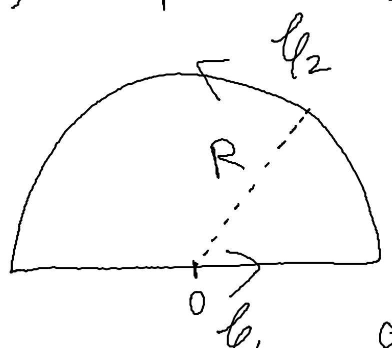
1) Compute $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



$f(z) = \frac{1}{z^4+1} = \frac{P(z)}{Q(z)}$

$P(z) = 1 \quad Q(z) = z^4+1$

$\deg Q = 4 \geq \underbrace{\deg(P)}_{=0} + 2 = 2$

then $\int_{\mathcal{C}_2} \frac{P(z)}{Q(z)} dz = \int_{\mathcal{C}_2} \frac{1}{z^4+1} dz \longrightarrow 0$ as $R \to \infty$

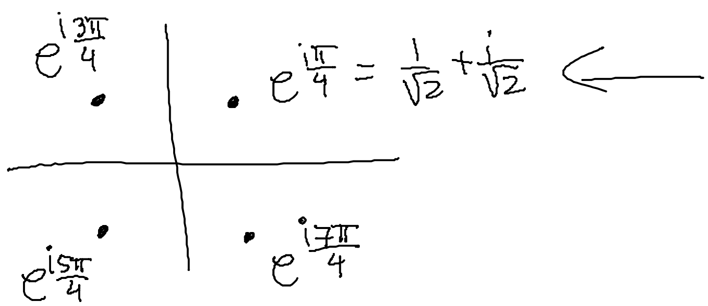
$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. then $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \lim_{R\to\infty} \int_{\mathcal{C}} \frac{dz}{z^4+1} =$

$= 2\pi i \sum_{\operatorname{Im}(z_k)>0} \operatorname{Res}\left(\frac{1}{z^4+1}, z_k\right)$

z_k singularity of $\frac{1}{z^4+1}$

$z^4+1=0 \qquad z^4=-1 \qquad z^4 = e^{i\pi} \qquad z = e^{i\left(\frac{\pi}{4}+\frac{2\pi k}{4}\right)}$

$0 \leq k \leq 3$



$\operatorname{Res}\left(\frac{1}{z^4+1}, e^{\frac{i\pi}{4}}\right) = \lim_{z\to e^{\frac{i\pi}{4}}} \frac{z-e^{\frac{i\pi}{4}}}{z^4+1} = \lim_{z\to e^{\frac{i\pi}{4}}} \frac{1}{4z^3} =$

l'hopital

$$= \frac{1}{4} e^{-\frac{i3\pi}{4}}$$

$$\mathrm{Res}\left(\frac{1}{z^4+1}, e^{\frac{i3\pi}{4}}\right) = \frac{1}{4} e^{-\frac{i9\pi}{4}} = \frac{e^{-\frac{i\pi}{4}}}{4}$$

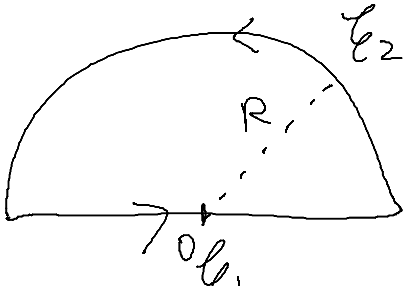
$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^4+1}} = 2\pi i \left\{ \frac{1}{4} e^{-\frac{i3\pi}{4}} + \frac{1}{4} e^{-\frac{i\pi}{4}} \right\} = \frac{\pi i}{2}\left\{ -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right\}$$

$$= \boxed{\frac{\pi}{\sqrt{2}}}$$

2) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+9} dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9} dx$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9} dx = \int_{-\infty}^{\infty} f(x) e^{iax} dx$$

$a > 0 \quad a = 1$ ✓ $\quad f(z) = \frac{1}{z^2+9}$

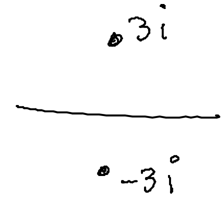


$$M_R = \sup_{z \in \mathscr{C}_2} |f(z)| = \max_{z \in \mathscr{C}_2} \left| \frac{1}{z^2+9} \right| =$$

$$= \frac{1}{R^2-9} \longrightarrow 0 \quad \text{as } R \to \infty$$

then $\int_{\mathscr{C}_2} f(z) e^{iaz} dz \longrightarrow 0$ as $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}\left(f(z)\, e^{iaz}, z_k\right)$$



$f(z) = \dfrac{1}{z^2+9}$ $\quad z^2+9=0 \Rightarrow z = \pm 3i$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9}\, dx = 2\pi i \operatorname{Res}\left(\frac{e^{ix}}{x^2+9}, 3i\right) =$$

$$2\pi i \lim_{z \to 3i} \frac{e^{iz}}{z^2+9}(z-3i) = 2\pi i \left.\frac{e^{iz}}{z+3i}\right|_{z=3i} = \frac{\pi}{3} e^{-3}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+9}\, dx = \frac{\pi}{3} e^{-3}$$

3) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\, dx$

$\displaystyle\int_{-\infty}^{\infty} f(z)\, e^{iaz}\, dz \qquad f(z) = \frac{1}{z(z^2-2z+2)}$

$a > 0 \quad a = 1$ ✓

$M_R = \max_{\substack{|z|=R \\ \operatorname{Im}(z) \geq 0}} f(z) \approx \dfrac{1}{R^3} \xrightarrow[R\to\infty]{} 0$ ✓

Singularities in real axis $x=0$, $x=0$ is a simple pole. ✓

$z(z^2-2z+2)=0$ then $\boxed{z=0}$ or

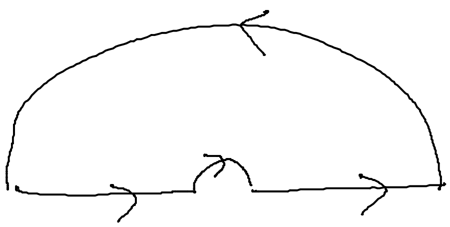
$z^2-2z+2=0 \qquad (z-1)^2+1=0 \qquad (z-1)^2=-1$

$z^2 - 2z + 2 = 0 \qquad (z-1)^2 + 1 = 0 \qquad (z-1)^2 = -1$

$z - 1 = \pm i \qquad \boxed{z = 1 \pm i}$

Singularity with positive imaginary part $1+i$

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx = \pi i \,\text{Res}\left(f(z)\, e^{iaz}, 0\right) + 2\pi i \,\text{Res}\left(\frac{e^{iz}}{z(z^2-2z+2)}, 1+i\right)$$



$$= \pi i \lim_{z\to 0} \frac{e^{iz}}{z(z^2-2z+2)} \cdot z + 2\pi i$$

$$\lim_{z\to 1+i} \frac{e^{iz}}{z(z^2-2z+2)}\left(z-(1+i)\right) =$$

$$= \frac{\pi i}{2} + 2\pi i \lim_{z\to 1+i} \frac{e^{iz}}{z(z-(1-i))} = \frac{\pi i}{2} + \frac{2\pi i\, e^{-1+i}}{(1+i)\,2i} =$$

$$= \frac{\pi}{2} i + \frac{\pi(1-i)}{2} e^{-1}(\cos 1 + i \sin 1) = \frac{\pi i}{2} + \frac{\pi}{2}e^{-1}\left(\cos 1 + \sin 1 + i(\sin 1 - \cos 1)\right)$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}\, dx = \frac{\pi}{2} i + \frac{\pi}{2} e^{-1}\left(\cos 1 + \sin 1 + i(\sin 1 - \cos 1)\right)$$

take Im

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\, dx = \frac{\pi}{2} + \frac{\pi}{2} e^{-1}(\sin 1 - \cos 1)$$

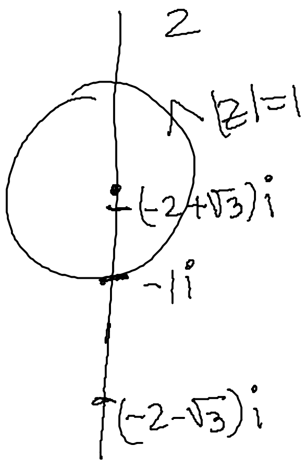
4) $$\int_0^{2\pi} \frac{d\theta}{1 + \frac{1}{2}\sin\theta} = \oint_{|z|=1} \frac{\left(-\frac{i}{z}\right) dz}{1 + \frac{1}{2}\left(\frac{z - \frac{1}{z}}{2i}\right)} =$$

$$z = e^{i\theta}$$
$$dz = i e^{i\theta}\, d\theta = i z\, d\theta$$
$$d\theta = \frac{-i}{z}\, dz$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$= (-i) \oint_{|z|=1} \frac{dz}{z + \frac{1}{2}\frac{(z^2-1)}{2i}} = 4 \oint_{|z|=1} \frac{dz}{z^2 + 4iz - 1} =$$

$$z^2 + 4iz - 1 = 0$$

$$\frac{-4i \pm \sqrt{-16+4}}{2} = \frac{-4i \pm i\sqrt{12}}{2} = (-2 \pm \sqrt{3})\, i$$



$$= 4\, 2\pi i \,\mathrm{Res}\left(\frac{1}{z^2+4iz-1}, (-2+\sqrt{3})\, i\right) =$$

$$= 8\pi i \lim_{z \to (-2+\sqrt{3})i} \frac{(z - (-2+\sqrt{3})i)}{z^2 + 4iz - 1} =$$

$$= 8\pi i \left. \frac{1}{(z - (-2-\sqrt{3})i)} \right|_{z = (-2+\sqrt{3})i} = \frac{8\pi i}{2\sqrt{3}\, i} = \boxed{\frac{4\pi}{\sqrt{3}}}$$