A non-linear homogenization problem motivated by composites

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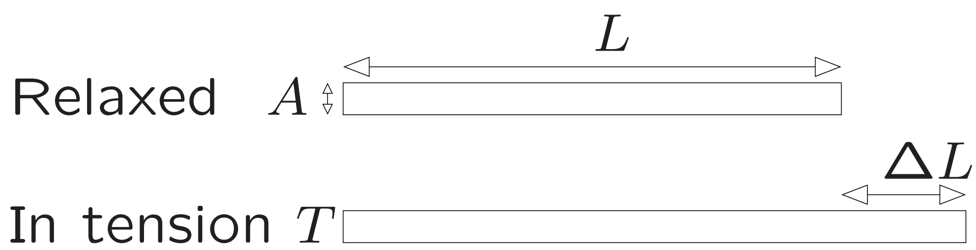
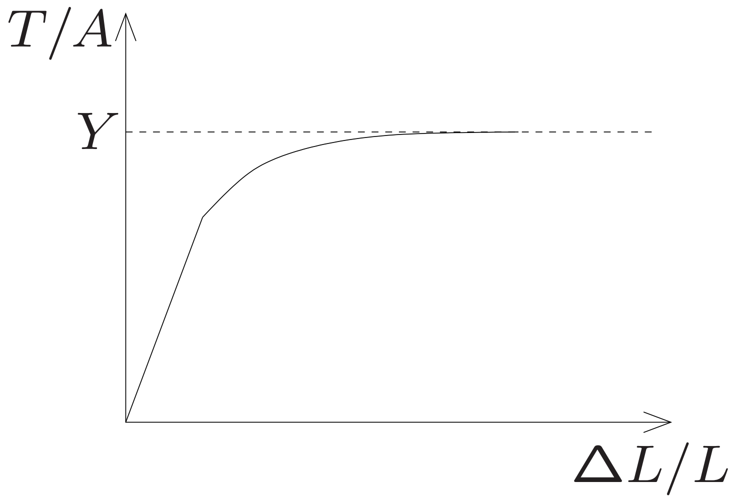
Perfectly plastic solids (1-D)





Always: $T/A \leq Y$.

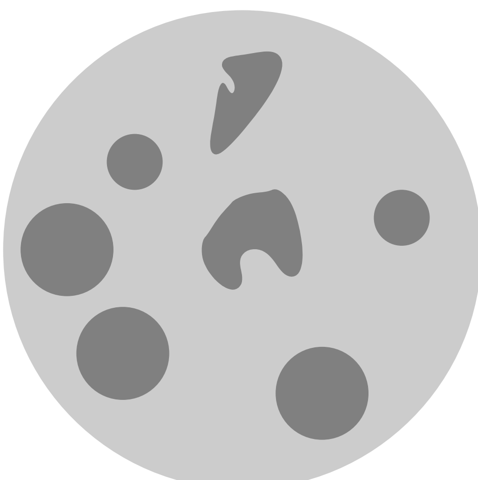
Y = **yield stress**.

Composites

Two pure solid phases bounded together. The inclusions, dark, inside the matrix, light.

The yield stress in the dark region is much larger than the yield stress in the light region.

The dark material is much stronger than the light material.



The toy mathematical problem

$$Y_{\mathbf{x}} = \begin{cases} Y_i & \text{if } \mathbf{x} \text{ in inclusion} \\ Y_m & \text{if } \mathbf{x} \text{ in matrix.} \end{cases}$$

Notation: $Q = [0,1]^2$.

$\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is admissible if it is $Q-$periodic; $|\sigma(\mathbf{x})| \leq Y_{\mathbf{x}}$; $\nabla\cdot\sigma = 0$.

Notation: If β is Q-periodic, then $\langle\beta\rangle = \int_Q \beta$.

$$K_{\text{hom}} = \{\tau = \langle\sigma\rangle : \sigma \text{ admissible}\}.$$

Goal: Find the set K_{hom}.

Obs: K_{hom} is convex.

Observations

Obs 1: K_{hom} depends on the function $Y_{\mathbf{x}}$. The geometry, Y_i and Y_m.

Obs 2: K_{hom} represent the forces the composite can withstand.

Def: $\widehat{n}$ unit vector. Strength in the direction $\widehat{n} = \sup\{\rho : \rho\widehat{n} \in K_{\mathsf{hom}}\}$

Obs 3: We want to reinforce a weak matrix with strong inclusions, $Y_i \gg Y_m$, to get a composite strong in all directions, i.e. K_{hom} to contain ball of radius Y, where $Y \gg Y_m$.

Obs 4: σ can be regarded as the velocity field of an incompressible fluid of constant density. Fluid can flow fast inside the inclusions, but must flow slow in the matrix

A bound on the weakest direction

$\rho^\star = \max\{\rho : B_\rho \subseteq K_{\text{hom}}\}$ (strength in the weakest direction).

ν = area occupied by inclusions in $[0,1]^2$ (volume fraction of inclusions).

I inclusion, then p_I = perimeter of I and a_I = area of I.

$$\rho^\star \leq Y_m \sqrt{\nu\eta + 1 - \nu}, \qquad \text{where} \qquad \eta = \frac{1}{2} \max \left\{ \frac{p_I^2}{a_I} : I \text{ inclusion} \right\}.$$

Obs: The bound seems to be pretty sharp

Observations on the bound

If I is a circular inclusion with radius R, then $p_I = 2\pi R$, $a_I = \pi R^2$. Thus, if all the inclusions are circular, $\eta = 2\pi$ and the bound gives

$$\rho^\star \leq Y_m \sqrt{\nu 2\pi + 1 - \nu}.$$

Circular inclusions can not reinforce much.

If I is rectangular with width w and length ℓ, then $p_I = 2(w + \ell)$, $a_I = w\ell$. If all the inclusions are rectangular with the same length to width ratio, then $\eta \sim 2\ell/w$ and the bound gives

$$\rho^\star \leq Y_m \sqrt{\nu 2\frac{\ell}{w} + 1 - \nu} \sim Y_m \sqrt{\nu 2\frac{\ell}{w}}.$$

Slender inclusions can reinforce the matrix.

Proof of the bound

Notation: $v^\perp = (-v_2, v_1)$ (rotate v $\pi/2$ counterclockwise)

Claim 1: Let σ and β be Q-periodic and divergence free. Then $\langle \sigma \cdot \beta^\perp \rangle = \langle \sigma \rangle \cdot \langle \beta \rangle^\perp$.

proof: $\sigma = \langle \sigma \rangle + (\nabla \phi)^\perp$ with ϕ Q-periodic. $\beta = \langle \beta \rangle + (\nabla \psi)^\perp$ with ϕ Q-periodic. Note $\int_Q \langle \sigma \rangle \cdot \nabla \psi = \int_Q \nabla \phi \cdot \langle \beta \rangle = 0$

$$|Q| \langle \sigma \cdot \beta^\perp \rangle = \int_Q \sigma \cdot \beta^\perp = |Q| \langle \sigma \rangle \cdot \langle \beta \rangle^\perp - \int_Q \nabla \psi \cdot (\nabla \phi)^\perp$$

$$\int_Q \nabla \psi \cdot (\nabla \phi)^\perp = \int_{\partial Q} \psi \hat{n} \cdot (\nabla \phi)^\perp = 0$$

Claim 2: Let I be and inclusion and σ and β admissible. Then $|\int_I \sigma \cdot \beta^{\perp}| \leq (1/2) Y_m^2 p_I^2$.

proof: $\sigma = (\nabla\phi)^{\perp}$; $\beta = (\nabla\psi)^{\perp}$; $\hat{t} = \hat{n}^{\perp}$. Parametrize ∂I with $\mathbf{x}(s)$ so that $\|\mathbf{x}'(s)\| = 1$ and $\psi(0) = 0$.

$$\begin{aligned}
\int_I \sigma \cdot \beta^{\perp} = -\int_I \nabla\psi \cdot (\nabla\phi)^{\perp} = -\int_I \nabla \cdot \left(\psi(\nabla\phi)^{\perp}\right) = \\
-\int_{\partial I} \hat{n} \cdot \left(\psi(\nabla\phi)^{\perp}\right) = \int_{\partial I} \psi \ \hat{t} \cdot \nabla\phi = \\
\int_0^p \mathrm{d}s \ \hat{t}(s) \cdot \nabla\phi(s) \int_0^s \mathrm{d}r \ \hat{t}(r) \cdot \nabla\psi(r) = \\
\int_0^p \mathrm{d}s \ \hat{n}(s) \cdot \sigma(s) \int_0^s \mathrm{d}r \ \hat{n}(r) \cdot \beta(r)
\end{aligned}$$

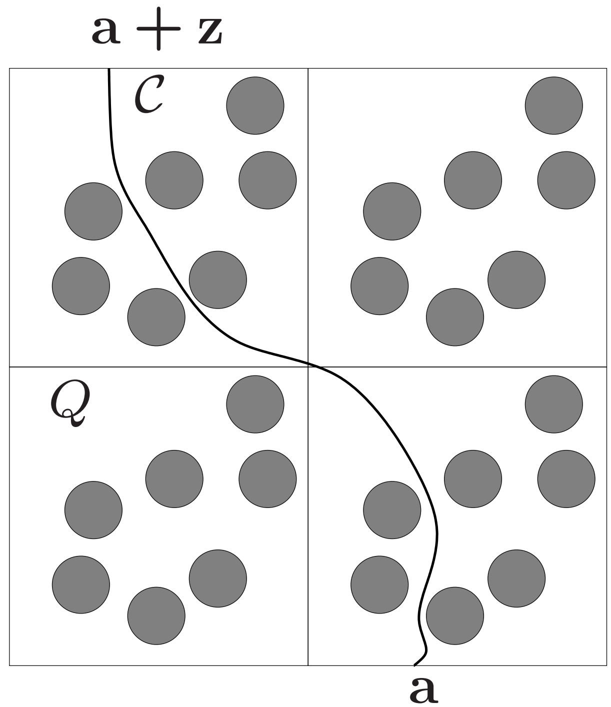
$$\left|\int_I \sigma \cdot \beta^{\perp}\right| \leq \int_0^p \mathrm{d}s \ Y_m^2 s = \frac{1}{2} Y_m^2 p^2$$

Claim 3: σ and β admissible. Then $\langle\sigma\rangle \cdot \langle\beta\rangle^{\perp} \leq (1-\nu)Y_m^2 + |Q|^{-1}(1/2)Y_m^2 \sum_I p_I^2$. (Sum over one I per period cell).

Claim 4: Let $\eta = (1/2)\max_I\{p_I^2/a_I\}$. σ and β admissible. Then $\langle\sigma\rangle \cdot \langle\beta\rangle^{\perp} \leq (1-\nu)Y_m^2 + Y_m^2\nu\eta$.

Claim 5: Select $\sigma = \rho^\star(1,0)$ and $\beta = \rho^\star(0,-1)$. Then $\rho^\star \leq Y_m\sqrt{(1-\nu)+\nu\eta}$

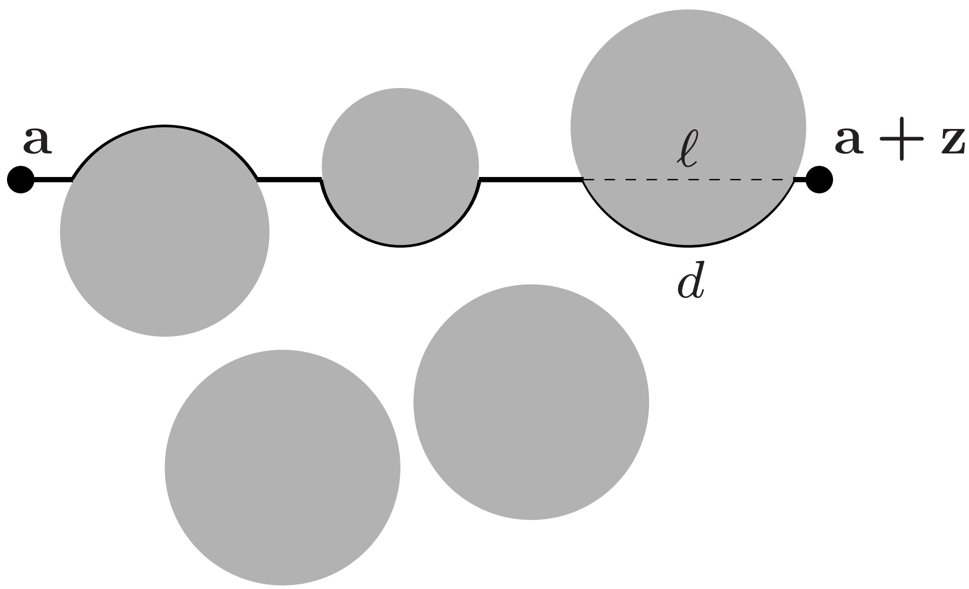
Flow across curves and bound on K_{hom}



If $\mathcal{C}$ included in matrix then,

$$\left|\langle\sigma\rangle\cdot\mathbf{z}^{\perp}\right| = \left|\int_{\mathcal{C}}\sigma\cdot\hat{n}\right| \leq Y_m \times \text{ length of } \mathcal{C}, \qquad \text{for all } \sigma \text{ admissible}$$

Circular rigid inclusions ($Y_i \gg Y_m$) can not reinforce much



$\mathcal{C}$ = solid curve

$\frac{d}{\ell} \leq \frac{\pi}{2}$

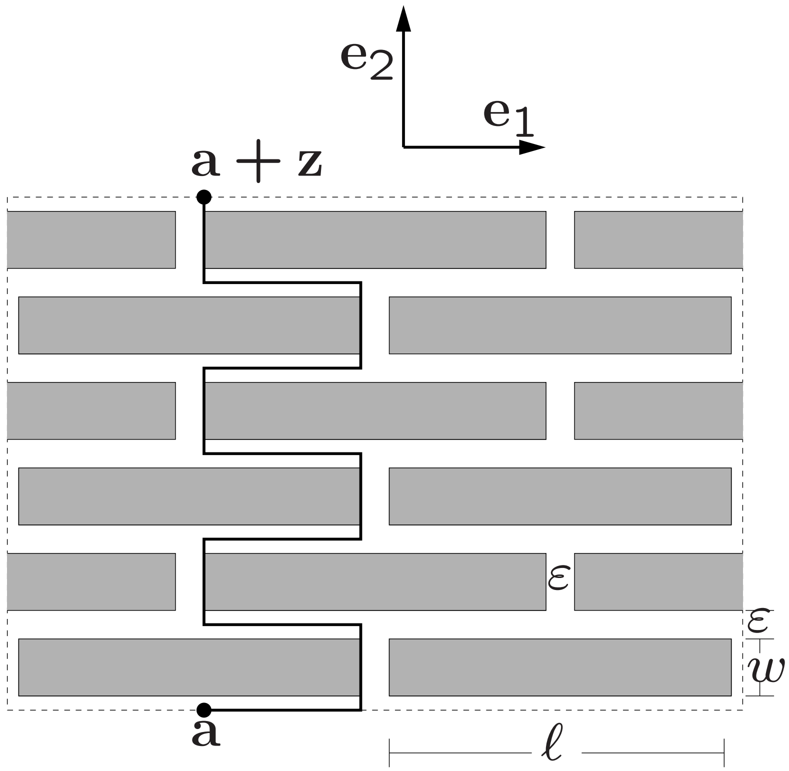
length of $\mathcal{C} \leq \frac{\pi}{2} \|\mathbf{z}\|$

$|\tau \cdot \mathbf{z}^\perp| \leq Y_m \frac{\pi}{2} \|\mathbf{z}\|$

$$K_{\text{hom}} \subseteq \left\{ \|\tau\| \leq \frac{\pi}{2} Y_m \right\}$$

No matter how hard are the inclusion, the yield stress is limited by the strength of the matrix.

Example 2. Thin long rectangular rigid inclusions.



$\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = 1$

$\mathcal{C}$ = solid curve

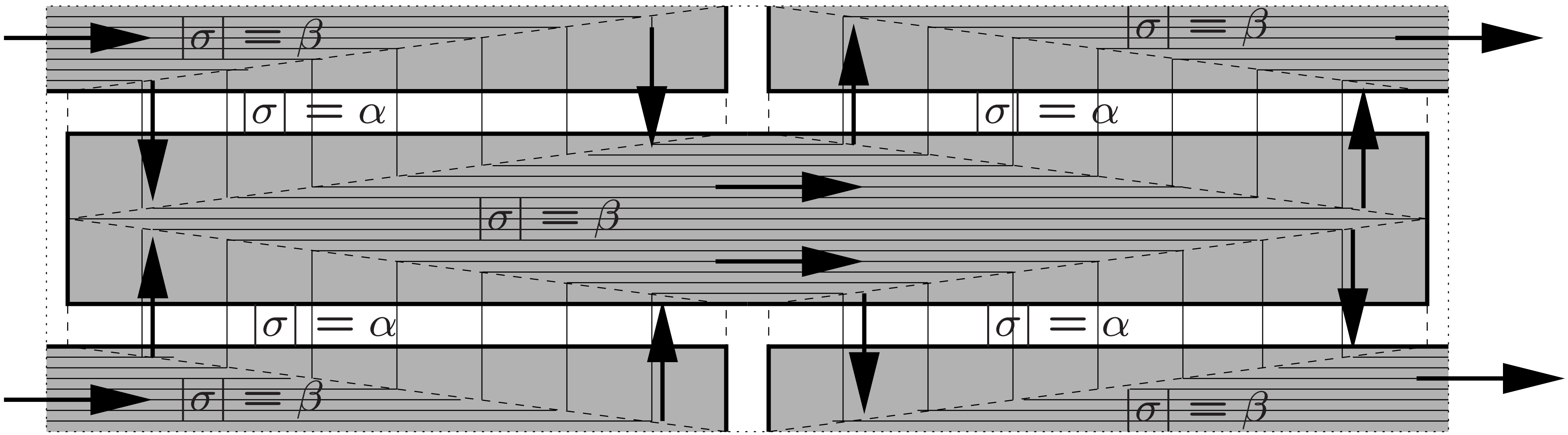
length of $\mathcal{C} \approx \left(\frac{\ell}{2w} + 1\right) \|z\|$

$|\tau \cdot \mathbf{e}_1| \le Y_m \left(\frac{\ell}{2w} + 1\right)$

$$K_{\text{hom}} \cdot \mathbf{e}_1 \subseteq \left[-\frac{\ell}{2w} Y_m, \frac{\ell}{2w} Y_m\right] \qquad \text{if } \varepsilon \ll w \ll \ell$$

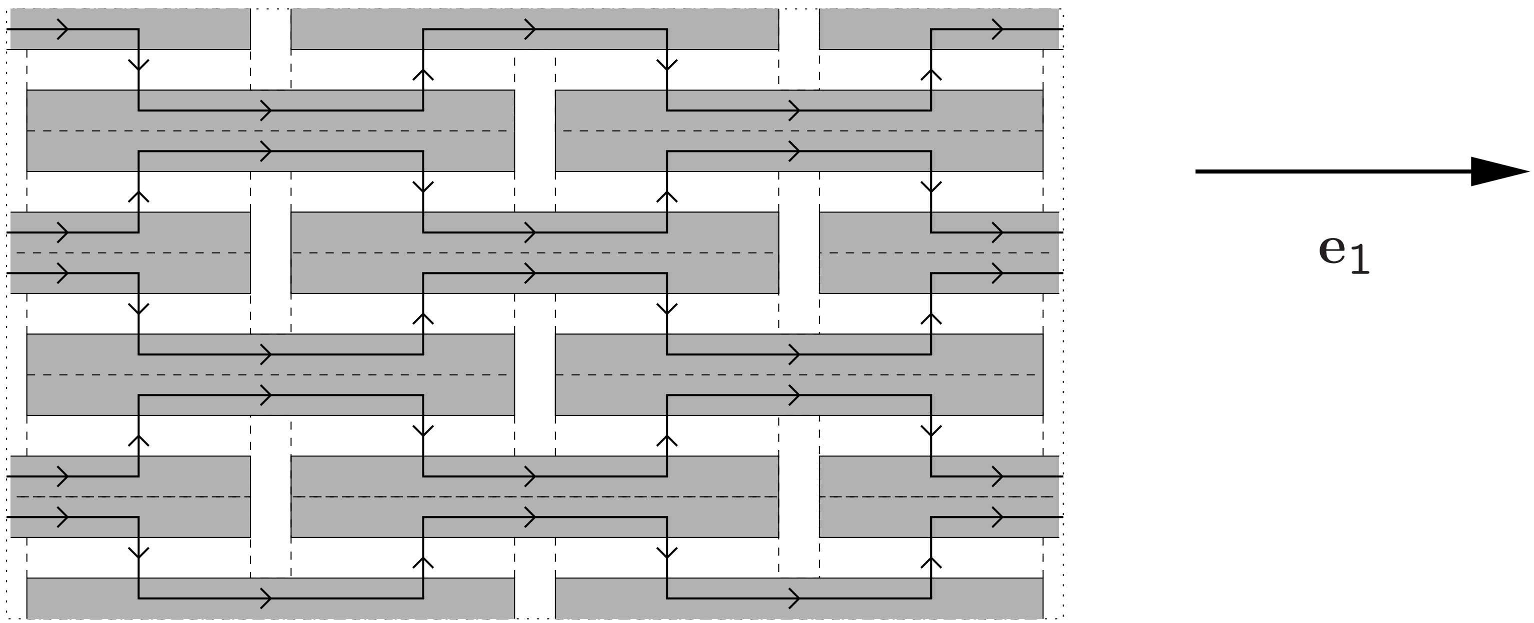
Same example 2. Flow showing that the bound is sharp

Flow in one and four quarter inclusions



$$\alpha = Y_m \quad \text{and} \quad \beta \approx \frac{\ell}{w} Y_m$$

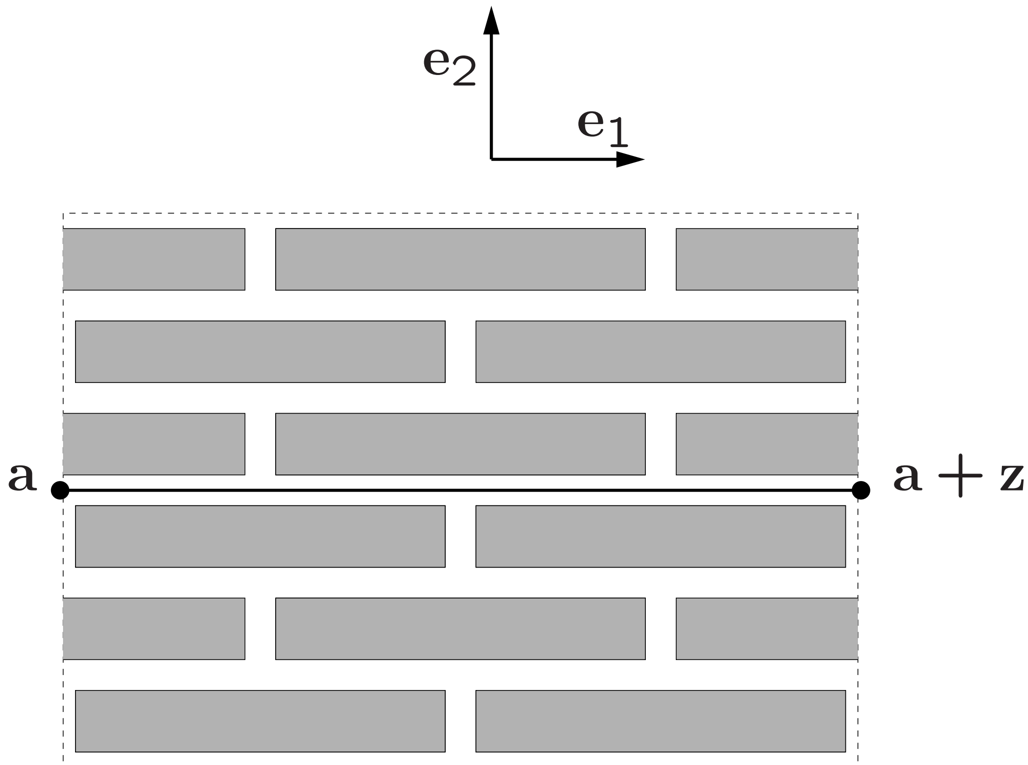
Same example 2. Flow showing that the bound is sharp



where $\|\mathbf{e}_1\| = 1$. Then

$$\tau = \langle \sigma \rangle \approx \frac{\ell}{2w} Y_m \, \mathbf{e}_1$$

Same example 2. The weakest direction



$\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = 1$

$\mathcal{C}$ = solid curve

length of $\mathcal{C} = \|z\|$

$|\tau \cdot \mathbf{e}_2| \leq Y_m$

$$K_{\text{hom}} \cdot \mathbf{e}_2 \subseteq [-Y_m, Y_m]$$

The weakest direction is weak. No good.

A bound on the weakest direction

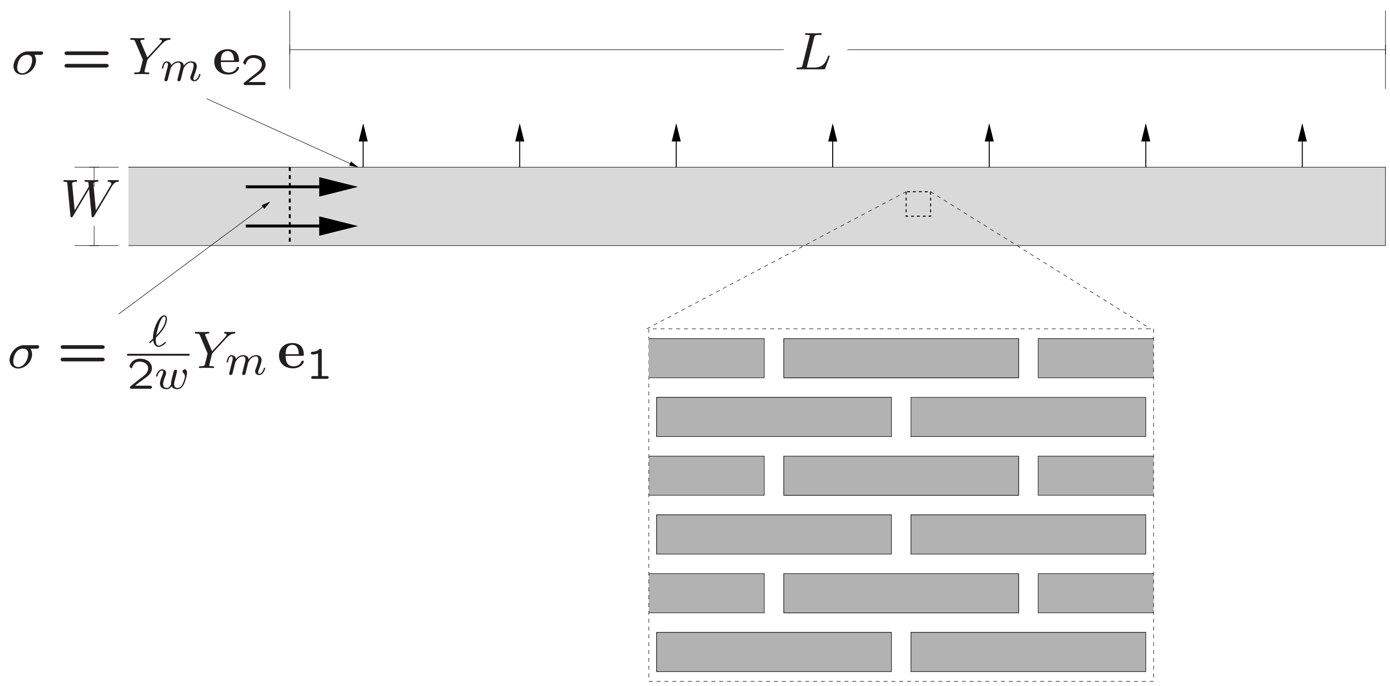
Bound: Let $\rho^\star$ be the strengh in the weakest direction, i.e.

$$\rho^\star = \max\{\rho : B_\rho \subseteq K_{\mathsf{hom}}\}.$$

$$\rho^\star \leq O\left(Y_m\sqrt{\frac{\ell}{w}}\right).$$

Reinforcing a weak matrix to get a composite strong in all directions

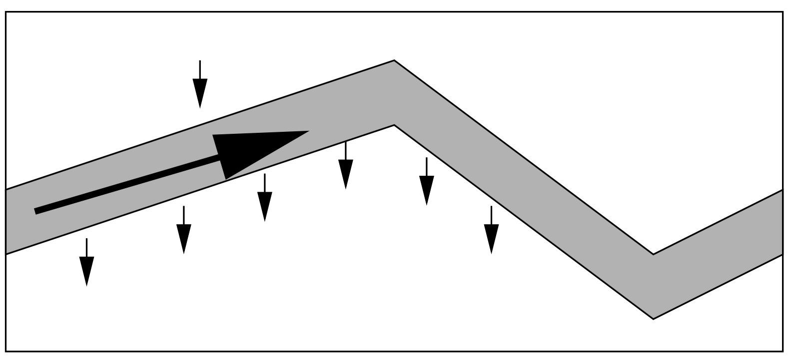
Stripe made of example 2 composite



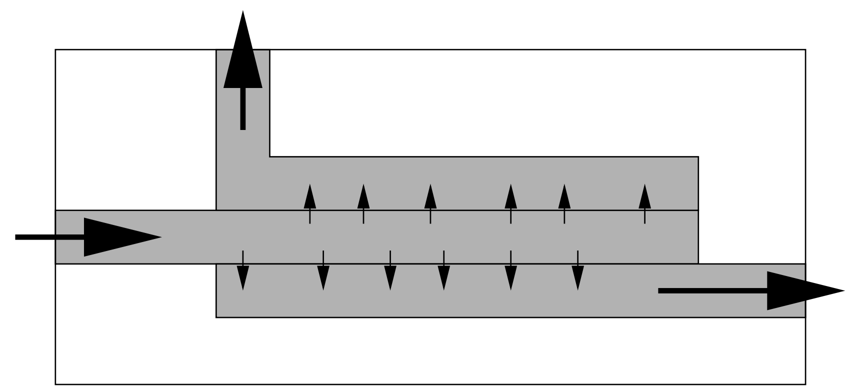
No flow reaches right end if $L \geq \frac{\ell}{2w} W$

Using previous stripes to build composites

Inifinitly long stripe. Flow limited in perpendicular direction

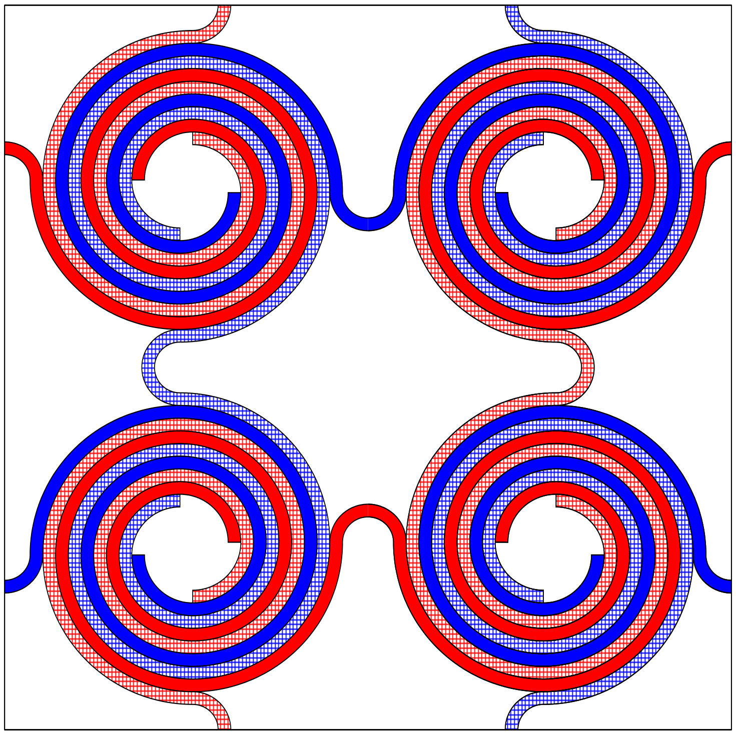


Using finite length stripes. Transfer of flow between them



Need large contact between stripes

Composite made with previous stripes. Strong in all directions



Flow transfer from same color stripes

Average flow $= \frac{Y_m}{2}\sqrt{\frac{\nu\ell}{2w}}(1,1)$

$\nu =$ volume fraction of inclusions
(can be made close to 1)

$$K_{\text{hom}} \supseteq \frac{Y_m}{2}\sqrt{\frac{\nu\ell}{2w}}\,[-1,1]^2$$

Proof of the bound

Observation 1: $v^{\perp} = (-v_2, v_1)$. Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be Q-periodic and divergence free. Then $\langle \sigma^{(1)} \cdot (\sigma^{(2)})^{\perp} \rangle = \langle \sigma^{(1)} \rangle \cdot \langle (\sigma^{(2)})^{\perp} \rangle$.

Observation 2: Let I be and inclusion and σ admissible. Let p_I be the perimeter of I. Then $|\int_I \sigma^{(1)} \cdot (\sigma^{(2)})^{\perp}| \leq (1/2) Y_m^2 p_I^2$.

Observation 3: $\langle \sigma^{(1)} \rangle \cdot \langle (\sigma^{(2)})^{\perp} \rangle \leq (1-\nu) Y_m^2 + |Q|^{-1} (1/2) Y_m^2 \sum_I p_I^2$. Let $\eta = (1/2) \max_I \{p_I^2 / a_I\}$. Then $\langle \sigma^{(1)} \rangle \cdot \langle (\sigma^{(2)})^{\perp} \rangle \leq (1-\nu) Y_m^2 + Y_m^2 \nu \eta$.

Observation 4: Select $\sigma^{(1)} = \rho^{\star}(1,0)$ and $\sigma^{(2)} = \rho^{\star}(0,-1)$. Then $\rho^{\star} \leq Y_m \sqrt{(1-\nu) + \nu\eta}$

Proof of Observation 1

Observation 1: $v^\perp = (-v_2, v_1)$. Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be Q-periodic and divergence free. Then $\langle \sigma^{(1)} \cdot (\sigma^{(2)})^\perp \rangle = \langle \sigma^{(1)} \rangle \cdot \langle (\sigma^{(2)})^\perp \rangle$.

$\sigma^{(i)} = \tau^{(i)} + (\nabla\phi^{(i)})^\perp$ with $\phi^{(i)}$ Q-periodic. Note $\int_Q \tau^{(1)} \cdot \nabla\phi^{(2)} = 0$

$$\int_Q \sigma^{(1)} \cdot (\sigma^{(2)})^\perp = |Q|\tau^{(1)} \cdot (\tau^{(2)})^\perp - \int_Q \nabla\phi^{(2)} \cdot (\nabla\phi^{(1)})^\perp$$

$$\int_Q \nabla\phi^{(2)} \cdot (\nabla\phi^{(1)})^\perp = \int_{\partial Q} \phi^{(2)} \hat{n} \cdot (\nabla\phi^{(1)})^\perp = 0$$

Proof of the Observation 2

Observation 2: Let I be and inclusion and σ admissible. Let p be the perimeter of I. Then $|\int_I \sigma^{(1)} \cdot (\sigma^{(2)})^\perp| \leq (1/2) Y_m^2 p^2$.

$\sigma^{(i)} = (\nabla\psi^{(i)})^\perp$ and $\hat{t} = \hat{n}^\perp$. Parametrize ∂I with $\mathbf{x}(s)$ so that $\|\mathbf{x}'(s)\| = 1$ and $\psi^{(2)}(0) = 0$.

$$\int_I \sigma^{(1)} \cdot (\sigma^{(2)})^\perp = -\int_I \nabla\psi^{(2)} \cdot (\nabla\psi^{(1)})^\perp = -\int_I \nabla \cdot \left(\psi^{(2)}(\nabla\psi^{(1)})^\perp\right) = \int_{\partial I} \psi^{(2)}\ \hat{t} \cdot \nabla\psi^{(1)} = \int_0^p \mathrm{d}s\ \hat{t}(s) \cdot \nabla\psi^{(1)}(s) \int_0^s \mathrm{d}r\ \hat{t}(r) \cdot \nabla\psi^{(2)}(r) = \int_0^p \mathrm{d}s\ \hat{n}(s) \cdot \sigma^{(1)}(s) \int_0^s \mathrm{d}r\ \hat{n}(r) \cdot \sigma^{(2)}(r)$$

$$\left|\int_I \sigma^{(1)} \cdot (\sigma^{(2)})^\perp\right| \leq \int_0^p \mathrm{d}s\ Y_m^2 s = \frac{1}{2} Y_m^2 p^2$$