note $x^{\ell} e^{ax} \cos bx = \mathrm{Re}\left(x^{\ell} e^{(a+ib)x}\right)$. First find a solution of $z^{(n)} + \dots + a_0 z = x^{\ell} e^{(a+bi)x}$

Then $y = \mathrm{Re}(z)$ is a solution of (3).

If we have $\sin bx$ instead of $\cos bx$, same steps, but $y = \mathrm{Im}(z)$

<u>Example</u>: $y' + 2y = \sin(3x) = \mathrm{Im}\left(e^{3ix}\right)$

First solve $z' + 2z = e^{3ix}$

$P(\lambda) = \lambda + 2$

$\ell = 0$

$\beta = 3i$

$k = 0$

$z = C e^{3ix}$

$3iC e^{3ix} + 2C e^{3ix} = e^{3ix}$

$C(2 + i3) = 1 \qquad C = \frac{1}{(2+3i)} \frac{(2-3i)}{(2-3i)}$

$C = \frac{2}{13} - \frac{3i}{13} \qquad z = \left(\frac{2}{13} - \frac{3}{13} i\right) e^{3ix}$

$$\boxed{y = \mathrm{Im}(z) = \frac{2}{13} \sin 3x - \frac{3}{13} \cos 3x}$$

Obs: If $y_1^{(n)} + \dots + a_0 y_1 = f_1$

Obs: If $y_1^{(n)} + \dots + a_0 y_1 = f_1$

and $y_2^{(n)} + \dots + a_0 y_2 = f_2$

then $y = c_1 y_1 + c_2 y_2$ satisfies

$$y^{(n)} + \dots + a_0 y = c_1 f_1 + c_2 f_2$$

Example: To find a solution of $\boxed{y'' + 2y = 2xe^{3x} - 4\sin x}$ (*).

First find y_1 solution of $y_1'' + 2y_1 = xe^{3x}$

then find y_2 solution of $y_2'' + 2y_2 = \sin x$

Then, $y = 2y_1 - 4y_2$ is a solution of (*)

Theorem: Let y_p be a solution of

(1) $y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \dots + a_0 y_p = f$

Then, y is a solution of eq. (1) $\iff y = y_p + y_h$,

where y_h is a solution of

(2) $y_h^{(n)} + a_{n-1} y_h^{(n-1)} + \dots + a_0 y_h = 0$

Example: $y'' - 3y' + 2y = xe^{x} \sin x$

1st Find a particular solution

1^{st} Find a particular solution

$$z'' - 3z' + 2z = xe^{(1+i)x} \qquad xe^x \sin x = \mathrm{Im}\left(xe^{(1+i)x}\right)$$

$$\ell = 1 \qquad \lambda^2 - 3\lambda + 2 = P(\lambda) \qquad z = (c_0 + c_1 x)\, e^{(1+i)x}$$

$$\beta = 1+i \qquad (1+i)^2 - 3(1+i) + 2 \neq 0$$

$$k = 0$$

$$z' = \left[c_1 + (c_0 + c_1 x)(1+i)\right] e^{(1+i)x}$$

$$z' = \left\{\left[c_1 + c_0(1+i)\right] + c_1(1+i)x\right\} e^{(1+i)x}$$

$$z'' = \left[c_1(1+i) + \left[c_1 + c_0(1+i)\right](1+i) + c_1(1+i)^2 x\right] e^{(1+i)x}$$

$$z'' = \left(2c_1(1+i) + 2ic_0 + 2ic_1 x\right) e^{(1+i)x}$$

$$z'' - 3z' + 2z = xe^{(1+i)x}$$

$$\left(\underline{2c_1(1+i)} + \underline{2ic_0} + \underline{2ic_1 x}\right) - 3\left[\left(\underline{c_1} + \underline{c_0(1+i)}\right) + c_1(1+i)x\right] + 2\left(\underline{c_0} + \underline{c_1 x}\right) = x$$

$$c_1(-1+2i) + c_0(-1-i) + (-1-i)c_1 x = x$$

$$\boxed{c_1 = \frac{-1}{1+i} = \frac{-(1-i)}{2} = -\frac{1}{2} + \frac{i}{2}}$$

$$c_1(-1+2i) + c_0(-1-i) = 0 \qquad c_0 = \frac{(-1+i)}{2}\frac{(-1+2i)}{1+i} \cdot \frac{(1-i)}{(1-i)}$$

$$\boxed{c_0 = \frac{(-1)}{4}(-2i)(-1+2i) = \frac{i}{2}(-1+2i) = -1 - \frac{i}{2}}$$

$$\boxed{c_0 = \frac{(-1)}{4}(-2i)(-1+2i) = \frac{i}{2}(-1+2i) = -1 - \frac{i}{2}}$$

$$z = (c_0 + c_1 x)\, e^{(1+i)x}$$

$$z = \left[\left(-1-\frac{i}{2}\right) + \left(-\frac{1}{2}+\frac{i}{2}\,x\right)\right] e^{x}(\cos x + i \sin x)$$

$$y_p = \operatorname{Im}(z) = e^{x}\left\{\left(1-\frac{1}{2}x\right)\sin x + \frac{(1+x)}{2}\cos x\right\}$$

all the solutions

$$\boxed{y = y_p + c_1 e^{x} + c_2 e^{2x}}$$